Some Equitably 2-colorable cycle decompositions of $K_v + I$ *

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Abstract

Let G be a graph in which each vertex has been colored using one of k colors, say $c_1, c_2, \cdots, c_k$. If an m-cycle C in G has n_i vertices colored c_i, $i = 1, 2, \cdots, k$, and $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \cdots, k\}$, then C is equitably k-colored. An m-cycle decomposition $\mathcal{C}$ of a graph G is equitably k-colorable if the vertices of G can be colored so that every m-cycle in $\mathcal{C}$ is equitably k-colored. For $m = 4$, 5 and 6, we completely settle the existence problem for equitably 2-colorable m-cycle decompositions of complete graphs with the edges of a 1-factor added.

Keywords: Graph coloring; cycle decomposition; equitable coloring.
AMS Subject Classification: 05B30

1. Introduction

Let G and H be graphs. A G-decomposition of H is a set $\mathcal{G}=\{G_1,G_2, \cdots,G_p\}$ such that G_i is isomorphic to G for $1 \leq i \leq p$ and $\mathcal{G}$ partitions the

*Research supported by the National Natural Science Foundation of China grant 10471093. Email: lsh751@163.com

edge set of H. Most commonly, $H = K_v$, the complete graph on v vertices. Other popular choices for H are $K_v - I$, the complete graph with the edges of a 1-factor removed, and $K_v + I$, the complete graph with the edges of a 1-factor added.

An m-cycle, denoted by $(x_1, x_2, \cdots, x_m)$, is the graph with vertex set $\{x_1, x_2, \cdots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_m, x_1\}\}$. An m-cycle system of H is a G-decomposition of H where G is an m-cycle. The existence problem for m-cycle system has recently been solved; see [2], [4] and [5].

A *coloring* of an m-cycle decomposition $\mathcal{C}$ of a graph G is an assignment of colors to the vertices of G. A *k-coloring* of $\mathcal{C}$ is a coloring in which k distinct colors are used. A *k-coloring* of an m-cycle decomposition $\mathcal{C}$ induces a coloring of each m-cycle in $\mathcal{C}$. If n_i is the number of vertices colored c_i in an m-cycle $C \in \mathcal{C}$, then C is *equitably k-colored* if $|n_i - n_j| \leq 1$ for any $i, j \in \{1, 2, \cdots, k\}$ and an m-cycle decomposition $\mathcal{C}$ is equitably k-colored if every $C \in \mathcal{C}$ is equitably k-colored. An m-cycle decomposition is *equitably k-colorable* if it can be equitably k-colored.

The existence question for equitably 2-colorable m-cycle decompositions of K_v and $K_v - I$, where $m \in \{4, 5, 6\}$, has been completely settled in [1]. In this paper, we consider the existence of equitably 2-colorable m-cycle decompositions of $K_v + I$. Throughout the paper, we use colors black and white, unless otherwise stated, b and w are used to denote the number of black and white vertices in $K_v + I$. Furthermore, an edge which connects two black (white) vertices is said to be a one-colored black (white) edge, and an edge which connects two differently colored vertices is said to be a two-colored edge. We make frequent use of the following important result.

Lemma 1.1 [5] An m-cycle decomposition of $K_v + I$ exists for all admissible v, that is, for all even v such that $3 \leq m \leq v$ and m divides the number of edges in $K_v + I$.

Our main result is the following theorem which completely settles the existence question for equitably 2-colored m-cycle decompositions of $K_v + I$ for $m \in \{4, 5, 6\}$.

Main Theorem There exists an equitably 2-colorable m-cycle decomposition of $K_v + I$, $m \in \{4, 5, 6\}$, for all *admissible* values of v.

2. Equitably 2-colorable 4-cycle decompositions

Every equitably 2-colored 4-cycle must contain two black and two white vertices, which may only be arranged in two formations; see Figure 1.

Type 1 Type 2

Figure 1: Possible equitable 2-colorings of 4-cycles.

Now we prove the following theorem:

Theorem 2.1 There exists an equitably 2-colorable 4-cycle decomposition of $K_v + I$ if and only if $v \equiv 0 \pmod 4$, $v \geq 4$.

Proof. From Lemma 1.1, a 4-cycle decomposition of $K_v + I$ exists if and only if $v \equiv 0 \pmod 4$, $v \geq 4$. As v is even, we color $v/2$ vertices black and $v/2$ vertices white. Note that there are $\frac{1}{8}v^2$ 4-cycles in any 4-cycle decomposition of $K_v + I$.

Let the vertex set of $K_v + I$ be $\bigcup_{i=0,1} \{0_i, 1_i, \cdots, \frac{1}{2}(v-2)_i\}$. Color the vertices with subscript 0 black and color the vertices with subscript 1 white. Let the edges in I be $\{j_k, (\frac{v-2}{2} - j)_{k+1}\}$, where $j = 0, 1, \cdots, (v-4)/4$, $k \in Z_2$. We obtain $v/4$ cycles of Type 2 (see Figure 1) by forming the 4-cycles

$$(j_0, j_1, (\frac{v-2}{2} - j)_0, (\frac{v-2}{2} - j)_1)$$

for $j = 0, 1, \cdots, (v-4)/4$. We generate the remaining $v(v-2)/8$ cycles of Type 1 (see Figure 1) by forming the 4-cycles

$$(j_0, (j+l)_0, j_1, (j+l)_1)$$

for $j = 0, 1, \cdots, (v-4)/2$ and $l = 1, 2, \cdots, \frac{v-2}{2} - j$. It can be easily checked that this gives an equitably 2-colored 4-cycle decomposition of $K_v + I$. $\Box$

3. Equitably 2-colorable 5-cycle decompositions

When considering cycles of odd length, if b vertices of a cycle are colored black and w vertices white, we cannot have $b = w$. Instead, each cycle within the decomposition must satisfy $|b - w| = 1$. We introduce the following definitions first.

Definition Let v and λ be given positive integers and K be a set of positive integers. A pairwise balanced design (v, K, λ)-PBD is an ordered pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a set of subsets of V (called blocks), such that $|B| \in K$ for each $B \in \mathcal{B}$ and each pair of distinct elements is contained in exactly λ blocks.

If $\lambda = 1$, we simply write (v, K)-PBD for $(v, K, 1)$-PBD. The integer v is the order of the PBD. We denote by $(v, \{k, s^*\})$-PBD a PBD of order v having one block of size s, and the other blocks of size k.

Definition A *group divisible design*, denoted GDD$(K, M; v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a v-set, $\mathcal{G}$ is a set of subsets (called groups) of X, $\mathcal{G}$ partitions X, $\mathcal{B}$ is a set of subsets (called blocks) of X such that

(1) $|G| \in M$ for each $G \in \mathcal{G}$,
(2) $|B| \in K$ for each $B \in \mathcal{B}$,
(3) $|B \cap G| \leq 1$ for each $B \in \mathcal{B}$ and each $G \in \mathcal{G}$,
(4) Each pair of elements of X from distinct groups is contained in a unique block.

When $K = \{k\}$ and $M = \{m\}$, we simply write GDD$(k, m; v)$ for GDD$(\{k\},\{m\};v)$. The following result is well-known, and is useful in proving the main theorem of this section.

Lemma 3.1 [3] For all positive integers x there exists a $(2x + 1, 3)$-PBD or a $(2x + 1, \{3, 5^*\})$-PBD.

Corollary 3.2 For all positive integers x there exists a GDD$(3, 2; 2x)$ or a GDD$(3,\{2, 4^*\}; 2x)$.

Proof. Take a $(2x+1, 3)$-PBD or a $(2x+1, \{3, 5^*\})$-PBD (see Lemma 3.1), and delete an element "a" (for the $(2x + 1, \{3, 5^*\})$-PBD, "a" is contained in the unique block of size 5) and take the truncated blocks as groups. This gives a GDD$(3,2;2x)$ or a GDD$(3,\{2,4^*\};2x)$ respectively. □

We also make use of the following existence results. We use $K_{p(n)}$ to denote the multipartite graph with p parts with n vertices in each part. Although there exists an equitably 2-colored 5-cycle decomposition of $K_{3(5)}$ [1], for convenience we reproduce it here, since this decomposition will be used later in this paper.

Lemma 3.3 [1] There exists an equitably 2-colored 5-cycle decomposition of $K_{3(5)}$.

Proof. Let the vertex set of $K_{3(5)}$ be $\bigcup_{i=1,2,3} \{0_i, 1_i, \cdots, 4_i\}$, with the obvious vertex partition. Color the vertices $0_i, 2_i$ and 4_i black for $i = 1,2,3$ and color the remaining vertices white. A suitable decomposition of $K_{3(5)}$ is given by the following cycles :

$$\begin{array}{lll}
(1_1,1_2,0_1,0_2,0_3), & (1_1,3_2,0_1,2_2,2_3), & (1_1,1_3,0_1,4_2,4_3),\\
(1_1,3_3,0_1,0_3,2_2), & (3_1,1_2,2_1,0_2,2_3), & (3_1,3_2,2_3,0_1,4_3),\\
(3_1,1_3,2_1,4_2,0_3), & (3_1,3_3,2_1,2_3,4_2), & (1_2,1_3,0_2,4_3,4_1),\\
(1_2,3_3,2_2,2_1,0_3), & (3_2,1_3,2_2,4_3,2_1), & (3_2,3_3,4_2,4_1,0_3),\\
(1_1,4_2,1_3,4_1,0_2), & (3_1,0_2,3_3,4_1,2_2), & (1_2,4_3,3_2,4_1,2_3).
\end{array}$$

□

Lemma 3.4 There exists an equitably 2-colored 5-cycle decomposition of $K_{10} + I$.

Proof. Let the vertex set of $K_{10} + I$ be Z_{10}. Color the vertices $0, 1, \cdots, 5$ black and color the vertices $6, 7, 8, 9$ white. Let the edges in I be $\{0,9\}$, $\{1,7\}$, $\{2,4\}$, $\{3,5\}$ and $\{6,8\}$. A suitable decomposition of $K_{10} + I$ is given by the following cycles:

$$\begin{array}{lllll}
(0,1,5,7,8), & (0,3,2,8,9), & (0,5,9,3,6), & (0,7,1,4,9), & (1,2,4,8,6),\\
(1,3,4,7,9), & (1,7,3,5,8), & (2,0,4,6,9), & (2,4,5,6,7), & (3,5,2,6,8).
\end{array}$$

□

Lemma 3.5 There exists an equitably 2-colored 5-cycle decomposition of $K_{20} + I$.

Proof. Let the vertex set of $K_{20} + I$ be Z_{20}. Color the vertices $0, 1, \cdots, 11$ black and color the vertices $12, 13, \cdots, 19$ white. Let the edges in I be $\{0,16\}$, $\{1,17\}$, $\{2,6\}$, $\{3,4\}$, $\{5,15\}$, $\{7,10\}$, $\{8,13\}$, $\{9,19\}$, $\{11,18\}$

and $\{12, 14\}$. A suitable decomposition of $K_{20} + I$ is given by the following cycles:

$(12,13,11,10,0)$,	$(13,14,0,2,5)$,	$(14,16,4,2,9)$,	$(15,5,17,2,8)$,
$(12,14,3,1,2)$,	$(13,15,0,4,6)$,	$(14,17,7,0,8)$,	$(16,17,4,5,11)$,
$(12,14,4,3,10)$,	$(13,16,2,6,10)$,	$(14,18,2,7,10)$,	$(16,18,5,7,8)$,
$(12,15,1,4,3)$,	$(13,17,1,6,9)$,	$(14,19,7,4,11)$,	$(16,19,6,8,10)$,
$(12,16,0,3,5)$,	$(13,18,3,11,1)$,	$(14,5,16,3,2)$,	$(16,0,11,18,1)$,
$(12,17,0,5,1)$,	$(13,19,1,7,3)$,	$(15,16,9,5,6)$,	$(16,7,10,17,6)$,
$(12,18,0,6,11)$,	$(13,0,9,19,8)$,	$(15,17,8,5,10)$,	$(17,18,11,7,9)$,
$(12,19,2,10,4)$,	$(13,2,6,18,4)$,	$(15,18,8,3,9)$,	$(17,19,10,9,11)$,
$(12,6,14,1,8)$,	$(13,7,18,9,8)$,	$(15,19,4,8,11)$,	$(17,3,19,9,1)$,
$(12,9,4,15,7)$,	$(14,15,3,6,7)$,	$(15,2,11,19,5)$,	$(18,19,0,1,10)$.

□

Theorem 3.6 There exists an equitably 2-colorable 5-cycle decomposition of $K_v + I$ if and only if $v \equiv 0 \pmod{10}$, $v \geq 10$.

Proof. By Lemma 1.1, a 5-cycle decomposition of $K_v + I$ exists if and only if $v \equiv 0 \pmod{10}$, $v \geq 10$. Let $v = 10x, x \geq 1$. By Corollary 3.2, we can take either a GDD$(3,2;2x)$ or a GDD$(3,\{2,4^*\};2x)$ and simultaneously construct $K_v + I$ and its equitably 2-colored 5-cycle decomposition as follows. Replace each element of the design with five vertices, coloring three vertices black and two vertices white. Within each set of vertices arising from a group of size 2 of the design, let there be two one-colored black edges and one one-colored white edge and two two-colored edges in I. Similarly, within any set of vertices arising from a group of size 4, let there be three one-colored black edges and one one-colored white edge and six two-colored edges in I.

By Lemma 3.3, we can place an equitably 2-colored 5-cycle decomposition of $K_{3(5)}$ on each set of vertices arising from a block of the design. Furthermore, by Lemmas 3.4 and 3.5, we can place an equitably 2-colored 5-cycle decomposition of $K_{10} + I$ or $K_{20} + I$ on g for each set of vertices g arising from a group of the design of size 2 or 4 respectively. It is not difficult to check that the result is an equitably 2-colored 5-cycle decomposition of $K_v + I$. □

4. Equitably 2-colorable 6-cycle decompositions

For 6-cycles, we proceed in much the same manner as for 4-cycles. We use the following existence results when proving Theorem 4.3.

Lemma 4.1 There exists an equitably 2-colored 6-cycle decomposition of $K_6 + I$.

Proof. Let the vertex set of K_6+I be Z_6. Color the vertices 0,1 and 2 black and color the vertices 3,4 and 5 white. Let the edges in I be $\{0,3\}, \{1,2\}$, and $\{4,5\}$. A suitable decomposition of $K_6 + I$ is given by the following cycles:

$$\{0,1,2,5,4,3\}, \quad \{0,2,4,1,3,5\}, \quad \{0,4,5,1,2,3\}.$$

□

Lemma 4.2 [1] There exists an equitably 2-colored 6-cycle decomposition of $K_{6,6}$.

Proof. Let the vertex set of $K_{6,6}$ be $\bigcup_{i=1,2} \{0_i, 1_i, \cdots, 5_i\}$, with the obvious vertex partition. Color the vertices $0_i, 2_i$ and 4_i black, for $i = 1,2$, and color the remaining vertices white. A suitable decomposition of $K_{6,6}$ is given by the following cycles:

$$(0_1,0_2,4_1,5_2,1_1,3_2), \quad (2_1,2_2,0_1,1_2,3_1,5_2), \quad (4_1,4_2,2_1,3_2,5_1,1_2),$$
$$(4_1,2_2,1_1,0_2,3_1,3_2), \quad (0_1,4_2,3_1,2_2,5_1,5_2), \quad (2_1,0_2,5_1,4_2,1_1,1_2).$$

□

Let G and H be graphs. The join of G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u,v\} | u \in V(G) \text{ and } v \in V(H)\}$. Now we prove the following:

Theorem 4.3 There exists an equitably 2-colorable 6-cycle decomposition of $K_v + I$ if and only if $v \equiv 0 \pmod 6$, $v \geq 6$.

Proof. By Lemma 1.1, a 6-cycle decomposition of $K_v + I$ exists if and only if $v \equiv 0 \pmod 6$, $v \geq 6$. Let $v = 6x, x \geq 1$. Let the vertex set of $K_v + I$ be $\bigcup_{i=1,\cdots,x} V_i$, where $V_i = \{0_i, 1_i, \cdots, 5_i\}$. Color the vertices $0_i, 1_i$ and 2_i black for $i = 1, 2, \cdots, x$, and color the remaining vertices white. Let the edges in I be $\{0_i, 3_i\}, \{1_i, 2_i\}$, and $\{4_i, 5_i\}$, for $i = 1, 2, \cdots, x$.

By Lemma 4.2 we can place an equitably 2-colored 6-cycle decomposition of $K_{6,6}$ on $V_i \vee V_j$, for $1 \leq i < j \leq x$. By Lemma 4.1, we can place an equitably 2-colored 6-cycle decomposition of $K_6 + I$ on V_i for $1 \leq i \leq x$. It is not difficult to check that the result is an equitably 2-colored 6-cycle decomposition of $K_v + I$. □

5. Conclusion

By Theorems 2.1, 3.6 and 4.3, we have the main result of this paper.

Theorem 5.1 For $m \in \{4, 5, 6\}$, there exists an equitably 2-colorable m-cycle decomposition of $K_v + I$ for all *admissible* values of v.

As a consequence of Theorem 5.1, we have the following corollary:

Corollary 5.2 For $m \in \{4, 5, 6\}$, there exists a 2-colorable m-cycle decomposition of $K_v + I$ for all *admissible* values of v.

Acknowledgements

The authors would like to thank Doctor Mary Waterhouse for her kindness in providing one of the references.

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