ONE RECURSION FORMULA OF SECOND-ORDER RECURRENT SEQUENCES

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Abstract: Let $\{w_n\}$ be a second order recurrence sequence. According to the definition and characteristics of the recurrent sequence, we proved a recursion formula for certain reciprocal sums whose denominators are products of consecutive elements of $\{w_n\}$.
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1 Introduction

Let $\mathbf{Z}$ and $\mathbf{R}$ denote the ring of the integers and the field of real numbers, respectively. For a field $\mathbf{F}$, we put $\mathbf{F}^* = \mathbf{F} \setminus \{0\}$. Fix $A \in \mathbf{R}$ and $B \in \mathbf{R}^*$, and let $\mathcal{L}(A, B)$ consist of all those second-order recurrent sequences $\{w_n\}_{n\in\mathbf{Z}}$ of complex numbers satisfying the recursion:

$$w_{n+2} = Aw_{n+1} - Bw_n(\; i.e. Bw_n = Aw_{n+1} - w_{n+2}) \quad \text{for } n = 0, \pm1, \pm2, \cdots . \tag{1}$$

For sequences in $\mathcal{L}(A, B)$, the corresponding characteristic equation is $x^2 - Ax + B = 0$, whose roots $(A \pm \sqrt{A^2 - 4B})/2$ are denoted by α and β. If $A \in \mathbf{R}$ and $\Delta = A^2 - 4B \geq 0$, then we have

$$\alpha = \frac{A - sg(A)\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A + sg(A)\sqrt{\Delta}}{2},$$

where $sg(A) = 1$ if $A > 0$, and $sg(A) = -1$ if $A < 0$.

The Lucas sequences $\{u_n\}_{n\in\mathbf{Z}}$ and $\{v_n\}_{n\in\mathbf{Z}}$ in $\mathcal{L}(A, B)$ take special values at $n = 0, 1$, namely,

$$u_0 = 0, \quad u_1 = 1, \quad v_0 = 2, \quad v_1 = A. \tag{2}$$

If $A = 1$ and $B = -1$, then those $F_n = u_n$ and $L_n = v_n$ are called Fibonacci numbers and Lucas numbers, respectively.

Let a, b, m, n, k be integers with $a \neq 0$ and let $f(n) = an + b$. If $w_{f(n)} \neq 0$ for all $n = 1, 2, \cdots$, the sum is defined as follows:

$$T_{m,k} = \sum_{n=1}^{\infty} \frac{B^{kf(n)}}{\prod_{i=0}^{m} w_{f(n+i)}}. \tag{3}$$

In [1] Brousseau proved $T_{2,-1} = \frac{5}{12} - \frac{3}{2}T_{4,0}$, $T_{4,0} = \frac{97}{2640} - \frac{40}{11}T_{6,1}$ and $T_{6,-1} = \frac{589}{1900080} - \frac{273}{29}T_{8,0}$ when $f(n) = n$ and $\{w_n\} = \{F_n\}$. Under same condition, Melham showed $T_{m,-1} = r_1 + r_2 T_{m+2,0}$ and $T_{m,0} = r_3 + r_4 T_{m+2,1}$ in [5], where r_i are rational numbers that depend on m. In this paper we obtain the following theorem.

Theorem 1.1 Let a, b, k be integers, and m, n be positive integers. Let $f(n) = an + b$ with $a \neq 0$. If $w_{f(n)} \neq 0$ for all $n = 1, 2, \cdots$,

$$\begin{aligned} T_{m+2,k+1} &= \frac{B^{kb}[B^{a(m-k+1)} w_{f(m+2)} - w_{f(2m+3)}]}{e u_{a(m+1)} u_{a(m+2)} \prod_{i=1}^{m+2} w_{f(i)}} \\ &- \frac{B^{ak} + B^{a(m-k+1)} - v_{a(m+1)}}{e B^{ak} u_{a(m+1)} u_{a(m+2)}} T_{m,k} \end{aligned} \tag{4}$$

where $e = w_0 w_2 - w_1^2$.

Remark 1.1 Theorem of Melham [5] is essentially our (4) in the special case $a = 1, b = 0, A = 1, B = -1, k = 0, k = 1$ and $\{w_n\} = \{F_n\}$.

2 Some Lemmas

To complete the proof of Theorem 1.1, we need the following two lemmas:

Lemma 2.1 Let m and n be non-negative integers, then we have

$$\begin{aligned} & w_{f(n+m)}w_{f(n+m+2)} - B^{ak}w_{f(n)}w_{f(n+m+1)} \\ = \; & B^{a(k-m-1)}u_a^{-1}u_{a(m+1)}w_{f(n+m+1)}w_{f(n+m+2)} \\ + \; & (1 - B^{a(k-m-1)}u_a^{-1}u_{a(m+2)})w_{f(n+m)}w_{f(n+m+2)} \\ + \; & eB^{f(n+k-1)}u_a u_{a(m+2)}. \end{aligned} \tag{5}$$

Proof The following identity is well known (see [4] and [7]) that

$$B^{a(m+1)}u_a w_{f(n)} = w_{f(n+m+1)}u_{a(m+2)} - w_{f(n+m+2)}u_{a(m+1)}, \tag{6}$$

$$w_{f(n)} = B^{-a(m+1)}u_a^{-1}[w_{f(n+m+1)}u_{a(m+2)} - w_{f(n+m+2)}u_{a(m+1)}] \tag{7}$$

and

$$w_{f(n+m+1)}^2 = w_{f(n+m)}w_{f(n+m+2)} - eB^{f(n+m)}u_a^2. \tag{8}$$

Thus, we find that

$$\begin{aligned} & B^{ak}w_{f(n)}w_{f(n+m+1)} \\ = \; & B^{ak}w_{f(n+m+1)}B^{-a(m+1)}u_a^{-1}(w_{f(n+m+1)}u_{a(m+2)} - w_{f(n+m+2)}u_{a(m+1)}) \\ = \; & B^{a(k-m-1)}u_a^{-1}(w_{f(n+m+1)}^2u_{a(m+2)} - w_{f(n+m+1)}w_{f(n+m+2)}u_{a(m+1)}) \\ = \; & B^{a(k-m-1)}u_a^{-1}(w_{f(n+m)}w_{f(n+m+2)}u_{a(m+2)} \\ & -eB^{f(n+m)}u_a^2u_{a(m+2)} - w_{f(n+m+1)}w_{f(n+m+2)}u_{a(m+1)}) \end{aligned} \tag{9}$$

and hence

$$\begin{aligned} & w_{f(n+m)}w_{f(n+m+2)} - B^{ak}w_{f(n)}w_{f(n+m+1)} \\ = \; & B^{a(k-m-1)}u_a^{-1}u_{a(m+1)}w_{f(n+m+1)}w_{f(n+m+2)} \\ + \; & (1 - B^{a(k-m-1)}u_a^{-1}u_{a(m+2)})w_{f(n+m)}w_{f(n+m+2)} \\ + \; & eB^{f(n+k-1)}u_a u_{a(m+2)}. \end{aligned}$$

This proves Lemma 2.1. □

Lemma 2.2 Let b, k be integers, and m, n, a be positive integers. Let $f(n) = an + b$. If $w_{f(n)} \neq 0$ for all $n = 1, 2, \cdots$,

$$\sum_{n=1}^{\infty} \frac{B^{f(n)}}{w_{f(n+m+1)}\prod_{i=n}^{n+m-1} w_{f(i)}} = \frac{-B^{am+kb}u_a}{u_{a(m+1)}\prod_{i=1}^{m+1} w_{f(i)}} + \frac{B^{a(m-k)}u_a + u_{am}}{u_{a(m+1)}}T_{m,k}. \tag{10}$$

Proof For k be an integer, and m and n be positive integers, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{B^{f(n)}}{w_{f(n+m+1)}\prod_{i=n}^{n+m-1}w_{f(i)}}-\frac{B^{a(m-k)}u_a+u_{am}}{u_{a(m+1)}}T_{m,k}\\
=&\sum_{n=1}^{\infty}\frac{B^{kf(n)}[w_{f(n+m)}u_{a(m+1)}-w_{f(n+m+1)}u_{am}-B^{a(m-k)}u_aw_{f(n+m+1)}]}{u_{a(m+1)}\prod_{i=n}^{n+m+1}w_{f(i)}}\\
=&\sum_{n=1}^{\infty}\frac{B^{kf(n)}[B^{am}u_aw_{f(n)}-B^{a(m-k)}u_aw_{f(n+m+1)}]}{u_{a(m+1)}\prod_{i=n}^{n+m+1}w_{f(i)}}\\
=&\frac{B^{am}u_a}{u_{a(m+1)}}\sum_{n=1}^{\infty}\frac{B^{kf(n)}w_{f(n)}-B^{kf(n-1)}w_{f(n+m+1)}}{\prod_{i=n}^{n+m+1}w_{f(i)}}\\
=&\frac{B^{am}u_a}{u_{a(m+1)}}[\sum_{n=1}^{\infty}\frac{B^{kf(n)}}{\prod_{i=n+1}^{n+m+1}w_{f(i)}}-\sum_{n=1}^{\infty}\frac{B^{kf(n-1)}}{\prod_{i=n}^{n+m}w_{f(i)}}]\\
=&\frac{-B^{am+kb}u_a}{u_{a(m+1)}\prod_{i=1}^{m+1}w_{f(i)}}.
\end{aligned} \tag{11}$$

This completes the proof. □

3 Proof of Theorem 1.1

Let k be an integer, and m be a positive integer. We define

$$\sum=\sum_{n=1}^{\infty}\frac{B^{kf(n)}}{w_{f(n+m+1)}\prod_{i=n}^{n+m-1}w_{f(i)}}-\sum_{n=1}^{\infty}\frac{B^{kf(n+1)}}{w_{f(n+m+2)}\prod_{i=n+1}^{n+m}w_{f(i)}}. \tag{12}$$

Then, we get

$$\sum=\frac{B^{ak+bk}}{w_{f(n+m+2)}\prod_{i=1}^{m}w_{f(i)}}. \tag{13}$$

By Lemma 2.1 and Lemma 2.2, we obtain

$$\sum=\sum_{n=1}^{\infty}B^{kf(n)}\frac{w_{f(n+m)}w_{f(n+m+2)}-B^{ak}w_{f(n)}w_{f(n+m+1)}}{\prod_{i=n}^{n+m+2}w_{f(i)}}$$

$$= \sum_{n=1}^{\infty} \frac{B^{kf(n)}}{\prod_{i=n}^{n+m+2} w_{f(i)}} [B^{a(k-m-1)} u_{a(m+1)} u_a^{-1} w_{f(n+m+1)} w_{f(n+m+2)}$$

$$+ w_{f(n+m)} w_{f(n+m+2)} (1 - B^{a(k-m-1)} u_a^{-1} u_{a(m+2)}) + eB^{f(n+k-1)} u_a u_{a(m+2)}]$$

$$= \frac{B^{a(k-m-1)} u_{a(m+1)}}{u_a} T_{m,k} + eB^{a(k-1)} u_a u_{a(m+2)} T_{m+2,k+1}$$

$$+ \frac{u_a - B^{a(k-m-1)} u_{a(m+2)}}{u_a} \sum_{n=1}^{\infty} \frac{B^{kf(n)}}{w_{f(n+m+1)} \prod_{i=n}^{n+m-1} w_{f(i)}}$$

$$= \frac{B^{a(k-m-1)} u_{a(m+1)}}{u_a} T_{m,k} + eB^{a(k-1)} u_a u_{a(m+2)} T_{m+2,k+1}$$

$$+ \frac{u_a - B^{a(k-m-1)} u_{a(m+2)}}{u_a} [\frac{-B^{am+kb} u_a}{u_{a(m+1)} \prod_{i=1}^{m+1} w_{f(i)}} + \frac{B^{a(m-k)} u_a + u_{am}}{u_{a(m+1)}} T_{m,k}]$$

$$= \frac{B^{a(k-m+1)} u_{a(m+1)}^2 - B^{a(k-m-1)} u_{am} u_{a(m+2)} + B^{a(m-k)} u_a^2}{u_a u_{a(m+1)}} T_{m,k}$$

$$- \frac{u_a [B^{-a} u_{a(m+1)} - u_{am}]}{u_a u_{a(m+1)}} T_{m,k} + eB^{a(k-1)} u_a u_{a(m+2)} T_{m+2,k+1}$$

$$- \frac{B^{am+kb} u_a - B^{ak-a+kb} u_{a(m+2)}}{u_{a(m+1)} \prod_{i=1}^{m+1} w_{f(i)}}$$

$$= \frac{u_a [B^{ak} + B^{a(m-k+1)} - v_{a(m+1)}]}{B^a u_{a(m+1)}} T_{m,k} - \frac{B^{am+kb} u_a - B^{ak-a+kb} u_{a(m+2)}}{u_{a(m+1)} \prod_{i=1}^{m+1} w_{f(i)}}$$

$$+ eB^{a(k-1)} u_a u_{a(m+2)} T_{m+2,k+1}. \tag{14}$$

Thus,

$$eB^{a(k-1)} u_a u_{a(m+2)} T_{m+2,k+1}$$

$$= \frac{B^{ak+bk}}{w_{f(m+2)} \prod_{i=1}^{m} w_{f(i)}} + \frac{B^{am+bk} u_a - B^{ak-a+bk} u_{a(m+2)}}{u_{a(m+1)} \prod_{i=1}^{m+1} w_{f(i)}}$$

$$- \frac{u_a [B^{a(k-1)} + B^{a(m-k)}] - B^{-a} u_a v_{a(m+1)}}{u_{a(m+1)}} T_{m,k}$$

$$= \frac{B^{bk}[B^{ak}w_{f(m+1)}u_{a(m+1)} - B^{ak-a}w_{f(m+2)}u_{a(m+2)} + B^{am}w_{f(m+2)}u_a]}{u_{a(m+1)}\prod_{i=1}^{m+2} w_{f(i)}}$$
$$- \frac{u_a[B^{a(k-1)} + B^{a(m-k)}] - B^{-a}u_a v_{a(m+1)}}{u_{a(m+1)}} T_{m,k}. \tag{15}$$

Now, using the identity

$$u_a w_{f(2m+3)} = u_{a(m+2)}w_{f(m+2)} - B^a u_{m+1} w_{f(m+1)}, \tag{16}$$

we have

$$eB^{a(k-1)}u_{a(m+2)}T_{m+2,k+1} + \frac{[B^{a(k-1)} + B^{a(m-k)}] - B^{-a}v_{a(m+1)}}{u_{a(m+1)}} T_{m,k}$$
$$= \frac{B^{am+bk}w_{f(m+2)} - B^{ak-a+bk}w_{f(2m+3)}}{u_{a(m+1)}\prod_{i=1}^{m+2} w_{f(i)}}. \tag{17}$$

So

$$T_{m+2,k+1} = \frac{B^{kb}[B^{a(m-k+1)}w_{f(m+2)} - w_{f(2m+3)}]}{eu_{a(m+1)}u_{a(m+2)}\prod_{i=1}^{m+2} w_{f(i)}} - \frac{B^{ak} + B^{a(m-k+1)} - v_{a(m+1)}}{eB^{ak}u_{a(m+1)}u_{a(m+2)}} T_{m,k}. \tag{18}$$

The proof is now completed. □

4 Corollaries of the Theorem 1.1

If $A, B \in R^*$, $A^2 \geq 4B$, $w_1 \neq \alpha w_0$, and $w_n \neq 0$ for all $n \geq 1$, by Theorem 2 of [4] we have

$$\begin{aligned} T_{1,1} &= \sum_{n=1}^{\infty} \frac{B^{f(n)}}{w_{f(n)}w_{f(n+1)}} = \frac{\alpha^b}{(w_1 - \alpha w_0)w_b u_a} - \frac{\beta^b}{w_b w_{a+b}} \\ &= \frac{\alpha^b}{w_b}\left(\frac{1}{w_1 u_a - \alpha w_0 u_a} - \frac{\beta^b}{w_{a+b}}\right). \end{aligned} \tag{19}$$

By Theorem 1.1 we obtain following results.

Corollary 4.1 If $A, B \in R^*$, $A^2 \geq 4B$, $w_1 \neq \alpha w_0$, and $w_{f(n)} \neq 0$ for all

$n = 1, 2, \cdots$, in the case $k = 1$ and $m = 1$, we have

$$\begin{aligned} & T_{3,2} \\ = & \sum_{n=1}^{\infty} \frac{B^{2f(n)}}{w_{f(n)} w_{f(n+1)} w_{f(n+2)} w_{f(n+3)}} \\ = & \frac{B^b [B^a w_{f(3)} - w_{f(5)}]}{e w_{f(1)} w_{f(2)} w_{f(3)} u_{2a} u_{3a}} - \frac{2B^a - v_{2a}}{e B^a u_{2a} u_{3a}} \left[\frac{\alpha^b}{(w_1 - \alpha w_0) w_b u_a} - \frac{\beta^b}{w_b w_{a+b}} \right] \end{aligned} \tag{20}$$

Remark 4.2 (3.10) of Melham [6] is essentially our (20) in the special case $f(n) = n$, $w_0 = 0$, $w_1 = 1$ and $w_n = 3w_{n-1} - w_{n-2} = F_{2n}$.

Example 4.3 Let $f(n) = n$, in the case $\{w_n\} = \{F_n\}$ and $\{w_n\} = \{L_n\}$, (20) turns out to be

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3}} = \frac{12 - 5\sqrt{5}}{4}, \tag{21}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1} L_{n+2} L_{n+3}} = \frac{5 - 2\sqrt{5}}{40}. \tag{22}$$

Corollary 4.4 If $A, B \in R^*$, $A^2 \geq 4B$, $w_1 \neq \alpha w_0$, and $w_n \neq 0$ for all $n = 1, 2, \cdots$, let $f(n) = n$, in the case $k = 2$ and $m = 3$, (4) says that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{B^{3(n-1)}}{\prod_{i=n}^{n+5} w_i} - \frac{B^2 w_5 - w_9}{e B^3 u_4 u_5 \prod_{i=1}^{5} w_i} \\ = & \frac{2B^2 - v_4}{e^2 B^5 w_1 \prod_{i=2}^{5} u_i} \times \left(\frac{2B - v_2}{\beta w_1 (w_1 - \alpha w_0)} - \frac{B w_3 - w_5}{w_1 w_2 w_3} \right). \end{aligned} \tag{23}$$

Example 4.5 In the case $\{w_n\} = \{F_n\}$ and $\{w_n\} = \{L_n\}$, (23) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=n}^{n+5} F_i} = \frac{421}{450} - \frac{5\sqrt{5}}{12}, \tag{24}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=n}^{n+5} L_i} = \frac{\sqrt{5}}{300} - \frac{41}{5544}. \tag{25}$$

In the case $\{w_n\} = \{F_{2n}\}$ and $\{w_n\} = \{L_{2n}\}$, (23) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=n}^{n+5} F_{2i}} = \frac{2301 - 700\sqrt{5}}{172480}, \tag{26}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=n}^{n+5} L_{2i}} = \frac{1}{385}\left(\frac{1741}{35532} - \frac{\sqrt{5}}{80}\right). \tag{27}$$

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