A note on the total domination vertex critical graphs*

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Abstract

In 2006, Mojdeh and Jafari rad [On the total domination critical graphs, Electronic Notes in Discrete Mathematics, 24 (2006), 89-92] gave an open problem: Does there exists a 3-γ_t-critical graph G of order $\Delta(G)+3$ with $\Delta(G)$ odd and $\delta(G)\geq 2$? In this paper, we positively answer that for each odd integer $n\geq 9$, there exists a 3-γ_t-critical graphs G_n of order $n+3$ with $\delta(G)\geq 2$. On contrary, we also prove that for $\Delta(G)=3,5,7$, there is no 3-γ_t-critical graph of order $\Delta(G)+3$ with $\delta(G)\geq 2$.

Key words: total domination number, total domination vertex critical.

1 Introduction

A domination and its variations in graphs have been studied extensively [3, 4]. The literature on this subject has been well surveyed in the two books by Haynes, Hedetniemi and Slater [1,2]. For notation and terminology in graph theory, we refer to [1]. Let $G=(V(G),E(G))$ be a simple graph of order $n(G)$. A subset $S\subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted

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by $\gamma(G)$, is the minimum cardinality of dominating sets. A subset $S \subseteq V$ is a *total dominating set* of G if every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of total dominating sets. A total dominating set of cardinality $\gamma_t(G)$ is called a *$\gamma_t(G)$-set.*

The degree, neighborhood and closed neighborhood of a vertex v in a graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. For a subset S of V, we set $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. The minimum degree and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *cycle*, *path* and *complete graph* on n vertices are denoted by C_n, P_n and K_n, respectively. A vertex of degree one is called a *leaf.* A vertex v of G is called a *support vertex* if it is adjacent to a leaf. Let $S(G)$ be the set of all support vertices of G. The *corona* of a graph H, denoted by $cor(H)$, is the graph obtained from H by adding a leaf edge to each vertex of H.

Goddard et al. introduced the concept of total domination critical graphs [3]. A graph G with no isolated vertex is *total domination vertex critical* if for any vertex v of G that is not adjacent to a leaf and the total domination number of $G - v$ is less than the total domination number of G. Such a graph is called *γ_t-critical.* If such a graph G has total domination number k, we call it *k-γ_t-critical.* They gave the following open problem:

Problem 1. *Characterize γ_t-critical graphs G with order $\Delta(G) + \gamma_t(G)$.*

Mojdeh and Jafari rad obtained some partial results on Problem 1 [4]. Furthermore, they suggested the following open problem.

Problem 2. *Does there exist a 3-γ_t-critical graph G of order $\Delta(G) + 3$ with $\Delta(G)$ odd and $\delta(G) \geq 2$?*

In this paper, we prove that there is no 3-γ_t-critical graph of order $\Delta(G) + 3$ with $\Delta(G) = 7$ and $\delta(G) \geq 2$. We find a family of 3-γ_t-critical graphs of order $\Delta(G) + 3$ with $\Delta(G) \geq 9$, $\delta(G) \geq 2$ and $\Delta(G)$ odd. Thus, the Problem 2 is positively answered.

2 Main results

To prove the main result, we review the following lemmas from [3].

Lemma 3 (Goddard et al. [3]). *If G is a γ_t-critical graph, then $\gamma_t(G-v) = \gamma_t(G) - 1$ for every $v \in V - S(G)$. Furthermore, a $\gamma_t(G - v)$-set contains no neighbor of v.*

Lemma 4 (Goddard et al. [3]). *If a graph G has nonadjacent vertices u and v with $v \notin S(G)$ and with $N(u) \subseteq N(v)$, then G is not γ_t-critical.*

Mojdeh and Jafari rad [4] found the following lemma about total domination vertex critical graph G of order $\Delta(G) + \gamma_t(G)$ with $\delta(G) \geq 2$.

Lemma 5 (Mojdeh and Jafari rad [4]). *There is no 3-γ_t-critical graph G of order $\Delta(G) + 3$ with $\Delta(G) = 3, 5$ and $\delta(G) \geq 2$.*

By combing these lemmas, first we obtain the following theorem.

Theorem 6. *There is no 3-γ_t-critical graph G of order $\Delta(G) + 3$ with $\Delta(G) \leq 7$, $\delta(G) \geq 2$ and $\Delta(G)$ odd.*

Proof. By Lemma 5, it suffices to show for the case of $\Delta(G) = 7$. To the contrary, suppose that G is a 3-γ_t-critical graph of order $\Delta(G) + 3$ with $\Delta(G) = 7$ and $\delta(G) \geq 2$. For any vertex $u \in V(G)$, let S_u be a $\gamma_t(G - u)$-set. Choose $v \in V(G)$ such that $d(v) = \Delta(G)$. Since $n(G) = \Delta(G) + 3$, we can assume that $V(G) - N[v] = \{u, w\}$. Since G is 3-γ_t-critical, by Lemma 3, it follows that $S_v = \{u, w\}$ and $N(u) \cup N(w) - \{u, w\} = N(v)$. Furthermore, $N(u) \cap N(w) = \emptyset$. Otherwise, say $t \in N(u) \cap N(w)$, then $\{v, t\}$ is a $\gamma_t(G)$-set, which is a contradiction. It divides into three cases depending on $|N(u) \cap N(v)|$.

Case 1. $|N(u) \cap N(v)| = 1$. *Let $N(u) \cap N(v) = \{u_1\}$ and $N(w) \cap N(v) = \{w_1, \cdots, w_6\}$. Then $u_1w_j \notin E(G)$ for $j = 1, 2, \cdots, 6$. Otherwise, if there exists a vertex w_j such that $u_1w_j \in E(G)$, then $\{w, w_j\}$ is a $\gamma_t(G)$-set, which is a contradiction. For any $\gamma_t(G - w_1)$-set S, by Lemma 3, it follows that $v, w \notin S$. In order to dominate u in graph $G - w_1$, we have $S = \{u, u_1\}$. Since $uw_2, u_1w_2 \notin E(G)$, w_2 is not dominated by S in $G - w_1$, which is a contradiction.*

Case 2. $|N(u) \cap N(v)| = 2$. *Let $N(u) \cap N(v) = \{u_1, u_2\}$ and $N(w) \cap N(v) = \{w_1, \cdots, w_5\}$. By Lemma 3, $S_{w_j} \cap \{u_1, u_2\} \neq \emptyset$ for $j = 1, 2, \cdots, 5$. By the Pigeonhole Principle, we can assume that $u_1 \in S_{w_1} \cap S_{w_2} \cap S_{w_3}$. It is obvious that $S_{w_j} \cap \{w_4, w_5\} \neq \emptyset$ for $j = 1, 2, 3$. By the Pigeonhole Principle, we can assume that $S_{w_1} = S_{w_2} = \{u_1, w_4\}$. Since $\{u_1, w_4\}$ is a $\gamma_t(G - w_1)$-set and $u_1w_2 \notin E(G)$, it follows that $w_2w_4 \in E(G)$. So, $\{u_1, w_4\}$ is not a $\gamma_t(G - w_2)$-set, which is a contradiction.*

Case 3. $|N(u) \cap N(v)| = 3$. *Let $N(u) \cap N(v) = \{u_1, u_2, u_3\}$ and $N(w) \cap N(v) = \{w_1, \cdots, w_4\}$. By Lemma 3 and the Pigeonhole Principle, we can assume that $u_1 \in S_{w_1} \cap S_{w_2}$. It is obvious that $S_{w_1} \cap \{w_3, w_4\} \neq \emptyset$, $S_{w_2} \cap \{w_3, w_4\} \neq \emptyset$ and $S_{w_1} \neq S_{w_2}$. Without loss of generality, we can assume that $S_{w_1} = \{u_1, w_3\}$ and $S_{w_2} = \{u_1, w_4\}$. So $w_2w_3, w_1w_4 \in E(G)$ and $u_1w_1, w_1w_3, u_1w_2, w_2w_4 \notin E(G)$. It divides into two subcases.*

Subcase 3.1. $w_1 \in S_{u_2} \cup S_{u_3}$ *or* $w_2 \in S_{u_2} \cup S_{u_3}$. *Assume that $w_1 \in S_{u_2}$. Then $S_{u_2} = \{u_3, w_1\}$. Since $u_1w_1, w_1w_3 \notin E(G)$, it follows that*

$u_1u_3, u_3w_3 \in E(G)$. *If* $u_2w_3 \in E(G)$ *then* $N(u) \subseteq N(w_3)$. *By Lemma 4, G is not a* γ_t*-critical, which is a contradiction. So* $u_2w_3 \notin E(G)$. *Since* $S_{w_1} = \{u_1, w_3\}$, $u_1u_2 \in E(G)$. *Since* $u_2w_1, u_2u_3 \notin E(G)$, *it follows that* $S_{w_3} = \{u_2, w_4\}$ *and* $u_3w_4 \in E(G)$. *So* $\{w, w_4\}$ *is a* $\gamma_t(G)$*-set, which is a contradiction.*

Subcase 3.2. $w_1, w_2 \notin S_{u_2} \cup S_{u_3}$. *Assume that* $w_3 \in S_{u_2}$. *Since* $u_1w_1, w_1w_3 \notin E(G)$, $\{u_1, w_3\}$ *is not a* $\gamma_t(G - u_2)$*-set. So* $u_3w_3 \in E(G)$. *By the similar way as Subcase 3.1,* $u_1u_2 \in E(G)$. *So* $w_4 \in S_{u_3}$ *and* $u_1u_3, u_2w_4 \in E(G)$. *Then* $S_{u_1} = \{w, w_1\}$ *or* $S_{u_1} = \{w, w_2\}$. *Without loss of generality, we can assume that* $S_{u_1} = \{w, w_1\}$. *Then* $u_2w_1, u_3w_1 \in E(G)$. *If* $u_3w_2 \in E(G)$, *then* $\{u_3, w_1\}$ *is a* $\gamma_t(G)$*-set, which is a contradiction. So* $u_3w_2 \notin E(G)$. *Hence* $S_{w_4} = \{u_3, w_3\}$. *Since* $u_2w_3 \notin E(G)$, $u_2u_3 \in E(G)$. *If* $u_2w_2 \in E(G)$, *then* $\{u_2, w_2\}$ *is a* $\gamma_t(G)$*-set, which is a contradiction. So* $u_2w_2 \notin E(G)$. *Hence* $S_{w_3} = \{u_2, w_1\}$ *and* $w_1w_2 \in E(G)$. *So* $\{u_3, w_1\}$ *is a* $\gamma_t(G)$*-set, which is a contradiction.*

This completes the proof. □

By a similar philosophy of Theorem 6, we have the following corollary.

Corollary 7. *Suppose that G is a graph of order* $\Delta(G) + 3$ *with* $\Delta(G)$ *odd and* $\delta(G) \geq 2$. *Let* $d(v) = \Delta(G)$ *and* $V(G) - N[v] = \{u, w\}$. *Say* $d(w) > d(u)$. *If* $d(u) = 2, 3$, *then G is not 3-*γ_t*-critical.*

To find a positive answer for Problem 2, we first prove the following lemma.

Lemma 8. *Let G be a connected graph with* $\Delta(G) = 9$ *or* $\Delta(G) \geq 11$. *Then there are positive integers* $t, x_1, x_2, \cdots, x_t$ *satisfying the following three conditions;*

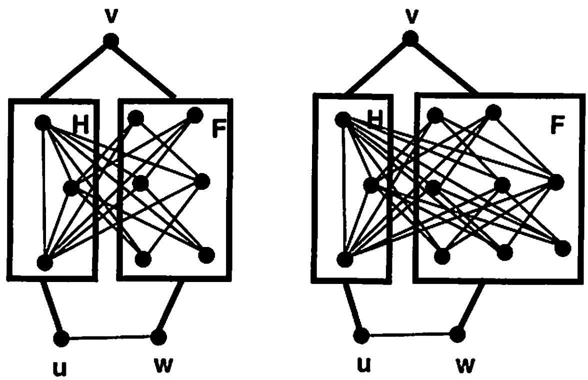
(1) $t + x_1 + x_2 + \cdots + x_t = \Delta(G)$

(2) $t \geq 3$

(3) $2 \leq x_1 \leq x_2 \leq \cdots \leq x_{t-2} \leq x_{t-1} = x_t$.

Now we construct a family of 3-γ_t-critical graphs of order $\Delta(G)+3$ with $\delta(G) \geq 2$ and $\Delta(G) = 9$ or $\Delta(G) \geq 11$.

Let H be a copy of K_t with $t \geq 3$. Let $V(H) = \{1, 2, \cdots, t\}$. Let H_i be a graph with a vertex set $V(H_i) = \{w_{i1}, w_{i2}, \cdots, w_{ix_i}\}$ for $i = 1, 2, \cdots, t$. Suppose that $2 \leq x_1 \leq x_2 \leq \cdots \leq x_{t-2} \leq x_{t-1} = x_t$. Let F be the graph obtained from $H_1 \cup H_2 \cup \cdots \cup H_t$ by adding edges $w_{ij}w_{i+1,k}$ for $i = 1, 2, \cdots, t-1$, $j = 1, 2, \cdots, x_i$, $k = 1, 2, \cdots, x_{i+1}$ and $j \neq k$. Let G be the graph obtained from $H \cup F$ and three new vertices v, u, w by adding edges iw_{jk} for $1 \leq i, j \leq t$, $i \neq j$ and $1 \leq k \leq x_j$, and then joining v to every vertex in $H \cup F$, joining u to every vertex in $H \cup \{w\}$ and joining w to every vertex in F. Then $\Delta(G) = t + x_1 + x_2 + \cdots + x_t$. Two figures in Figure 1 are examples of 3-γ_t-critical graphs with $\Delta(G) = 9, 11$.



$\Delta(G) = 9$ $\Delta(G) = 11$

Figure 1

Theorem 9. *Let G be the graph defined as above. Then G is a 3-γ_t-critical graph.*

Proof. It is obvious that $\gamma_t(G) = 3$. So we only prove that G is γ_t-critical graph. First, $\{v, w_{11}\}$, $\{v, 1\}$ and $\{u, w\}$ is a total dominating set of $G - u$, $G - w$ and $G - v$ respectively. For any vertex $i \in V(G)$, $\{w, w_{11}\}$ is a total dominating set of $G - i$. For any vertex $w_{jk} \in V(G)$, $\{j, w_{j+1,k}\}$ is a total dominating set of $G - w_{jk}$, where $j = 1, 2, \cdots, t-1$ and $k = 1, 2, \cdots, x_j$. For any vertex $w_{tk} \in V(G)$, $\{t, w_{t-1,k}\}$ is a total dominating set of $G - w_{tk}$, where $k = 1, 2, \cdots, x_t$. In general, for any vertex $v \in V(G)$, $\gamma_t(G - v) = 2$. So G is a 3-γ_t-critical graph. □

Note that if we add an edge $e = w_{ij}w_{i\ell}$ between the vertices w_{ij} and $w_{i\ell}$ in $V(H_i)$ for $1 \leq j,\ \ell \leq x_i$ and $j \neq \ell$, then $G \cup \{e\}$ is also a 3-γ_t-critical graph. By Lemma 8 and Theorem 9, we have the following theorem.

Theorem 10. *There is a family of 3-γ_t-critical graphs G of order $\Delta(G)+3$ with $\Delta(G) \geq 9$, $\delta(G) \geq 2$ and $\Delta(G)$ odd.*

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