Dynamical Behavior of Symmetric Weighted Median mapping on Two-dimensional Real Sequences Space R_s *

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Abstract In this paper, we obtain a fundamental result on dynamical behavior of symmetric weighted mapping for two-dimensional real sequences space R_s.

Keywords: symmetric weighted median mapping, root, recurrent sequence.

Linear filters have been the primary tool for signal processing for some time. They are easy to design, and in most cases, they offer excellent performance. This is particularly true for spectral separation where the desired signal spectrum is significantly different from that of the interference. Not all signal processing problems can be satisfactorily addressed through the use of linear filters. Linear filters tend to blur sharp edges, fail to remove heavy tailed distribution noise effectively, and perform poorly in the presence of signal-dependent noise([1],[2],[3]). Median-type filters have been subject to growing interesting since the discovery of the standard median filter by Tukey[4], who applied it to the smoothing of statistical data. Pratt[5] was the first to use median filter in image processing. Later median-type filters have shown their usefulness in many one-and two-dimensional applications. The success of median-type filters is based on two intrinsic properties: edge preservation and efficient noise attenuation with robustness against impulsive-type noise. As median-type filters are nonlinear, they cannot be analyzed with classical linear techniques. They have been characterized statistically, using order statistics[6]-[8], and syntactically, using the idea of roots(invariant signals)[9]-[13]. The syntactic

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analysis has been quite successful, as it has shown which signals are invariant to these filters. For finite length signals, Gallagher and Wise[9] proved arbitrary finite length signal is a median filter root if it consists of constant neighborhoods and edges only(see also Tyan[14]). They also proved that repeated median filtering of any finite length signal will result in a root signal after a finite number of passes. The property of median filters is very significant and is called the "convergence of property", the tightest known bound on the number of passes of the median filter necessary to reach a root can be found in[15]. Wendt[22] proved that symmetric weighted threshold filters make any of finite length signals converge to a root or to cycle of period 2, which means that the output of the filter oscillates between two signals on successive passes of the filter. Center weighted median filters[16], which give more weight only to the center value of the window, are possess the convergence property. Zheng[18],[19] considers weighted median filters, with symmetric nondecreasing weights. In the usual case a root consists only of CNs (constant neighborhoods) and edges. For infinite length signals, Astola *et al* [17]showed that the median, recursive median and hybrid filters may have oscillatory infinitely long root signals. Eberly *et al*[20] proved that the set of roots with no monotone segment of length N+1 is finite, and each root is periodic. Given N, the authors give a constructive procedure able to list all such roots. Mao *et al*[21] presented a class of infinite length signals associated to the width of window of the median filter, and this median filter can make any signal in this class converge to a root. To my knowledge, there has been little research on two-dimensional symmetric weighted filters, particularly for infinite two-dimensional signals. In this paper, we obtain a fundamental result on dynamical behavior of symmetric weighted filter. Our method is effective to one-dimensional infinite length signals or finite two-dimensional signals by modifying sightly.

In this paper, k is a fixed positive integer, Z is the set of integers, and N is the set of nonnegative integers. R is the set of real numbers.

Suppose that

$w(i,j) \in N, i,j = 0,\pm1,\pm2,\cdots,\pm k$, $w(i,j) = w(-i,-j), w(0,0)$ is odd, and $H = \frac{1}{2}(1+\sum_{i,j=-k}^{k} w(i,j))$. Let $x = \{x(m,n)\}$ is a two-dimensional sequence, and $x^{(1)} = \{x^{(1)}(m,n)\}$, for $m,n \in Z$,

$x^{(1)}(m,n) =$

$\text{Median}\{w(i-m,j-n)\Diamond x(i,j) : m-k \le i \le m+k, n-k \le j \le n+k\}$, which is denoted by $wx[m,n:k]$, where $n\Diamond x = \underbrace{x,\cdots,x}_{n}$. In particular,

$0\Diamond x = \emptyset$.

Let $R_s = \{\{x(m,n)\} | x(m,n) \in R, m,n \in Z\}$ and a mapping

$$WF_k : R_s \mapsto R_s,$$

where for each $x = \{x(m,n)\} \in R_s$, let $WF_k(x) = x^{(1)} = \{x^{(1)}(m,n)\} \in R_s$, we call WF_k symmetric weighted mapping on the space R_s, and

$$(WF_k)^p(x) = x^{(p)} = \{x^{(p)}(m,n)\}, \text{for} \quad p \geq 1.$$

For a fixed $x = \{x(m,n)\} \in R_s$,if, for any $m, n \in Z$, $\lim_{p\to\infty} x^{(p)}(m,n) = r(m,n)$ is a real number, we say that x is convergent with respect to the mapping WF_k , denoted by $x^{(p)} \to r(p \to \infty)$, where $r = \{r(m,n)\}$.

1 Main result and propositions

The main result in this paper is the following theorem.

Theorem 1 *Suppose that $x = \{x(m,n)\} \in Rs$. Then, when $p \to \infty$, both $x^{(2p)}$ and $x^{(2p-1)}$ are convergent. Moreover, if let $x^{(2p)} \to \alpha$ and $x^{(2p-1)} \to \beta$, then $(MF_k)^2(\alpha) = \alpha, (MF_k)^2(\beta) = \beta$, and $MF(\alpha) = \beta, MF_k(\beta) = \alpha$.*

In the section, suppose below that $x = \{x(m,n)\} \in R_s$ with $x(m,n) \in \{-1,1\}$ for $m, n \in Z$.

We next give some relevant propositions for proving the above theorem.

Proposition 1 *Suppose that the real sequence $v = \{v(n)\}_{n\in Z}$ satisfies the following three conditions*

(i) $v(n) = v(-n), n \in Z$;

(ii) $1 = v(0) > v(1) > \cdots > v(2k) = \sqrt{\frac{H}{H+1}}$;

(iii) $v(r) = \sqrt{\frac{H}{H+1}}v(r-2k)$ *for each* $r > 2k$.

Then $v(n) > 0$ and $\sum_{m,n\in Z} v(m)v(n) < \infty$. Moreover, for each $(m,n), m,n \in Z$, we have

$$\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} v(i)v(j)w(i,j)u(i,j) \cdot \sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)u(i,j) > 0, \quad (1)$$

where $u(m,n) = 1$ or $-1, m,n \in Z$.

Proof $v(n) > 0$ and $\sum_{m,n\in Z} v(m)v(n) < \infty$ are trivial. Now we prove (1).

Since $\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)|u(i,j)| = 2H+1$,

$$\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)u(i,j) \neq 0.$$

If

$$\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)u(i,j) = \sum_{u(i,j)=1} w(i,j) - \sum_{u(i,j)=-1} w(i,j) > 0,$$

we have

$$\sum_{u(i,j)=1} w(i,j) \geq H+1 \quad \text{and} \quad \sum_{u(i,j)=-1} w(i,j) \leq H.$$

Thus

$$\begin{aligned}
&\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} v(i)v(j)w(i,j)u(i,j)\\
&= \sum_{u(i,j)=1} v(i)v(j)w(i,j) - \sum_{u(i,j)=-1} v(i)v(j)w(i,j)\\
&> \min v[m-k,m+k] \min v[n-k,n+k] \sum_{u(i,j)=1} w(i,j)\\
&\quad - \max v[m-k,m+k] \max v[n-k,n+k] \sum_{u(i,j)=-1} w(i,j)\\
&\geq (H+1)\min v[m-k,m+k]\min v[n-k,n+k]\\
&\quad -(H)\max v[m-k,m+k]\max v[n-k,n+k],
\end{aligned}$$

where $\min v[i,j] = \min\{v(i), v(i+1), \cdots, v(j)\}$, $\max v[i,j] = \max\{v(i), v(i+1), \cdots, v(j)\}$. By the definition of sequence v, for any $n-k \leq i \leq n+k$

$$v(n-k) \geq v(i) \geq v(n+k) \text{ and } v(n+k) = \sqrt{\tfrac{H}{H+1}}v(n-k) \quad \text{for } n \geq k,$$

$$v(n-k) \leq v(i) \leq v(n+k) \text{ and } v(n-k) = \sqrt{\tfrac{H}{H+1}}v(n+k) \quad \text{for } n \leq -k,$$

$$\sqrt{\tfrac{H}{H+1}} \leq v(i) \leq 1 \quad \text{for } |n| < k.$$

So we have

$$(H+1)\cdot \min v[m-k,m+k]\min v[n-k,n+k]$$
$$-H\cdot \max v[m-k,m+k]\max v[n-k,n+k] \geq 0.$$

Thus $\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} v(i)v(j)w(i,j)u(i,j) > 0$. (1) holds.

If $\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)u(i,j) < 0$, then $\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} w(i,j)(-u(i,j)) > 0$.

By a similar argument, we can get $\sum_{i=m-k}^{m+k}\sum_{j=n-k}^{n+k} v(i)v(j)w(i,j)(-u(i,j)) > 0$. Therefore (1) holds. □

Secondly by the definition of MF_k, we immediately have

Proposition 2 *Suppose that $x = \{x(m,n)\} \in R_s$ with $x(m,n) = 1$ or -1 for each $(m,n), m,n \in Z$. Then*

$$x^{(1)}(m,n) = \begin{cases} 1, & \text{if } \sum_{i=m-k}^{m+k} \sum_{j=n-k}^{n+k} w(i,j)x(i,j) > 0, \\ -1, & \text{if } \sum_{i=m-k}^{m+k} \sum_{j=n-k}^{n+k} w(i,j)x(i,j) < 0, \end{cases} \quad m,n \in Z.$$

Now let $\{v(n)\}_{n\in Z}$ be the sequence defined in Proposition 1. For each $x = \{x(m,n)\} \in R_s, x(m,n) = 1$ or -1, let

$$E(x) = \sum_{m,n;s,t\in Z} v(m)v(n)v(s)v(t)b(m,n;s,t)x^{(1)}(m,n)x(s,t),$$

where

$$b(m,n;s,t) = \begin{cases} w(m-s,n-t), & \text{if } |m-s|,|n-t| \le k, \\ 0, & \text{otherwise,} \end{cases} \quad m,n,s,t \in Z.$$

Proposition 3 *Suppose that $x = \{x(m,n)\} \in R_s$ with $x(m,n) = 1$ or -1 for each $m,n \in Z$. Then for each $p \in N$*
$E(x^{(p)}) - E(x^{(p-1)}) =$

$$\sum_{m,n\in Z} v(n)v(m)\cdot[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)x^{(p)}(s,t)]\cdot[x^{(p+1)}(m,n)-x^{(p-1)}(m,n)], \tag{2}$$

where items in the series are non-negative, and $\{E(x^{(p-1)})\}_{p>1}$ is a bounded increasing sequence.

Proof Now we first prove that $\{E(x^{(p-1)})\}_{p>1}$ is a bounded sequence. In fact, by Proposition 1,2, and the definition of $E(x)$,

$$\begin{aligned} &|E(x^{(p-1)})| \\ = &\ |\sum_{m,n,s,t\in Z} v(m)v(n)v(s)v(t)b(m,n:s,t)x^{(p)}(m,n)x^{(p-1)}(s,t)| \\ \le &\sum_{m,n,s,t\in Z} v(m)v(n)v(s)v(t)b(m,n:s,t) \\ = &\sum_{m,n\in Z} v(m)v(n)[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)] \end{aligned}$$

$$< \sum_{m,n\in Z} v(m)v(n) \sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} w(s,t)$$

$$\leq (2H+1)\sum_{m\in Z} v(m) \sum_{n\in Z} v(n) < \infty.$$

Secondly, we prove that (2) holds.

By $b(m,n;s,t) = b(s,t;m,n)$, we have

$E(x^{(p)}) - E(x^{(p-1)})$

$= \sum_{m,n,s,t\in Z} v(m)v(n)v(s)v(t)b(m,n;s,t)x^{(p+1)}(m,n)x^{(p)}(s,t)$

$- \sum_{s,t,m,n\in Z} v(s)v(t)v(m)v(n)b(s,t;m,n)x^{(p)}(s,t)x^{(p-1)}(m,n)$

$= \sum_{m,n,s,t\in Z} v(m)v(n)v(s)v(t)b(m,n;s,t)x^{(p)}(s,t)$

$\cdot [x^{(p+1)}(m,n) - x^{(p-1)}(m,n)]$

$= \sum_{m,n\in Z} v(m)v(n)[\sum_{s,t\in Z} v(s)v(s)b(m,n;s,t)x^{(p)}(s,t)]$

$\cdot [x^{(p+1)}(m,n) - x^{(p-1)}(m,n)]$

$= \sum_{m,n\in Z} v(m)v(n)[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t)]$

$\cdot [x^{(p+1)}(m,n) - x^{(p-1)}(m,n)]$.

If

$$\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t) > 0,$$

by (1)

$$\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} w(s,t)x^{(p)}(s,t) > 0.$$

By Proposition 2, $x^{(p+1)}(m,n) = 1$, so

$x^{(p+1)}(m,n) - x^{(p-1)}(m,n) \geq 0$,

thus

$v(m)v(n)[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t)]$

$\cdot[x^{(p+1)}(m,n) - x^{(p-1)}(m,n)] \geq 0$;

If

$$\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t) < 0,$$

by (1)

$$\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} w(s,t)x^{(p)}(s,t) < 0.$$

By Proposition 2, $x^{(p+1)}(m,n) = -1$, so

$x^{(p+1)}(m,n) - x^{(p-1)}(m,n) \leq 0,$

thus

$v(m)v(n)[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t)]$
$\cdot[x^{(p+1)}(m,n) - x^{(p-1)}(m,n)] \geq 0.$

This shows that the items in the series (2) are non-negative. Therefore $\{E(x^{(p-1)})\}_{p>1}$ is an increasing sequence.

Now let $x = \{x(m,n)\}_{m,n\in Z} \in R_s$ with $x(m,n) = 1$ or -1 for each $(m,n), m,n \in Z$. By Proposition 3, for each $(m,n), m,n \in Z$, we have

$$\lim_{p\to\infty} \{[\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t)]\cdot[x^{(p+1)}(m,n) - x^{(p-1)}(m,n)]\} = 0. \tag{3}$$

Since $x^{(p)}(m,n) = 1$ or -1 for any $p \in N$ and any $m,n \in Z$, by Proposition 1

$$\min_{p>1} |\sum_{s=m-k}^{m+k} \sum_{t=n-k}^{n+k} v(s)v(t)w(s,t)x^{(p)}(s,t)| > 0.$$

Then by (3), $x^{(p+1)}(m,n) - x^{(p-1)}(m,n) \to 0$ as $p \to \infty$. Thus $x^{(p+1)}(m,n) = x^{(p-1)}(m,n)$ if p is large enough. Therefore both $\{x^{(2p)}\}_{p\geq 1}$ and $\{x^{(2p-1)}\}_{p\geq 1}$ are convergent.

We write the above result as the following.

Proposition 4 *Suppose that $x = \{x(m,n)\} \in R_s$ with $x(m,n) = 1$ or -1 for each $(m,n), m,n \in Z$. Then both $\{x^{(2p)}\}_{p\geq 1}$ and $\{x^{(2p-1)}\}_{p\geq 1}$ are convergent.*

2 Proof of main result

We first define some notations.

Let $x = \{x(m,n)\}, y = \{y(m,n)\} \in R_s$, and let λ be a real number. In this case, $x + y$ and λx are defined as

$$(x+y)(m,n) = x(m,n) + y(m,n), \quad (\lambda x)(m,n) = \lambda x(m,n), \quad m,n \in Z;$$

x_+ and x_- are defined as

$$x_+(m,n) = \max\{0, x(m,n)\}, \quad x_-(m,n) = \max\{0, -x(m,n)\}, \quad m,n \in Z;$$

x_+ and x_- are called the positive part and the negative part of x respectively and both of them are non-negative sequences.

$\{x(i,j)\}_{m-k\leq i\leq m+k, n-k\leq j\leq n+k}$ are denoted by $x[m,n:k]$;
$\min x[m,n:k] = \min\{x(i,j) : m-k \leq i \leq m+k, n-k \leq j \leq n+k\}$;
$\max x[m,n:k] = \max\{x(i,j) : m-k \leq i \leq m+k, n-k \leq j \leq n+k\}$.

If $x(i,j) \le y(i,j)$ for integers $i, j, m-k \le i \le m+k, n-k \le j \le n+k$, we write $x[m,n:k] \le y[m,n:k]$. In particular, we write $x \le y$ if $x(m,n) \le y(m,n)$ for all integers $m, n \in Z$.

Lemma 1 *Suppose that $x = \{x(m,n)\}, y = \{y(m,n)\} \in R_s$. Then*

(i) $\max x^{(p+1)}[m,n:k] \le \max x^{(p)}[m,n:k]$ *for any integers $m, n \in Z$ and $p \in N$;*

(ii) *If $x[m,n:k] \le y[m,n:k]$ for some $m, n \in Z$, then $x^{(1)}(m,n) \le y^{(1)}(m,n)$. Particularly $x^{(p)} \le y^{(p)}$ for each $p \in N$ if $x \le y$;*

(iii) *If $y = \{y(m,n)\} \in R_s$ is a constant sequence, then $(x+y)^{(p)} = x^{(p)} + y^{(p)}$ for each $p \in N$;*

(iv) *If $\lambda > 0$, then $(\lambda x)^{(p)} = \lambda x^{(p)}$ for each $p \in N$;*

(v) *For each $p \in N, x_+^{(p)} = (x^{(p)})_+, x_-^{(p)} = (x^{(p)})_-$, and $x^{(p)} = x_+^{(p)} - x_-^{(p)}$.*

(vi)*For some integers m, n, if $x^{(1)}(m,n) > a$, then there are at least $H+1$ items in $wx[m,n:k]$ larger than a; if $x^{(1)}(m,n) < a$, then there are at least $H+1$ items in $wx[n-k, n+k]$ less than a.*

Proof (i) For each $i, j, m-k \le i \le m+k, n-k \le j \le n+k$, there are at least $H+1$ items in $wx[i,j:k]$ no larger than $\max x[m,n:k]$, so $x^{(1)}(i,j) \le \max x[m,n:k]$. Thus $\max x^{(1)}[m,n:k] \le \max x[m,n:k]$. For general $p \in N$, the argument are similar.

(ii), (iii),(iv) and (vi) are trivial.

(v) Fix $m, n \in Z$. Suppose $x^{(1)}(m,n) \ge 0$. Then there are at least $H+1$ items in $wx[m,n:k]$ no less than 0 and

$$x_+^{(1)}(m,n) = x^{(1)}(m,n) = (x^{(1)})_+(m,n), \quad x_-^{(1)}(m,n) = 0 = (x^{(1)})_-(m,n).$$

Similarly, when $x^{(1)}(m,n) < 0$,

$$x_+^{(1)}(m,n) = 0 = (x^{(1)})_+(m,n), x_-^{(1)}(m,n) = -x^{(1)}(m,n) = (x^{(1)})_-(m,n).$$

Thus (v) holds for $p = 1$. By the similar argument, we can prove that (v) holds for $p \in N$.

Proposition 5 *Suppose that $x = \{x(m,n)\} \in R_s$. Then both $\{x^{2p}\}_{p \ge 1}$ and $\{x^{(2p-1)}\}_{p \ge 1}$ are convergent.*

Proof Assume that for any integers m, n, $x(m,n) = a$ or b, $a < b$. Setting

$$y(m,n) = \frac{2}{b-a} x(m,n) - (\frac{2a}{b-a} + 1), \quad m, n \in Z.$$

Then $y(m,n) \in \{-1,1\}, m,n \in Z$, and by Lemma 1 (iii) and(iv)

$$y^{(p)}(m,n) = \frac{2}{b-a}x^{(p)}(m,n) - (\frac{2a}{b-a}+1), \quad m,n \in Z, p \geq 1.$$

Since $\{y^{(2p)}\}_{p\geq 1}$ and $\{y^{(p)}\}_{p\geq 1}$ are convergent by Proposition 4, $\{x^{(2p)}\}_{p\geq 1}$ and $\{x^{(2p-1)}\}_{p\geq 1}$ are convergent.

Suppose that $x = \{x(m,n)\} \in R_s$ is a nonnegative real sequence and $t > 0$. Let

$$x_t(m,n) = \begin{cases} 1, & \text{if } x(m,n) \geq t, \\ 0, & \text{if } x(m,n) < t, \end{cases} \quad n \in Z.$$

Then $x_t = \{x_t(m,n)\} \in R_s$ is a binary sequence, and

$$x(m,n) = \int_0^{x(m,n)} x_t(m,n)dt, \quad m,n \in Z.$$

Proposition 6 *Suppose that $x = \{x(m,n)\} \in R_s$ is a nonnegative sequence. Then both $\{x^{(2p)}\}_{p\geq 1}$ and $\{x^{(2p-1)}\}_{p\geq 1}$ are convergent.*

Proof Fix any $m,n \in Z$. If $t > x^{(1)}(m,n)$, by Lemma 1(vi), then there are at least $H+1$ items in $wx[m,n:k]$ less than t, so there are at least $H+1$ items in $wx_t[m,n:k]$ equal to 0, this implies that $x_t^{(1)}(m,n) = 0$; If $t < x^{(1)}(m,n)$, by Lemma 1(vi), then there are at least $H+1$ items in $wx[n-k,n+k]$ larger than t, so there are at least $H+1$ items in $wx_t[n-k,n+k]$ equal to 1, this implies that $x_t^{(1)}(m,n) = 1$. Hence, if we let $L(m,n) = \max x[m,n:k]$, by Lemma 1 (i), then

$$x^{(1)}(m,n) = \int_0^{L(m,n)} x_t^{(1)}(m,n)dt, \quad m,n \in Z$$

Similarly we have

$$x^{(p)}(m,n) = \int_0^{L(m,n)} x_t^{(p)}(m,n)dt, \quad n \in Z, p \geq 1.$$

Fix any $m,n \in Z$, by Proposition 5, both $\{x_t^{(2p)}(m,n)\}$ and $\{x_t^{(2p-1)}(m,n)\}$ are convergent, therefore, by Lebesgue theorem on dominated convergence, $\{x^{(2p)}(m,n)\}$ and $\{x^{(2p-1)}(m,n)\}$ are convergent.

Proof of Theorem 1 Suppose that $x = \{x(m,n)\} \in R_s$. Then x_+ and x_- are nonnegative sequences. By Proposition 6 and Lemma (v) $x^{(2p)} = (x_+)^{(2p)} - (x_-)^{(2p)}$ and $x^{(2p-1)} = (x_+)^{(2p-1)} - (x_-)^{(2p-1)}$ are convergent as $p \to \infty$. Let

$$\lim_{p\to\infty} x^{(2p)}(m,n) = \alpha(m,n), \quad \lim_{p\to\infty} x^{(2p-1)}(m,n) = \beta(m,n), \quad \forall m,n \in Z.$$

Fix $m, n \in Z$ and $\varepsilon > 0$. Then there is $P \geq 1$ such that

$$x^{(2p)}(i,j) - \varepsilon \leq \alpha(i,j) \leq x^{(2p)}(i,j) + \varepsilon, \forall |i-n|, |j-m| \leq k, \forall p > P.$$

By Lemma (ii) and (iii)

$$x^{(2p+1)}(m,n) - \varepsilon \leq \alpha^{(1)}(m,n) \leq x^{(2p+1)}(m,n) + \varepsilon, \forall p > P.$$

Let $p \to \infty$,

$$\beta(m,n) - \varepsilon \leq \alpha^{(1)}(m,n) \leq \beta(m,n) + \varepsilon.$$

Thus $\alpha^{(1)}(m,n) = \beta(m,n)$, since ε is arbitrary. Hence $\alpha^{(1)} = \beta$. Similarly $\beta^{(1)} = \alpha$. Moreover, $\alpha(2) = \alpha, \beta^{(2)} = \alpha$. Therefore $(MF_k)^2(\alpha) = \alpha, (MF_k)^2(\beta) = \beta$, and $MF(\alpha) = \beta, MF_k(\beta) = \alpha$.

This completes proof of Theorem 1.

If $x^{(1)} = x, x$ is called a root ;

If $x^{(1)} \neq x$ and $x^{(s)} = x$ for some $s \geq 2$, x is called a recurrent sequence.

Suppose that x is a recurrent sequence ,i.e., $x^{(1)} \neq x$, and there exists a positive integer $q > 1$, such that $x^{(q)} = x$. Then we have

$$x^{(mq)} = x, \quad \forall m \in N.$$

Now let $x^{(2p)}$ and $x^{(2p-1)}$ converge to the recurrent sequence α and β, respectively.

Let m be even and $m \to \infty$, then $x^{(mq)}$ converges to α, so we have $x = \alpha$. By Theorem 1, $x^{(2)} = \alpha^{(2)} = \alpha = x$. This is the following

Corollary 1 *If x is a recurrent sequence, then $(MF)^2(x) = x$.*

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