Dominating Broadcasts of Caterpillars

Suzanne M. Seager[1]

Mount Saint Vincent University, Halifax, NS, Canada

Abstract. A *dominating broadcast* of a graph G of diameter d is a function $f : V(G) \to \{0, 1, 2, ..., d\}$ such that for all $v \in V(G)$ there exists $u \in V(G)$ with $d(u, v) \leq f(u)$. We investigate dominating broadcasts for caterpillars.

1 Introduction

Think of the vertices of a dominating set for a graph as consisting of locations on which to build broadcast towers. Each vertex with a tower can broadcast to all of its neighbours, and the goal is to minimize the number of towers while still ensuring every vertex can receive a broadcast. Erwin [3] generalized this idea to towers having different broadcasting power, so that a tower of power k can broadcast to all vertices within distance k while incurring cost k. The goal of such a *dominating broadcast* is to minimize the total cost such that every vertex can receive a broadcast from some tower. We formalize these concepts below.

Let $G = (V, E)$ be a graph. We assume throughout that G is connected and nontrivial. For any vertex v, let $N(v)$ be the set of neighbours of v and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ be the *degree* of v. Let $N[S] = \bigcup_{v \in S} N[v]$. A *dominating set* for G is a set $D \subseteq V$ with $V \subseteq N[D]$. The *domination number* $\gamma(G) = \min\{|D| : D \text{ is a dominating set for } G\}$.

The *distance* $d(u, v)$ from vertex u to vertex v is the minimum length of a path from u to v. The *eccentricity* of v is $e(v) = \max_{u \in V} d(u, v)$, the *diameter* of G is $\text{diam}(G) = \max_{v \in V} e(v)$, and the *radius* is $\text{rad}(G) = \min_{v \in V} e(v)$. A *broadcast* is a function $f : V \to \{0, 1, 2, ..., \text{diam}(G)\}$ such that for every v, $f(v) \leq e(v)$. The set of *broadcast dominators* for f is $V_f = \{v : f(v) > 0\}$. For any vertex v, the set of vertices that v can *hear* is $H_f(v) = \{u \in$

[1]Research supported in part by an MSVU Internal Grant

$V_f : d(u,v) \leq f(v)\}$. The *cost* of a broadcast f incurred by a set $S \subseteq V$ is $f(S) = \Sigma_{v \in S} f(v)$; the *cost of* f is $f(V)$. A broadcast f is *dominating* if $|H(v)| \geq 1$ for all $v \in V$, and *efficient* if $|H(v)| = 1$ for all $v \in V$. Note that the characteristic function χ_D of a dominating set D is a dominating broadcast with cost $|D|$. The *dominating broadcast number* $\gamma_b(G) = \min\{f(V) : f$ is a dominating broadcast for $G\}$. An *optimal* broadcast is a dominating broadcast f with $f(V) = \gamma_b(G)$. A dominating broadcast f is *radial* if $|V_f| = 1$; a radial broadcast can always be defined by choosing $v \in V$ with $e(v) = \text{rad}(G)$, and setting $f(v) = \text{rad}(G)$, $f(u) = 0$ for all $u \neq v$. A *radial* graph is a graph with a radial optimal broadcast, so $\gamma_b(G) = \text{rad}(G)$.

Erwin [3] determined upper bounds for $\gamma_b(G)$.

Theorem 1 [3] *For any graph G,* $\gamma_b(G) \leq \min\{rad(G), \gamma(G)\}$.

Dunbar et. al. [2] proved that dominating broadcasts could be made efficient. Their proof actually gives a more general result, so we sketch the proof below.

Theorem 2 [2] *Every graph G has an optimal broadcast which is efficient.*

Theorem 3 *Let f be an optimal broadcast of a graph G with $H_f(v) > 1$ for some vertex v. Then there exists an optimal broadcast g of G with $H_g(v) < H_f(v)$, $|V_g| < |V_f|$, and such that $d(x) \geq 2$ for all $x \in V_g - V_f$.*

Sketch of Proof [2] If f is a non-efficient dominating broadcast of G then there exists $v \in V$ and $u, w \in V_f$ such that $d(u,v) \leq f(u)$ and $d(w,v) \leq f(w)$. Assume without loss of generality that $f(u) \leq f(w)$ and let x be the vertex at distance $f(w) - f(u)$ from v on the shortest path from v to w. Define a broadcast g by $g(x) = f(u) + f(w), g(u) = g(w) = 0$, and $g(y) = f(y)$ for $y \neq u, w, x$. □

Dunbar et al. [2] also defined graphs to be of Type I if $\gamma_b(G) = \gamma(G)$, Type II if $\gamma_b(G) = \text{rad}(G)$, and Type III otherwise, and asked which graphs belong to each type. We answer this question for caterpillars.

2 Caterpillars

A *caterpillar* is a tree such that removing all leaves results in a path called the *spine*. We call a vertex which is adjacent to a leaf a *stem* and a vertex of degree 2 which is not adjacent to a leaf a *trunk*, so all vertices in a caterpillar are either leaves, stems, or trunks, and the first and last spine vertices are stems. Thus for a caterpillar C with r spine vertices, $\operatorname{diam}(C) = r + 1$ and $\operatorname{rad}(C) = \lceil \frac{r+1}{2} \rceil$.

We assume throughout that all dominating broadcasts f are chosen such that V_f contains no leaves, since every leaf in V_f can be replaced by its stem without changing the cost (and similarly we will assume dominating sets contain no leaves). For a caterpillar C we call the pattern of stems and trunks along the spine its *form*, using s to denote a stem and t to denote a trunk. So, for example, $K_{1,m}$ has form s for $m \geq 2$, and P_6 has form $stts$. The proof of the following lemma is clear.

Lemma 4 *If caterpillars C and C' have the same form, where the sequence of spine vertices of C is $v_1v_2...v_q$ and of C' is $w_1w_2...w_q$, then for any dominating broadcast f of C, the function f' defined by $f'(u) = 0$ for all leaves $u \in V(C')$ and $f'(w_i) = f(v_i)$ for $i = 1, 2, ..., q$ is a dominating broadcast of C' with $f(V(C)) = f'(V(C'))$. Similarly, for any dominating set D of C which contains no leaves, the set D' defined by $w_i \in D'$ if and only if $v_i \in D$ is a dominating set of C'.*

Thus for any caterpillar C with form F we can define $\gamma_b(F) = \gamma(C)$ and $\gamma(F) = \gamma(C)$. For example, $\gamma(sststtsssts) = 7$, since, by assumption, every leaf must be dominated by its stem. Moreover, although any trunk adjacent to a stem can be dominated by that stem, for any three trunks in a row, one trunk must be a dominator; so, for example, $\gamma(stttsttsstttttttts) = 8$ since it has five stems and three non-overlapping sequences of form ttt.

Lemma 5 *Let C be a caterpillar. Then $\gamma(C) = \sigma + \tau$, where σ is the number of stems in C and τ is the maximum number of non-overlapping sequences of three trunks in a row.*

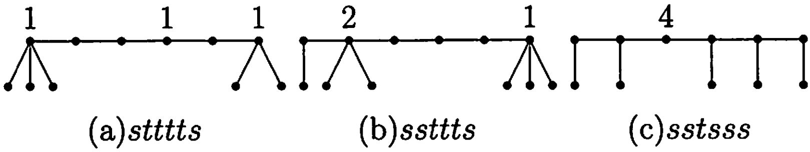

Figure 1: Three caterpillars of form $s?t??s$.

We use exponential notation to indicate repeated sequences in caterpillars, so $ststststts$ will be written $(st)^3ts$. We use ? to indicate a spine vertex which may be either a stem or a trunk. The three caterpillars in Figure 1 all have form $s?t??s$ and thus radius 4; (a) is Type I with $\gamma_b = \gamma = 3$, (b) is Type III with $\gamma_b = 3$ and $\gamma = 4$, and (c) is Type II with $\gamma_b = 4$ and $\gamma = 5$.

3 Caterpillars of Type I

In this section we classify those caterpillars for which $\gamma_b = \gamma$.

Lemma 6 *If form F contains a subsequence of form $F' = sss$, $ss(tttts)^k s$, $ss(tttts)^k ttts$ (or its mirror image $sttt(sttttt)^k ss$), or $sttt(stttt)^k sttts$, where $k \geq 0$, then $\gamma_b(F) < \gamma(F)$.*

Proof. Let D be a minimum dominating set. For each F' define a dominating broadcast f on F by listing the values of f in order for each spine vertex in F' and setting $f(u) = \chi_D(u)$ for $u \in F - F'$: $f(sss) = 2$ for 020; $f(ss(tttts)^k s) = 2k+2$ for $02(00002)^k 0$; $f(ss(tttts)^k ttts) = 2k+3$ for $02(00002)^k 0001$; and $f(sttt(stttt)^k sttts) = 2k+4$ for $1000(20000)^k 20001$. By Lemma 5, $\chi_D(sss) = 3$, $\chi_D(ss(tttts)^k s) = 2k+3$, $\chi_D(ss(tttts)^k ttts) = 2k+4$, and $\chi_D(sttt(stttt)^k sttts) = 2k+5$. Thus, in each case, $f(V) < \chi_D(V)$, and so $\gamma_b(F) < \gamma(F)$. □

Now let f be an optimal broadcast for a caterpillar C, and $z \in V_f$. We will say z *covers* a sequence of spine vertices of C if z dominates all of the spine vertices and all of their leaves.

Lemma 7 *Let f be an optimal broadcast for caterpillar C, $z \in V_f$, F the form of the maximum sequence of spine vertices z can* *cover, and k the length of F. If $F = t?^{k-2}t$ then $f(z) = \lceil \frac{k-1}{2} \rceil$; if $F = s?^{k-2}t$ then $f(z) = \lceil \frac{k}{2} \rceil$; and if $F = s?^{k-2}s$ then $f(z) = \lceil \frac{k+1}{2} \rceil$. Conversely, say $F = F_1?F_2$ where ? is the form of z. If $k = 2f(z)+1$ then F_1, F_2 both have $f(z)$ vertices and a t on the end away from z; if $k = 2f(z)$ then one of F_1, F_2 has length $f(z)$ and a t on the end away from z and the other has length $f(z)-1$; and if $k = 2f(z) - 1$ then F_1, F_2 both have length $f(z) - 1$.*

Proof. Suppose that the maximum sequence of spine vertices z can cover is $u_p u_{p-1} \ldots u_1 z v_1 v_2 \ldots v_q$. If u_p is a stem then z must dominate its leaf so $p = f(z) - 1$, while if u_p is a trunk, then $p = f(z)$ or $p = f(z) - 1$, depending on the type of vertex to the left of u_p; and similarly for v_q. The proof follows immediately from this observation. □

In what follows, when we use the same superscript for ? in two caterpillars, for example $?^p$, we mean that the p vertices indicated by $?^p$ have the same form in each caterpillar. We say that a sequence $?^p?^q$ of spine vertices *splits* into subsequences $?^p$ and $?^q$ if there exists an optimal broadcast f with $y, z \in V_f, y \neq z$, such that y covers the right end vertex of $?^p$ and z covers the left end vertex of $?^q$. Splits into more than two subsequences are defined recursively.

Lemma 8 *For any $p, q \geq 0$, the form $?^p stt?^q$ splits into either $?^p st$ and $t?^q$ or $?^p stt$ and $t?^{q-1}$ (the latter case requiring $q \geq 1$).*

Proof. Let $v_0 v_1 v_2$ denote the sequence stt, and let f be an optimal broadcast. Choose $z \in V_f$ such that z covers the stem v_0; then z also covers the trunk v_1. If z does not cover v_2, then f splits $?^p stt?^q$ into $?^p st$ and $t?^q$, so assume z covers v_2. Let k be the length of the maximum sequence $?^{p'} stt?^{q'}$ that z can cover, so $k = p' + q' + 3$, where $p' \leq p, q' \leq q$. Then, by Lemma 7, $k = 2f(z) + 1, 2f(z)$, or $2f(z) - 1$.

Suppose first $q' = 0$, and let v_3 be the right neighbour of v_2. If v_3 is a stem, then whatever covers v_3 also covers trunk v_2, so f splits $?^{p'} stt?^{q'}$ into $?^{p'} st$ and $t?^{q'}$. Otherwise v_3 is a trunk and f splits $?^{p'} stt?^{q'}$ into $?^{p'} stt$ and $t?^{q'-1}$.

So assume $q' \geq 1$. If p' is odd, let x be the centre vertex of

$?^{p'}st$, and if p' is even, let x be the leftmost of the two centre vertices. Similarly, if q' is even, let y be the centre vertex of $t?^{q'}$, and if q' is odd, let y be the rightmost of the two centre vertices. We define a dominating broadcast f' such that x covers v_0 and v_1, y covers v_2, $f'(x)+f'(y) = f(z)$, $f'(z) = 0$, and $f'(u) = f(u)$ for $u \neq x, y, z$; so f splits $?^{p'}stt?^{q'}$ into $?^{p'}st$ and $t?^{q'}$:

If $k = 2f(z)+1$ then $p'+q' = 2f(z)-2$ is even, and, by Lemma 7, the form must be $t?^{p'-1}stt?^{q'-1}t$. Define $f'(x) = \lceil\frac{p'+1}{2}\rceil$ and $f'(y) = \lceil\frac{q'}{2}\rceil$. Then, whether p', q' are both odd or both even, $f'(x) + f'(y) = \frac{k-1}{2} = f(z)$.

If $k = 2f(z)$ then $p' + q' = 2f(z) - 3$ is odd, and, by Lemma 7, the form must be either $t?^{p'-1}stt?^{q'}$ or $?^{p'}stt?^{q'-1}t$. In the first case, define $f'(x) = \lceil\frac{p'+1}{2}\rceil$ and $f'(y) = \lceil\frac{q'+1}{2}\rceil$. Since one of p', q' is odd and the other even, $f'(x)+f'(y) = \frac{k}{2} = f(z)$. The second case follows by symmetry.

If $k = 2f(z) - 1$ then $p' + q' = 2f(z) - 4$ is even. Define f' by $f'(x) = \lceil\frac{p'+2}{2}\rceil$ and $f'(y) = \lceil\frac{q'+1}{2}\rceil$. Then, whether p' and q' are both odd or both even, $f'(x) + f'(y) = \frac{k+1}{2} = f(z)$. □

Lemma 9 *Suppose $?^pt^ks?^q$ splits into $?^p$ and $t^ks?^q$, for $k \geq 3, p \geq 1, q \geq 0$. Then $t^ks?^q$ splits further into $\frac{k-3}{3}$ copies of ttt followed by tt and $ts?^q$ if $k \equiv 0 (mod\ 3)$; into $\frac{k-1}{3}$ copies of ttt followed by $ts?^q$ if $k \equiv 1 (mod\ 3)$; and into $\frac{k-2}{3}$ copies of ttt followed by $tts?^q$ if $k \equiv 2 (mod\ 3)$.*

Proof. Let f be the optimal broadcast for the original split, and let $v_0v_1v_2...v_kv_{k+1}$ denote the sequence $?t^ks$. Then there exist $y, z \in V_f, y \neq z$, such that y covers v_0 and z covers v_1. Let z' be the right neighbour of z. Define f' by $f'(z') = f(z) - 1$, $f'(z) = 0$, $f'(v_2) = 1$, and $f'(u) = f(u)$ otherwise. If $k = 3$ then whatever covers stem v_4 also covers trunk v_3, so f' splits $ttts?^q$ into tt and $ts?^q$. If $k > 3$ then f' splits $t^ks?^q$ into ttt and $t^{k-3}s?^q$ and the result follows by induction. □

Lemma 10 *If $?^p?^q$ splits into $?^p$ and $?^q$, for $p, q \geq 1$, then $\gamma_b(?^p?^q) = \gamma_b(?^p) + \gamma_b(?^q)$.*

Proof. Let f be the optimal broadcast for the split. Then $f|_{?^p}$

and $f|_{?^q}$ are dominating broadcasts of $?^p$ and $?^q$ respectively, so $\gamma_b(?^p?^q) = f|_{?^p}(?^p) + f|_{?^q}(?^q) \geq \gamma_b(?^p) + \gamma_b(?^q)$.

Conversely, let f_1, f_2 be optimal broadcasts of $?^p, ?^q$ respectively. Then $f' = f_1 \cup f_2$ is a dominating broadcast of $?^p?^q$ so $\gamma_b(?^p?^q) \leq f'(?^p?^q) = \gamma_b(?^p) + \gamma_b(?^q)$. □

Lemma 11 $\gamma_b(?^p st(ttt)^k ts?^q) = \gamma_b(?^p st) + k + \gamma_b(ts?^q)$, *for* $p, q, k \geq 0$.

Proof. By Lemma 8 and Lemma 9, the sequence splits either into $?^p st$, k copies of ttt, and $ts?^q$, so that the result follows from Lemma 10 (using $\gamma_b(ttt) = 1$), or into $?^p stt$, $k - 1$ copies of ttt, tt, and $ts?^q$. The latter case is thus a split into $?^p st(ttt)^k$ and $ts?^q$, so, by Lemma 9 in mirror image, can be resplit into the former case. □

Lemma 12 $\gamma_b(?^p sts?^q) = \gamma_b(?^p s?^q) + 1$, *for* $p, q, k \geq 0$.

Proof. Let C be a caterpillar of form $?^p sts?^q$, with $v_1v_2v_3$ the spine vertices of form sts. Let C' be a caterpillar of form $?^p s?^q$ with v' the spine vertex of form s. For $x \in V(C) - \{v_1v_2v_3\}$, let x' be the corresponding vertex in $V(C') - \{v'\}$.

Let f be an optimal broadcast of C and choose $z \in V_f$ such that z covers stem v_1; then z also covers trunk v_2. Suppose first z does not cover stem v_3. Choose $y \in V_f$ such that y covers v_3; then y also covers trunk v_2. But then, by Theorem 3, there exist an optimal broadcast g and $x \in V_g$ such that x covers v_1, v_2, v_3. So we may assume z covers v_1, v_2, v_3. Let y be the right neighbour of z if z is to the right of v_1, and the left neighbour otherwise. If $z = v_2$, define f' on $V(C')$ by $f'(v') = f(z) - 1$ and $f'(x') = f(x)$ otherwise. If $z \neq v_2$, define f' on $V(C')$ by $f'(v') = 0$, $f'(z) = 0$ if $z \neq v_1, v_3$, $f'(y) = f(z) - 1$, and $f'(x') = f(x)$ otherwise. Either way, f' is a dominating broadcast for C' so $\gamma_b(C) = f(V(C) = f'(V(C')) + 1 \geq \gamma_b(C') + 1$.

Conversely, let f' be an optimal broadcast on C' and choose $z' \in V'_f$ such that z' covers v' in C'. Let y' be the right neighbour of z' if z' is to the left of v', and the left neighbour otherwise. If $y', z' \neq v'$, define f on $V(C)$ by $f(y) = f'(z')+1$, $f(v_1) = f(v_2) = f(v_3) = 0$, and $f(x) = f'(x')$ otherwise. If $z' = v'$, define f on

$V(C)$ by $f(v_2) = f'(z')+1, f(v_1) = f(v_3) = 0$, and $f(x) = f'(x')$ otherwise. If $y' = v'$ and y' is to the left of z', define f on $V(C)$ by $f(v_3) = f'(z') + 1, f(v_1) = f(v_2) = 0$, and $f(u) = f'(u')$ otherwise. If $y' = v'$ and y' is to the right of z', define f on $V(C)$ by $f(v_1) = f'(z')+1, f(v_2) = f(v_3) = 0$, and $f(x) = f'(x')$ otherwise. In each case f is a dominating broadcast for C, so $\gamma_b(C) \leq f(V(C)) = f'(V(C')) + 1 = \gamma_b(C') + 1$. □

Lemma 13 $\gamma_b(?^p st^k s?^q) = \gamma_b(?^p st^{k-3} s?^q)+1$, *for* $p, q \geq 0, k \geq 5$.

Proof. Let C be a caterpillar of form $?^p st^k s?^q$, with $v_0 v_1 ... v_k v_{k+1}$ the spine vertices of form $st^k s$, and let C' be a caterpillar of form $?^p st^{k-3} s?^q$, with $v'_0 v'_1 ... v'_{k-2}$ the spine vertices of form $st^{k-3}s$. For $x \in V(C) - \{v_0 v_1 ... v_{k+1}\}$, let x' be the corresponding vertex in $V(C') - \{v'_0 v'_1 ... v'_{k-2}\}$.

By Lemma 8 and Lemma 9, $?^p st^k s?^q$ splits as either $?^p st$, ttt, and $t^{k-4} s?^q$; or $?^p stt$, ttt, and $t^{k-5} s?^q$. Let f be the optimal broadcast for such a split, and choose $z \in V_f$ such that z covers trunk v_3; then z covers $v_{r-1} v_r v_{r+1}$ of form ttt for either $r = 3$ or $r = 4$, and $f(z) = 1$. Define f' on $V(C')$ by $f'(v'_i) = f(v_i)$ for $0 \leq i < r - 1$, $f'(v'_i) = f(v_{i-3})$ for $r + 1 < i \leq k - 2$, and $f'(x') = f(x)$ otherwise. Then f' is a dominating broadcast for C', so $\gamma_b(C) = f(V(C)) = f'(V(C')) + 1 \geq \gamma_b(C') + 1$.

Conversely, by Lemma 8, $?^p st^{k-3} s?^q$ splits as either $?^p st$ and $t^{k-4} s?^q$, or $?^p stt$ and $t^{k-5} s?^q$. Let f' be the optimal broadcast for such a split, and choose $y', z' \in V'_f, y' \neq z'$ such that y' covers stem v'_0, and thus also trunk v'_1, and z' covers trunk v'_3. Let $p = 2$ if z' covers v'_2 and $p = 3$ otherwise. Define f on $V(C)$ by $f(v_{p-1}) = f(v_{p+1}) = 0, f(v_p) = 1, f(v_i) = f'(v'_{i-3})$ for $0 \leq i < p-1$ and $p+1 < i \leq k+1$, and $f(x) = f'(x')$ otherwise. Then f is a dominating broadcast for C, so $\gamma_b(C) \leq f(V(C)) = f'(V(C')) + 1 = \gamma_b(C') + 1$. □

For any form F we define the *b−decomposition* of F as follows:
1. For all $k \geq 5$, replace all sequences $st^k s$ with $sttts$ if $k \equiv 0 \pmod 3$, $stttts$ if $k \equiv 1 \pmod 3$, and $stts$ if $k \equiv 2 \pmod 3$.
2. For all $k \geq 1$, replace all maximal sequences $(st)^k s$ with s.
3. Remove tt from each $stts$ sequence.

The $b-$*decomposition* of F consists of the resulting components $F_1, F_2, ...F_p$. For example, $stssstttttstttttttstststststtts \longrightarrow stssstt$ $stttststststtts \longrightarrow ssstttsttttsttts \longrightarrow sss$ and $stttttsttts$. Note that whatever covers a stem also covers a neighbouring trunk; so, for example, $\gamma_b(ssst) = \gamma_b(sss)$. This observation, along with Lemmas 11, 12, and 13, gives us the following result:

Lemma 14 *Let $F_1, F_2, ...F_p$ be the $b-$decomposition of form F. Then $\gamma_b(F) = \gamma(F)$ if and only if $\gamma_b(F_i) = \gamma(F_i)$ for $i = 1, 2, ..., p$.*

The next five lemmas characterize the Type I caterpillars. Note that the only possible st^ks sequences ($k \geq 0$) in any component of a $b-$decomposition are ss, $sttts$ and $stttts$.

Lemma 15 *Let F' be a component of a $b-$decomposition. If F' contains two ss sequences, two $sttts$ sequences, or one ss and one $sttts$ sequence (with possible overlap on one s, giving sss, $ssttts$, $stttss$, or $stttsttts$), then $\gamma_b(F') < \gamma(F')$.*

Proof. Choose two such sequences in F' with no other such sequence between them. Either they overlap, they are adjacent, or the sequence between them is $(tttts)^k tttt$ for some $k \geq 0$. In each case, by Lemma 6, $\gamma_b(F') < \gamma(F')$. □

Lemma 16 *Let F' be a component of a $b-$decomposition. If F' contains no $sttts$ or ss sequences, then $\gamma_b(F') = \gamma(F')$.*

Proof. The only possibility for F' is $s(tttts)^k$, for some $k \geq 0$, so by Lemma 5, $\gamma(F') = 2k+1$. If $k = 0$ then $F' = s$, and $\gamma_b(s) = 1 = \gamma(s)$. So suppose $k \geq 1$. By Lemma 8 in mirror image, F' splits as either $sttt$ and $ts(tttts)^{k-1}$, or stt and $tts(tttts)^{k-1}$. In the first case, by Lemma 10, $\gamma_b(F') = \gamma_b(sttt)+\gamma_b(ts(tttts)^{k-1})$. Then $\gamma_b(sttt) = 2$ and $\gamma_b(ts(tttts)^{k-1}) = \gamma_b(s(tttts)^{k-1}) = 2(k-1)+1$ by induction, so $\gamma_b(F') = 2+2(k-1)+1 = 2k+1$. In the second case, $\gamma_b(stt) = 2 = \gamma_b(sttt)$ so we may assume that whatever covers the last t of stt also covers the first t of $tts(tttts)^{k-1}$, reducing the second case to the first case. □

Lemma 17 *Let F' be a component of a $b-$decomposition. If F' contains no $sttts$ sequences and exactly one ss sequence, then $\gamma_b(F') = \gamma(F')$.*

Proof. The only possibility for F' is $s(tttts)^p s(tttts)^q$, for some $p, q \geq 0$. Let $k = p + q$. Then, by Lemma 5, $\gamma(F') = 2k + 2$. If $k = 0$ then $F' = ss$, and $\gamma_b(ss) = 2 = \gamma(ss)$. So suppose $k \geq 1$. If $p \geq 1$, then F' has the form $sttts(tttts)^{p-1}s(tttts)^q$; apply the same inductive argument as in Lemma 16. If $q \geq 1$, apply the same inductive argument in mirror image. □

Lemma 18 *Let F' be a component of a b–decomposition. If F' contains no ss sequences and exactly one $sttts$ sequence, then $\gamma_b(F') = \gamma(F')$.*

Proof. The only possibility for F' is $s(tttts)^p ttts(tttts)^q$, for some $p, q \geq 0$. Let $k = p+q$. Then, by Lemma 5, $\gamma(F') = 2k+3$. If $k = 0$ then $F' = sttts$, and $\gamma_b(sttts) = 3 = \gamma(sttts)$. If $k \geq 1$, apply the same argument as in Lemma 17. □

The last five lemmas prove the following theorem.

Theorem 19 *Let F be the form of a caterpillar. Then $\gamma_b(F) = \gamma(F)$ if and only if each component of the b–decomposition of F contains at most one ss or $sttts$ sequence in total, including overlaps.*

This can be rewritten in terms of the original caterpillar.

Theorem 19′ *Let C be a caterpillar. Then $\gamma_b(C) = \gamma(C)$ if and only if between any two sequences of form $st^p s$ and $st^q s$, where $p \equiv q \equiv 0$ (mod 3), there is at least one (possibly overlapping) sequence of form $st^r s$, where $r \equiv 2$ (mod 3).*

4 Caterpillars of Type II

Next we determine when a caterpillar C is radial. Note that $\mathrm{rad}(C) = \lceil \frac{diam(C)}{2} \rceil$, so C is radial if and only if $\gamma_b(C) = \lceil \frac{diam(C)}{2} \rceil$. Also, given any optimal broadcast f for which V_f contains no leaves, repeated application of Theorem 3 results in an efficient optimal broadcast g for which V_g contains no leaves.

Lemma 20 *Let C be a non-radial caterpillar and choose an efficient optimal broadcast f for C such that V_f contains no* *leaves and $|V_f| = min\{|V_g| : g$ is an efficient optimal broadcast and V_g contains no leaves$\}$. Let $V_f = \{z_1, z_2, ..., z_r\}$, the z_i's listed in the order they occur along the spine. Let C_i be the maximum sub-caterpillar covered by z_i for $i = 1, 2, ..., r$, and $u_i v_i$ the edge between C_i and C_{i+1} for $i = 1, 2, ..., r-1$; then u_i and v_i are both trunks, and $diam(C_i)$ is even, for all i.*

Proof. Since C is not radial, $r > 1$. Note that z_i dominates u_i and z_{i+1} dominates v_i. If, say, u_i is a stem with leaf w then either z_i dominates w and thus also v_i, or z_{i+1} dominates w and thus also u_i; either way contradicting efficiency. So u_i and v_i are trunks for $i = 1, 2, ..., r-1$.

By Lemma 10, $\gamma_b(C) = \sum_i \gamma_b(C_i)$ and $f|_{C_i}$ is optimal on C_i for $i = 1, ..., r$, so each C_i is radial and $\gamma_b(C_i) = \lceil \frac{diam(C_i)}{2} \rceil$. Suppose diam$(C_i)$ is odd for some i; then either $i > 1$ or $i < r$ so without loss of generality suppose $i < r$. Let $C' = C_i \cup \{u_i v_i\} \cup C_{i+1}$. Since u_i and v_i are trunks, diam(C') = diam(C_i)+ 1+diam(C_{i+1}). By Lemma 10, $\gamma_b(C') = \gamma_b(C_i) + \gamma_b(C_{i+1}) = \lceil \frac{diam(C_i)}{2} \rceil + \lceil \frac{diam(C_{i+1})}{2} \rceil$. If diam$(C_{i+1})$ is also odd, then $\gamma_b(C') = \frac{diam(C_i)+1}{2} + \frac{diam(C_{i+1})+1}{2} = \frac{diam(C')+1}{2} = \text{rad}(C')$. If diam$(C_{i+1})$ is even, then $\gamma_b(C') = \frac{diam(C_i)+1}{2} + \frac{diam(C_{i+1})}{2} = \frac{diam(C')}{2} = \text{rad}(C')$. In either case, C' has an optimal broadcast f' which is radial. Extending f' to $V(C)$ by $f'(u) = f(u)$ for $u \in V(C) - V(C')$ gives an optimal broadcast f' of C with $|V_f'| < |V_f|$. By repeated application of Theorem 3, C has an efficient optimal broadcast g with $|V_g| \leq |V_{f'}| < |V_f|$, a contradiction. Thus diam(C_i) is even for $i = 1, 2, ..., r$. □

Theorem 21 *A caterpillar with odd diameter is radial if and only if when the edge between any two adjacent trunks is removed at least one of the resulting two components has odd diameter. A caterpillar with even diameter is radial if and only if when the edges between any two disjoint pairs of adjacent trunks are removed, at least one of the resulting three components has odd diameter.*

Proof. Suppose first C is not radial. Let f, u_i, v_i, and C_i be as in Lemma 20, so u_i and v_i are trunks and diam(C_i) is even. If diam(C) is odd, then since diam(C) = diam(C_1)+1+diam($C - C_1$), removing u_1v_1 gives two components of even diameter. Similarly, if diam(C) is even then removing u_1v_1 and u_2v_2 gives three components C_1, C_2 and $C - C_1 - C_2$, all of even diameter.

Conversely, suppose that diam(C) is odd and there exist two adjacent trunks on the spine of C such that when the edge between them is removed, each of the two components C_1 and C_2 has even diameter. Then since diam(C) is odd, rad(C) = $\frac{diam(C)+1}{2}$. But rad(C_1)+rad(C_2) = $\frac{diam(C_1)+diam(C_2)}{2}$ = $\frac{diam(C)-1}{2}$ < rad(C), so the union of radial broadcasts on each of C_1 and C_2 forms a dominating broadcast with smaller cost than a radial broadcast on C, and thus C is not radial.

Now suppose that diam(C) is even and there exist two disjoint pairs of adjacent trunks on the spine of C such that when the edge between each pair is removed, all three of the resulting components C_1, C_2, C_3 have even diameter. Since diam(C) is even, rad(C) = $\frac{diam(C)}{2}$. But rad(C_1)+rad(C_2)+rad(C_3) = $\frac{diam(C_1)+diam(C_2)+diam(C_3)}{2}$ = $\frac{diam(C)-2}{2}$ < rad(C), so the union of radial broadcasts on C_1, C_2 and C_3 forms a dominating broadcast with smaller cost than a radial broadcast on C, and thus C is not radial. □

5 Conclusion

We have completely characterized caterpillars of Type I, Type II, and Type III, so the next step would be to characterize trees. We give one partial result.

Theorem 22 *If T is a tree with $\gamma_b(T) = \gamma(T)$ then no vertex of T has more than three non-leaf neighbours.*

Proof. Let D be a minimum dominating set for T. Suppose for some $v \in V(T)$ there exist $v_1, v_2, v_3, v_4 \in N(v)$ with $d(v_i) \geq 2$ for $i = 1, 2, 3, 4$. Let $w_i \in N(v_i) - \{v\}$ for $i = 1, 2, 3, 4$. If $v \in D$, then v does not dominate any w_i, so there exist distinct $x_1, x_2, x_3, x_4 \in D$ such that x_i dominates w_i for $i = 1, 2, 3, 4$. But then $d(v, x_i) \leq 3$ for $i = 1, 2, 3, 4$. Define f on V by $f(v) = 4$,

$f(v_i) = f(w_i) = f(x_i) = 0$ for $i = 1, 2, 3, 4$ and $f(u) = \chi_D(u)$ otherwise. Then f is a dominating broadcast with $f(V) = |D| - 1$, so $\gamma_b(T) < \gamma(T)$.

So suppose $v \notin D$. Then some non-leaf neighbour of v must dominate v, so without loss of generality we may assume $v_1 \in D$. Then v_1 does not dominate v_2, v_3, v_4, so there exist distinct $w_2, w_3, w_4 \in D$ such that w_i dominates v_i for $i = 2, 3, 4$. But then $d(v, w_i) \leq 2$ for $i = 2, 3, 4$. Define f on V by $f(v) = 3$, $f(v_i) = 0$ for $i = 1, 2, 3, 4$, $f(w_i) = 0$ for $i = 2, 3, 4$ and $f(u) = \chi_D(u)$ otherwise. Then f is a dominating broadcast with $f(V) = |D| - 1$, so $\gamma_b(T) < \gamma(T)$. □

Blair et al. [1] give a $O(nr)$ algorithm for determining γ_b for a tree with n vertices and radius r (as well as algorithms for interval graphs and series-parallel graphs).

References

1. J. R. S. Blair, P. Heggernes, S. Horton, and F. Manne, Broadcast domination algorithms for interval graphs, series-parallel graphs, and trees, preprint.

2. J. E. Dunbar, D. J. Erwin, T. W. Haynes, S. M. Hedetniemi, and S. T. Hedetniemi, Broadcasts in graphs, preprint.

3. D. J. Erwin, Dominating broadcasts in graphs, *Bull. Inst. Comb. Appl.* **42** (2004) 89-105.