A Note on Star Arboricity of Crowns

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ABSTRACT. The star arboricity $sa(G)$ of a graph G is the minimum number of star forests which are needed to decompose all edges of G. For integers k and n, $1 \leq k \leq n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}\}$ and edge set $\{a_ib_j : i = 0, 1, \cdots, n-1;\ j \equiv i+1, i+2, \cdots, i+k \pmod n\}$. In [2], Lin et al. conjectured that for every k and n, $3 \leq k \leq n-1$, the star arboricity of the crown $C_{n,k}$ is $\lceil k/2 \rceil + 1$ if k is odd and $\lceil k/2 \rceil + 2$ otherwise. In this note we show that the above conjecture is not true for the case $n = 9t$ (t is a positive integer) and $k = 4$ by showing that $sa(C_{9t,4}) = 3$.

A *star forest* is a forest whose components are stars. The *star arboricity* $sa(G)$ of a graph G is the minimum number of star forests which are needed to decompose the edges of G. The star arboricity of regular graphs can be bounded below as follows:

Proposition 1 [1, 3] *Suppose that G is a d-regular graph with $d \geq 2$. Then $sa(G) \geq \lceil d/2 \rceil + 1$.* □

For integers k and n, $1 \leq k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}\}$ and edge set $\{a_ib_j : i = 0, 1, \cdots, n-1;\ j \equiv i+1, i+2, \cdots, i+k \pmod n\}$.

In [2] the star arboricity of crowns were investigated and the following conjecture was proposed.

Conjecture. *For every k and n, $3 \le k \le n-1$,*

$$sa(C_{n,k}) = \begin{cases} \lceil k/2 \rceil + 1 & \text{if } k \text{ is odd,} \\ \lceil k/2 \rceil + 2 & \text{if } k \text{ is even.} \end{cases}$$

It was proved in [2] that the above conjecture holds for (1) $k = 3, 5$, (2) $k = n-1, n-2, n-3 \ge 3$, (3) $k \ge 3$ is odd and $n = a\lceil k/2 \rceil + b$ where a and b are nonnegative integers with $a \ge b$.

In this note we show that the above conjecture is not true for the case $n = 9t$ (t is a positive integer), $k = 4$ by proving the following

Theorem 2 *For every positive integer t, $sa(C_{9t,4}) = 3$.*

Proof. By Proposition 1, $sa(C_{9t,4}) \ge 3$. To show $sa(C_{9t,4}) \le 3$ we need to decompose $C_{9t,4}$ into 3 star forests. For a subset S of $E(C_{n,k})$ and an integer p, we denote by $S+p$ the set $\{a_{i+p}b_{j+p} : a_i b_j \in S\}$. The subscripts of a_i and b_j are always taken modulo n.

Let S_1 be the subset $\{a_0b_1, a_0b_2, a_1b_3, a_1b_4, a_2b_5, a_2b_6, a_3b_7, a_4b_7, a_5b_8, a_6b_8, a_7b_9, a_8b_9\}$ of $E(C_{9t,4})$, and let F_1 be the subgraph of $C_{9t,4}$ induced by $S_1 \cup (S_1+9) \cup (S_1+18) \cup \cdots \cup (S_1+9(t-1))$.

Let S_2 be the subset $\{a_0b_4, a_2b_4, a_1b_5, a_4b_5, a_3b_6, a_5b_6, a_6b_7, a_6b_9, a_7b_8, a_7b_{10}, a_8b_{11}, a_8b_{12}\}$ of $E(C_{9t,4})$, and let F_2 be the subgraph of $C_{9t,4}$ induced by $S_2 \cup (S_2+9) \cup (S_2+18) \cup \cdots \cup (S_2+9(t-1))$.

Let S_3 be the subset $\{a_0b_3, a_2b_3, a_7b_{11}, a_1b_2, a_3b_4, a_3b_5, a_4b_6, a_4b_8, a_5b_7, a_5b_9, a_6b_{10}, a_8b_{10}\}$ of $E(C_{9t,4})$, and let F_3 be the subgraph of $C_{9t,4}$ induced by $S_3 \cup (S_3+9) \cup (S_3+18) \cup \cdots \cup (S_3+9(t-1))$.

We can see that each F_i ($i = 1, 2, 3$) is a star forest of $C_{9t,4}$, and that $C_{9t,4}$ is decomposed into F_1, F_2, F_3. This completes the proof. □

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