The total chromatic number of some bipartite graphs*

C. N. Campos† C. P. de Mello†

Abstract

The total chromatic number $\chi_T(G)$ is the least number of colours needed to colour the vertices and edges of a graph G such that no incident or adjacent elements (vertices or edges) receive the same colour. This work determines the total chromatic number of grids, particular cases of partial grids, near-ladders, and of k-dimensional cubes.

1 Introduction

Let $G := (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An *element* of G is a vertex or an edge of G. An edge $\{u, v\}$ is denoted by uv or vu. For a vertex $v \in V(G)$, $N(v)$ is the set of vertices of G that are adjacent to v.

For $S \subseteq V(G) \cup E(G)$ and C a set of colours, a *partial total colouring* of G is a mapping $\phi : S \to C$ such that, for each pair of adjacent or incident elements $x, y \in S$, we have $\phi(x) \neq \phi(y)$. If $S = V(G) \cup E(G)$, then ϕ is a *total colouring.* If $|C| = k$, then the mapping ϕ is called a *(partial) k-total colouring.* If $\phi(x) = c$ or there exists an element y incident with or adjacent to x such that $\phi(y) = c$, then we say that c *occurs* in x; otherwise c *is missing in* x. If $S \subseteq E(G)$, then ϕ is a *(partial) edge colouring* and if $S \subseteq V(G)$, then ϕ is a *(partial) vertex colouring.*

The *total chromatic number* of G, $\chi_T(G)$, is the least integer k for which G admits a k-total colouring. Clearly, $\chi_T(G) \geq \Delta(G) + 1$. Sánchez-Arroyo [11] showed that deciding whether $\chi_T(G) = \Delta(G) + 1$ is NP-complete. McDiarmid and Sánchez-Arroyo [9] showed that even the problem of determining the total chromatic number of k-regular bipartite graphs

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†Instituto de Computação, UNICAMP, Caixa Postal 6176, 13083-970, Campinas, SP, Brasil. {campos, celia}@ic.unicamp.br.

is NP-hard, for each fixed $k \geq 3$. The *Total Colouring Conjecture (TCC)*, posed independently by Behzad [1] and Vizing [14], states that every simple graph G has $\chi_T(G) \leq \Delta(G) + 2$. If $\chi_T(G) = \Delta(G) + 1$, then G is a *type 1* graph; if $\chi_T(G) = \Delta(G) + 2$, then G is a *type 2* graph.

In this work we study the total chromatic number of some subclasses of bipartite graphs. Behzad et. al. [2] determined the total chromatic number of complete graphs, including the bipartite case. A *k-partite graph* is a generalization of bipartite graphs in which the vertex set is partitioned into k sets. A *complete k-partite graph* is a k-partite graph where every vertex of one part is adjacent to every vertex of all other parts and a *balanced k-partite graph* is a k-partite graph with all parts of the same size. Bermond [3] determined the total chromatic number of all balanced complete k-partite graphs. Yap [15] extended a previous result of Rosenfeld [10] showing that every complete k-partite graph verifies the TCC. Chew and Yap [7] and Hoffman and Rodger [8] showed that every complete k-partite graph having odd number of vertices is type 1.

Almost all graphs analysed in this work are planar graphs. The TCC was verified for planar graphs with maximum degree 7 in [12]; the total chromatic number was determined for planar graphs with large girth in [5]; and with maximum degree greater than 11 in [4]. Moreover, Zhang et. al. [17] showed that outerplanar graphs with maximum degree greater than or equal to 3 are type 1.

Section 2 determines the total chromatic number of grids and of some particular cases of partial grids. Section 3 shows that near-ladder graphs with $|V(G)/2|$ even are type 1; otherwise are type 2. Section 4 shows that Q_k, the k-dimensional cube, is type 1.

2 Grids and partial grids

A simple graph $G_{m \times n}$, with vertex set the cartesian product of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, that is $V(G_{m \times n}) := \{(i,j)$, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}\}$, and edge set $E(G_{m \times n}) := \{(i,j)(k,l) : |i-k| + |j-l| = 1$, $(i,j),(k,l) \in V(G_{m \times n})\}$, is called an $m \times n$ *grid*. In fact, $G_{m \times n}$ is a cartesian product of P_m and P_n, path graphs on m and n vertices respectively. It is easy to see that grids are planar and bipartite. A *partial grid* is an arbitrary subgraph of a grid. We consider only connected partial grids.

In this section we prove that $G_{m \times n}$, with $m, n \geq 2$ and different from C_4, is type 1 and determine $\chi_T(G)$ for some particular cases of partial grids. Partial grids are harder to work with than grids; for instance, recognition of grids is polynomial, but is an open problem for partial grids ([6]).

THEOREM 1
Each graph $G_{m \times n}$, with $m, n \geq 2$ and different from C_4, is type 1.

Proof: First we consider the case when $m > 2$ and $n > 2$. Let $G := G_{m \times n}$ be a grid. Let π be a colour assignment for G that uses 5 colours defined as:

$$\pi((i,j)) := (2j + i - 2) \bmod 3; \quad (1)$$
$$\pi((i,j)(i,j+1)) := (2j + i - 1) \bmod 3; \quad (2)$$
$$\pi((i,j)(i+1,j)) := 4 - (i \bmod 2). \quad (3)$$

Now, we prove that π is a total colouring for G. In order to do this we show that the colour of each element of G is different from the colours of each of its adjacent and incident elements.

We start by considering edges $(i,j)(i+1,j)$, coloured in (3). By construction, these edges have colours 3 or 4 and these colours do not occur in (1) or (2). Moreover, adjacent edges coloured in (3) have colours with different parities. We conclude that (3) is an edge colouring for the subgraph of G induced by these edges.

Now, we analyse the vertices of G. Let (i,j) be a vertex of G. By construction, $\pi((i,j)) = (2j + i - 2) \bmod 3$. First, we consider the vertices of G that are adjacent to (i,j). These are, when they exist, $(i,j-1)$, $(i,j+1)$, $(i-1,j)$, and $(i+1,j)$, which have colours $(2j+i-1) \bmod 3$, $(2j+i) \bmod 3$, $(2j+i) \bmod 3$, and $(2j+i-1) \bmod 3$, respectively. Note that each colour is of the form $(a-b) \bmod 3$, where $a = 2j + i$ and $b \in \{0,1\}$. Moreover, vertex (i,j) has $b = 2$, differing from the others by at least 1 unit and at most 2 units. Therefore, the colours of its adjacent vertices are different from $\pi((i,j))$.

Consider now the edges incident with (i,j), that are, when they exist, edges $(i,j-1)(i,j)$, $(i,j)(i,j+1)$, $(i-1,j)(i,j)$, and $(i,j)(i+1,j)$, which have colours $(2j+i) \bmod 3$, $(2j+i-1) \bmod 3$, $4-(i-1) \bmod 2$, and $4 - i \bmod 2$, respectively. The colours of the first two edges differ from the colour of (i,j) by the same reasons of the previous case and the last two use colours 3 and 4, which are not used in the vertices of G.

In order to finish the proof of this case we have to show that two adjacent edges whose colour was given by (2) have different colours. To see this, consider an edge $(i,j)(i,j+1)$ and its two adjacent edges $(i,j-1)(i,j)$ and $(i,j+1)(i,j+2)$ whose colours are $(2j+i-1) \bmod 3$, $(2j+i) \bmod 3$, and $(2j+i+1) \bmod 3$, respectively. Again, these three colours are different and we are done.

Now, we assume that one of $\{m,n\}$ is 2. By symmetry, we can assume that $m = 2$. These graphs have maximum degree 3 because $n > 2$ and their colourings can be obtained directly from the previous colouring π.

Note that all edges whose colour was assigned in (3) have the same colour. We conclude that only four colours are used and the result follows. □

Let G be a connected partial grid. If $\Delta(G) = 0$ then G is composed by only one vertex, a type 1 graph. If $\Delta(G) = 1$, then $G \cong K_2$, a type 2 graph. If $\Delta(G) = 2$, then it is a path of length at least 2, a type 1 graph, or a cycle that is type 1 when $|V(G)| \equiv 0 \bmod 3$, and type 2 otherwise ([16]). If $\Delta(G) = 4$, then G is type 1 since it is a subgraph of a $G_{m\times n}$ with $m, n > 2$ that preserves the maximum degree and those grids are type 1. Therefore, the remaining case is $\Delta(G) = 3$. For these graphs we determine the total chromatic number of some cases.

THEOREM 2
Let G be a connected partial grid with maximum degree 3. If the length of the largest induced cycle of G is 4, then G is type 1.

Proof: First, we need an additional definition and two auxiliary results stated in Lemma 3 and Lemma 4. We define a *ladder graph*, L_n, as a $G_{2\times n}$, $n \geq 2$, and call its four vertices of degree 2 *corners*.

LEMMA 3
Every tree is type 1, except for K_2 that is type 2.

Proof: Let T be a tree. If T has no edges, then T is type 1. If T is K_2, then T is type 2. Suppose now that $\Delta(T) \geq 2$.

Let $u \in V(T)$ be a vertex of degree 1. Let $T' := T - u$. If T' is K_2, then it is type 2 and we can easily extend any 3-total colouring of T' to T without adding new colours. Now, we can assume that T' is not isomorphic to K_2. By induction hypothesis, there exists a $(\Delta(T') + 1)$-total colouring for T'.

Let v be the vertex of T adjacent to u. If $\Delta(T') = \Delta(T)$, then v is not a vertex of maximum degree in T'. Therefore, there exists a colour missing in v. Thus, assign this missing colour to edge uv. If $\Delta(T') = \Delta(T) - 1$, then v is a vertex of maximum degree in T'. Therefore, we assign a new colour to edge uv. Finally, in both cases, we assign to vertex u a colour different from the colours of uv and v. □

LEMMA 4
If G is a connected partial grid with maximum degree 3 having largest induced cycle with length 4, then G can be decomposed in connected subgraphs each of which is isomorphic to a ladder or a tree. Moreover, there exists an ordering of these subgraphs $G_1, \ldots, G_k$, where, for each G_i, $i > 1$, there exists exactly one G_j, such that $j < i$ and $V(G_i) \cap V(G_j) \neq \emptyset$. In particular, $|V(G_i) \cap V(G_j)| = 1$.

Proof: Let F be the subgraph induced by the edges of G that do not belong to ladders. Note that F is a forest since a largest induced cycle has length 4. Let G^* be the intersection graph of the maximal ladders of G and the connected components of F. Two maximal ladders are always vertex disjoint because $\Delta(G) = 3$. Therefore, if two vertices of G^* are adjacent, then one of them represents a maximal ladder and the other a connected component of F. Clearly G^* is connected. Moreover, we claim that it is a tree; otherwise there would exist in G a cycle of length greater than 4 or a vertex of degree greater than 3.

Now, choosing a vertex to be the root, we perform a depth-first-search in G^* labeling the vertices $1, \ldots, k$ in the order that they are visited. The subgraph represented by vertex i is called G_i.

By construction, G_i and G_j, $i \neq j$, have at most one vertex in common. Moreover, for each G_i there exists only one G_j such that $V(G_i) \cap V(G_j) \neq \emptyset$ that is the father of i in the depth-first tree. Therefore, $j < i$. □

Let G be a graph as stated in the hypothesis. Let $G_1, \ldots, G_k$ be the ordering of the connected subgraphs of G stated in Lemma 4. Note that each connected subgraph G_i has a 4-total colouring, either by Lemma 3, or by Theorem 1. Let π_i be such a 4-total colouring for G_i.

Starting from π_2 and following the order, we adjust the colours of π_i as follows to ensure that $\cup_{i=1}^{k} \pi_i$ is a total colouring for G. Let G_i be the next graph in the ordering. By Lemma 4, there exists only one G_j, with $j < i$, such that $V(G_i) \cap V(G_j) \neq \emptyset$ and, in particular, $|V(G_i) \cap V(G_j)| = 1$. Adjust the colours of π_i so that: (i) $v \in V(G_i) \cap V(G_j)$ has the same colour in π_i as in π_j; (ii) the edges of G_i that are incident with v have colours missing in v in G_j. Note that by Lemma 4 and because the maximum degree of v in G is 3, these adjustments of colours are always possible. □

THEOREM 5

Let G be a connected partial grid with maximum degree 3. If G has at most three vertices of degree 3, then G is type 1.

Proof: We consider three cases depending on the number of vertices of degree 3.

CASE 1 *Graph G has exactly one vertex of degree 3.*

We prove the assertion by induction. Since there exists a vertex of degree 3, graph G has at least 4 vertices. Moreover, there exists only one vertex of degree 3; thus, we conclude that there exists at least one vertex of degree 1. If $|V(G)| = 4$, then G is isomorphic to $K_{1,3}$, a type 1 graph.

Let G be a graph as in the hypothesis and let v be a vertex of degree 1. Let $G' := G - v$. If $\Delta(G') = 2$, then G' is a path or a cycle. So, there exists a 4-total colouring π' for G'. If $\Delta(G') = 3$, then G' has a 4-total colouring π' by induction hypothesis.

We construct π, a 4-total colouring for G, from π'. Let u be the vertex adjacent to v. The degree of u in G' is at most 2. Therefore, there exists a colour that can be assigned to edge uv. Moreover, vertex v is adjacent only to u and it is incident only with uv. Therefore, there exist two colours that can be assigned to v and the result follows.

CASE 2 *Graph G has exaclty two vertices of degree 3.*

Let u and v be the two vertices of degree 3. Suppose first that there exists a vertex w of degree 1 adjacent to one of $\{u, v\}$. Let $G' := G - w$. Graph G' has exactly one vertex of degree 3. By Case 1, G' has a 4-total colouring π', which can be easily expanded to a 4-total-colouring of G as described there.

Thus, we can assume that every vertex adjacent to u or v has degree at least two. Suppose first that there exists an induced path $P := (x_1, x_2, x_3, x_4)$, such that $x_1 \in \{u, v\}$, $x_2, x_3 \notin \{u, v\}$, and, if possible, $x_4 \notin \{u, v\}$. Let $G' := G - \{x_2, x_3\}$. Graph G' has at most one vertex of degree 3. Therefore, by previous cases, G' has a 4-total colouring π'. It is easy to see that π' can be expanded to a 4-total colouring of G, with perhaps a few minor colour adjustments.

Finally, if the previous cases do not apply, we claim that G is isomorphic to one of the type 1 graphs exhibited in Figure 1.

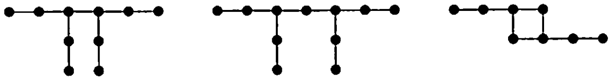


Figure 1: Each case has exactly two vertices of degree 3 and is type 1.

CASE 3 *Graph G has exactly three vertices of degree 3.*

We prove this case by induction. Since G has three vertices of degree 3, $|V(G)| \geq 7$. If $|V(G)| = 7$, then G is not a tree and the size of a largest induced cycle is 4. Therefore, by Theorem 2, G is type 1.

Graph G has at least one vertex of degree 1, say v. Let $G' := G - v$. Graph G' has two or three vertices of degree 3 depending on the degree of the vertex adjacent to v. If G' has two vertices of degree 3, then there exists a 4-total colouring π' for G' by Case 2. If G' has three vertices of degree 3, then there exists a 4-total colouring π' for G' by induction hypothesis. For each case we construct a 4-total colouring for G from π' as it was done in Case 1. □

3 Near-ladder graphs

The *near-ladder graph*, B_k, is a 3-regular bipartite connected graph with bipartition (X_k, Y_k), $X_k := \{x_0, \ldots, x_{k-1}\}$ and $Y_k := \{y_0, \ldots, y_{k-1}\}$, such that for each $x_i \in X_k$, $N(x_i) := \{y_i, y_{(i+1) \bmod k}, y_{(i+2) \bmod k}\}$.

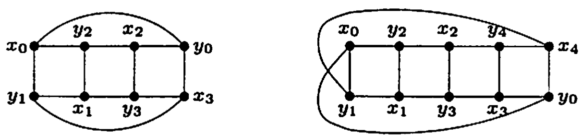
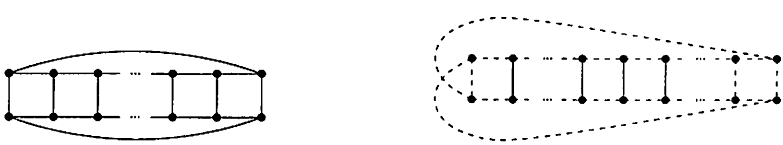


Figure 2: Graphs B_4 and B_5.

Near-ladders have several automorphisms. We remark two of them: (i) the σ-automorphism, in which graph B_k is rotated once along the vertical axis, is defined as $\sigma(x_i) := y_{i+2}$ and $\sigma(y_i) := x_i$; (ii) the τ-automorphism, in which graph B_k is flipped along the horizontal axis, is defined as $\tau(x_i) := y_{i+1}$ and $\tau(y_i) := x_{i-1}$. All operations on indexes are modular.

Near-ladders with k of different parities have important differences in their structures. Graphs B_k, k even, are planar graphs (yet not outerplanar) and B_k, k odd, are not. Figure 3 shows drawings that manifest this property.



(a) Planar drawing.

(b) Dashed edges induces a subdivision of $K_{3,3}$.

Figure 3: Near ladder graphs:(a) k even; (b) k odd.

For B_k and elements x_i, y_{i+1}, $x_i y_{i+2}$, $y_{i+1} x_{i+1}$, the pairs x_i, $y_{i+1} x_{i+1}$ and y_{i+1}, $x_i y_{i+2}$ are called *equivalent pairs.* The edges of an equivalent pair are called a *parallel pair.*

LEMMA 6
Let $G := B_k$ and let π be a 4-total colouring for the subgraph $G_{2\times k}$ obtained by removing exactly one parallel pair from G. Then, for each remaining equivalent pairs x_i, $y_{i+1}x_{i+1}$ and y_{i+1}, x_iy_{i+2}: (i) the edges of parallel pair x_iy_{i+2} and $y_{i+1}x_{i+1}$ have different colours; (ii) either $\pi(x_i) = \pi(y_{i+1}x_{i+1})$ or $\pi(y_{i+1}) = \pi(x_iy_{i+2})$, but not both.

Proof: In order to prove (i) item, suppose that $\pi(x_iy_{i+2}) = \pi(y_{i+1}x_{i+1})$. Elements y_{i+2}, x_{i+1}, and $x_{i+1}y_{i+2}$ have distinct colours and different from $\pi(x_iy_{i+2})$. Moreover, $\pi(x_{i+1}) = \pi(y_{i+2}x_{i+2})$ and $\pi(y_{i+2}) = \pi(x_{i+1}y_{i+3})$. Therefore, elements x_{i+2}, y_{i+3}, and $x_{i+2}y_{i+3}$ have only two colours assigned: $\pi(x_iy_{i+2})$ and $\pi(x_{i+1}y_{i+2})$, a contradiction. We conclude that $\pi(x_iy_{i+2}) \neq \pi(y_{i+1}x_{i+1})$.

Now, we prove (ii) item. First note that $\pi(x_i) \neq \pi(y_{i+1})$, $\pi(x_i) \neq \pi(x_iy_{i+2})$, and $\pi(y_{i+1}) \neq \pi(y_{i+1}x_{i+1})$ because they are adjacent or incident. Moreover, we have already proved that $\pi(x_iy_{i+2}) \neq \pi(y_{i+1}x_{i+1})$. Suppose that $\pi(x_i) \neq \pi(y_{i+1}x_{i+1})$ and $\pi(y_{i+1}) \neq \pi(x_iy_{i+2})$. We conclude that $\pi(x_i)$, $\pi(y_{i+1})$, $\pi(y_{i+1}x_{i+1})$, and $\pi(x_iy_{i+2})$ are pairwise distinct. Edge x_iy_{i+1} is incident with or adjacent to all these four elements. Therefore, $\pi(x_iy_{i+1})$ must be different from each one, contradiction. We conclude that either $\pi(x_i) = \pi(y_{i+1}x_{i+1})$ or $\pi(y_{i+1}) = \pi(x_iy_{i+2})$.

Suppose now that $\pi(x_i) = \pi(y_{i+1}x_{i+1})$ and $\pi(y_{i+1}) = \pi(x_iy_{i+2})$. Then, $\pi(x_{i+1})$, $\pi(y_{i+2})$, and $\pi(x_{i+1}y_{i+2})$ are different from $\pi(x_i)$ and $\pi(y_{i+1})$, a contradiction since only four colours are allowed and x_{i+1}, y_{i+2}, $x_{i+1}y_{i+2}$ are adjacent to or incident with each other. □

Let π be a partial 4-total colouring for B_k. Consider the equivalent pairs x_i, $y_{i+1}x_{i+1}$ and y_{i+1}, x_iy_{i+2}. If $\pi(x_i) = \pi(y_{i+1}x_{i+1})$, then we say that for these equivalent pairs the *anchor* is x_i; otherwise y_{i+1} is said to be the anchor.

LEMMA 7
Let $G := B_k$ and let π be a 4-total colouring for the subgraph $G_{2\times k}$ obtained by removing exactly one parallel pair from G. If x_i is an anchor, then y_i and y_{i+2} are the anchors of their respective equivalent pairs. Otherwise, that is if y_{i+1} is the anchor, x_{i-1} and x_{i+1} are the anchors of their respective equivalent pairs.

Proof: Suppose that x_i is an anchor; then $\pi(x_i) = \pi(y_{i+1}x_{i+1})$. We first prove that y_{i+2} is an anchor. By Lemma 6, either $\pi(y_{i+2}) = \pi(x_{i+1}y_{i+3})$ or $\pi(x_{i+1}) = \pi(y_{i+2}x_{i+2})$. Suppose that $\pi(x_{i+1}) = \pi(y_{i+2}x_{i+2})$. Since x_{i+1} is incident with $y_{i+1}x_{i+1}$, $\pi(x_{i+1}) \neq \pi(y_{i+1}x_{i+1})$. Therefore, $\pi(x_iy_{i+2})$, $\pi(y_{i+2})$, and $\pi(x_{i+1}y_{i+2})$ are different from $\pi(x_i)$ and $\pi(x_{i+1})$. We conclude that there exist only two colours in $\{\pi(x_iy_{i+2}), \pi(y_{i+2}), \pi(x_{i+1}y_{i+2})\}$, a contradiction since these elements are adjacent to and incident with each other. If y_i is not an anchor, then x_{i-1} is an anchor by Lemma 6. Then, y_{i+1} is an anchor by our previous argument, but this contradicts Lemma 6 since x_i is an anchor. Now, the case in which y_{i+1} is an anchor follows from τ-simmetry. □

THEOREM 8
Let $G := B_k$, k odd. Then, G is type 2.

Proof: We first prove that G is not type 1. Suppose the contrary and let π be a 4-total colouring for G. By τ-automorphism, we assume that $\pi(x_0) = \pi(y_1x_1)$. Applying Lemma 7 successively, we have that all vertices x_i, y_i with i even are anchors. Therefore, x_{k-1} and y_0 are anchors, which implies that $\pi(x_{k-1}) = \pi(y_0x_0)$ and $\pi(y_0) = \pi(x_{k-1}y_1)$, contradicting Lemma 6. We conclude that there is no 4-total colouring for B_k, with k odd. Moreover, Rosenfeld [10] and Vijayaditya [13] proved that $\chi_T(G) \leq 5$ for cubic graphs. Therefore, $\chi_T(B_k) = 5$, a type 2 graph. □

Let $B_k := (X_k, Y_k)$ and $B_\ell := (X_\ell, Y_\ell)$ be two near-ladder graphs. It is easy to check that the *glueing operation*, defined below, generates $B_{k+\ell} = (X_{k+\ell}, Y_{k+\ell})$ from B_k and B_ℓ.

(i) relabel the vertices of $X_\ell \cup (Y_\ell \setminus \{y_0\})$ adding k in each of its indexes, that is $X_\ell := \{x_k, x_{k+1}, \ldots, x_{k+\ell-1}\}$ and $Y_\ell := \{y_0, y_{k+1}, \ldots, y_{k+\ell-1}\}$;

(ii) relabel vertex $y_0 \in Y_k$ with y_k;

(iii) let $X_{k+\ell} := X_k \cup X_\ell$, $Y_{k+\ell} := Y_k \cup Y_\ell$, and $E(B_{k+\ell}) := (E(B_k) \cup E(B_\ell) \cup E_{in}) \setminus E_{out}$, where $E_{in} := \{x_0y_0, y_kx_k, y_1x_{k+\ell-1}, x_{k-1}y_{k+1}\}$ and $E_{out} := \{x_0y_k, y_1x_{k-1}, x_ky_0, y_{k+1}x_{k+\ell-1}\}$.

Figure 4 shows an example of the glueing operation.

THEOREM 9
Each B_k with k even is type 1.

Proof: The proof is by induction. For the basis case we construct 4-total colourings π_4 and π_6 for B_4 and B_6, respectively, shown in Figure 5.

By induction hypothesis, there exists a 4-total colouring for B_{k-4}, $k \geq 8$. Adjust π_{k-4} so that x_0 is the anchor of equivalent pairs x_0, y_1x_1 and y_1, x_0y_2, and so that $\pi_{k-4}(x_0) = \pi_4(x_0)$, $\pi_{k-4}(x_0y_0) = \pi_4(x_0y_0)$, $\pi_{k-4}(x_0y_1) = \pi_4(x_0y_1)$, and $\pi_{k-4}(x_0y_2) = \pi_4(x_0y_2)$. Note that, by Lemma 6, these adjustments imply that $\pi_{k-4}(y_1) = \pi_4(y_1)$ and $\pi_{k-4}(y_1x_{k-5}) = \pi_4(y_1x_3)$.

Graph B_k, k even and $k \geq 8$, can be obtained by glueing B_4 and B_{k-4}. A 4-colour assignment π for B_k can be constructed from π_4 and π_{k-4} as follows.

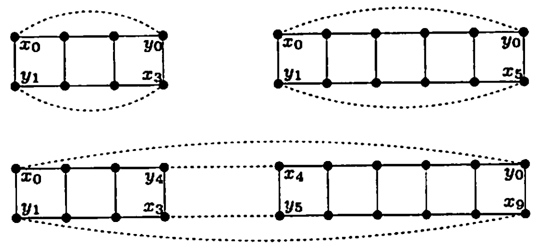


Figure 4: Glueing of B_4 and B_6. The dashed edges in B_4 and B_6 are the edges of E_{out} and the dashed edges of B_{10} are edges of E_{in}.

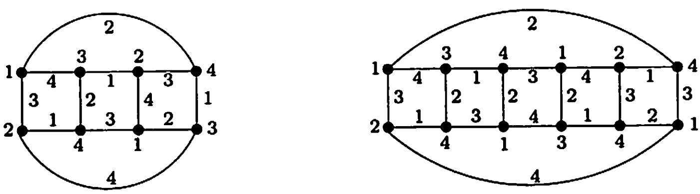


Figure 5: 4-total colouring of B_4 and B_6.

(i) if e is an element of B_k corresponding to an element of $B_i \setminus E_{out}$, $i = 4, k-4$, then $\pi(e) := \pi_i(e)$;

(ii) colour edges of E_{in} as follows: $\pi(x_0y_0) := \pi_4(x_0y_0)$, $\pi(y_4x_4) := \pi_4(x_0y_0)$, $\pi(x_3y_5) := \pi_4(y_1x_3)$, and $\pi(y_1x_{k-1}) := \pi_4(y_1x_3)$.

Now, we show that π is a total colouring for B_k. By construction of π, each element of B_k received a colour. Colourings of the two subgraphs induced by $S := \{x_0, \dots x_3, y_1, \dots, y_4\}$ and by $V(B_k) \setminus S$ are partial total colourings of B_k since the colours of their elements came from π_4 and π_{k-4}. Since π_4 is a total colouring, $\pi(x_0) \neq \pi(y_4)$ (remember that $y_4 \in V(B_k)$ corresponds to vertex $y_0 \in V(B_4)$) and $\pi(y_1) \neq \pi(x_3)$. Analogously, since π_{k-4} is a total colouring, $\pi(x_4) \neq \pi(y_0)$ and $\pi(y_5) \neq \pi(x_{k-1})$. By previous adjustments in π_{k-4}, $\pi(x_0) = \pi(x_4)$ and $\pi(y_1) = \pi(y_5)$. We conclude that $\pi(x_0) \neq \pi(y_0)$, $\pi(y_4) \neq \pi(x_4)$, $\pi(y_1) \neq \pi(x_{k-1})$, and $\pi(x_3) \neq \pi(y_5)$.

In order to conclude the proof, we have to analyse the edges of E_{in}. Let uv be an edge of E_{in}. Without loss of generality, by the glueing operation, u is a vertex from B_4, v from B_{k-4} and there exist exactly two edges in E_{out}, uw_1 and w_2v corresponding to edges of B_4 and B_{k-4} that do not exist in B_k. By the adjustments done in π_{k-4} we conclude that these three edges have the same colour and the result follows. □

4 k-dimensional cube

In this section we show that k-dimensional cubes are type 1 graphs. A *k-dimensional cube* Q_k, $k \geq 1$, or *k-cube* for short, is a graph whose set of vertices is comprised by the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. For a vertex v of Q_k we denote v by $(b_1 b_2 \ldots b_k)$, where $b_i \in \{0,1\}$ and $(b_1 b_2 \ldots b_k)$ is its ordered k-tuple. It is not difficult to see that the k-cube is bipartite, k-regular, with 2^k vertices and $k2^{k-1}$ edges.

It is well known that Q_k can be recursively constructed. Let G_0 and G_1 be two graphs isomorphic to Q_k. Then, Q_{k+1} can be obtained from G_0 and G_1 in the following way: (i) for each vertex $v \in V(G_i)$ that corresponds to vertex $(b_1 \ldots b_k)$ of Q_k, denote v by $(b_1 \ldots b_k\ i)$ ($(b_1 \ldots b_k\ 0)$ and $(b_1 \ldots b_k\ 1)$ are called a *corresponding pair*); (ii) $V(Q_{k+1}) := V(G_0) \cup V(G_1)$ and $E(Q_{k+1}) := E(G_0) \cup E(G_1) \cup M$, where $M := \{uv : u \in V(G_0),\ v \in V(G_1)$ and u, v is a corresponding pair$\}$.

We show that $\chi_T(Q_k) = \Delta(Q_k) + 1$, for each $k \geq 3$. Note that Q_1 is isomorphic to K_2 and Q_2 is isomorphic to C_2, that are both type 2 graphs.

THEOREM 10
For Q_k, $k \geq 3$, there exists a $(k+1)$-total colouring of Q_k such that only four colours occur in its vertex set.

<u>Proof:</u> We prove the assertion by induction. For the basis case we construct an explicit 4-total colouring for the 3-cube, shown in Figure 4. We call these four colours *base colours.*

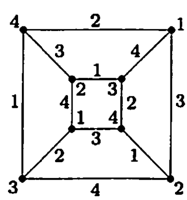


Figure 6: 4-total colouring of Q_3.

We construct π, a colour assignment for Q_{k+1} that uses $k+2$ colours, from two previously coloured copies of Q_k. The following algorithm describes the construction procedure:

(i) Let G_0 and G_1 be two copies of Q_k. By induction hypothesis there exists a $(k+1)$-total colouring π_i for G_i, $i = 0, 1$ such that only four colours occur in its vertex set. Let $1, \ldots, k+1$ be the used colours and let $1, \ldots, 4$ be the base colours. Adjust the colours so that corresponding pairs have the same colours.

(ii) For G_1, exchange colours $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$.

(iii) Construct Q_{k+1} from G_0 and G_1, by using the previous recursive procedure.

(iv) Assign colour $k+2$ to the edges of perfect matching M that join the two copies.

We show that π is a $(k+2)$-total colouring of $G := Q_{k+1}$. Clearly, π uses $k+2$ colours and each element of Q_{k+1} received a colour. Moreover, the colouring of each subgraph H_i induced by vertices $\{v : v = (b_1 \ldots b_k, i)\}$, $i = \{0, 1\}$, is a partial $(k+1)$-total colouring. Note that there do not exist incident edges e_0 and e_1 such that $e_i \in H_i$. Moreover, the edges of (iv) received a new colour.

In order to finish the proof we have to show that the ends of edges coloured in (iv) have different colours. These edges join the corresponding pairs in H_0 and H_1. Let v_0, v_1 be a corresponding pair, where $v_i = (b_1 \ldots b_k, i)$. From (i), $\pi(v_0) = \pi(v_1)$ and from (ii) $\pi(v_1) \neq \pi(v_0)$ and the result follows. □

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