Connectivity of Bi-Cayley Graphs

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Abstract : Let G is a finite group and S is a subset (possibly, contains the identity element) of G, we define the Bi-Cayley graph $X{=}BC(G,S)$ to be the bipartite graph with vertices $G{\times}\{0,1\}$ and edges $\{\{(g,0),(sg,1)\} : g{\in}G, s{\in}S\}$. In this paper, we show that if $X{=}BC(G,S)$ is connected, then $\kappa(X){=}\delta(X)$.

Key words: Bi-Cayley graph; connectivity; atom

1. Introduction

Let $X{=}(V,E)$ be a simple connected graph, with $V(X)$ the set of vertices and $E(X)$ the set of edges. A vertex disconnecting set of X is a subset U of V such that the subgraph $X \backslash U$ induced by $V \backslash U$ is either trivial or not connected. The connectivity $\kappa(X)$ of a nontrivial connected graph X is the minimum cardinality of all vertex disconnecting sets of X. If we denote by $\delta(X)$ the minimum degree of X, then $\kappa(X){\leq}\delta(X)$.

We denote by Aut(X) the automorphism group of X. The graph X is said to be *vertex transitive* if Aut(X) acts transitively on $V(X)$, and to be *edge transitive* if Aut(X) acts transitively on $E(X)$. It is proved that these two kinds of graphs usually have high connectivity. For instance, connected vertex transitive graphs have maximum edge connectivity[1], and connected edge transitive graphs have maximum vertex connectivity[8].

For a group G, and a subset S of G such that $1_G{\notin}S$ and $S^{-1}{=}S$, the *Cayley graph* $C(G,S)$ is a graph with vertex set G and edge set $\{(x,sx)|x{\in}G, s{\in}S\}$. For each element $g{\in}G$, it is easy to see that the right translation R(g), defined by R$(g)(x){=}xg$ for all $x{\in}G$, is an automorphism of $C(G,S)$. All these right translations R(g) form a subgroup R(G) of

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Aut$[C(G,S)]$, which acts transitively on G. Thus, Cayley graphs are vertex transitive, and so the edge connectivity of any connected Cayley graph attains its regular degree. Therefore, the research on the connectivity of the Cayley graph is focused on the vertex connectivity. Results on this subject are referred to [4,7,8].

For studying semisymmetric graphs, which are regular edge transitive but not vertex transitive, Xu defined the so-called *Bi-Cayley graph*[2]. For a finite group G and a subset S (possibly, contains the identity element) of G, the Bi-Cayley graph $X{=}BC(G,S)$ of G with respect to S is defined as the bipartite graph with vertex set $G\times\{0,1\}$ and edge set $\{\{(g,0),(sg,1)\}|g\in G, s\in S\}$. Clearly, the translation BR(g), defined by $(x,0)\rightarrow(xg,0), (x,1)\rightarrow(xg,1)$ for any $x\in G$, is an automorphism of X. Since all these automorphisms form a subgroup BR(G) of Aut(X), which acts transitively on $G\times\{0\}$ and $G\times\{1\}$ respectively, thus Aut(X) has at most two orbits, and these two orbits are a bipartition of X. Generally, Bi-Cayley graphs are not definitely vertex transitive[9]. But if $S^\sigma{=}S^{-1}g$ for some $\sigma\in$Aut(G) and $g\in G$, then $BC(G,S)$ is vertex transitive[9].

So far, the research on the Bi-Cayley graph is primarily focused on its isomorphisms[2,9], few results, if any, are known on graphic properties of Bi-Cayley graphs. In this paper, we study the vertex connectivity of Bi-Cayley graphs, and we will prove that the vertex connectivity of any connected Bi-Cayley graph is its regular degree.

2. Atom and connectivity

Let X be a (simple and undirected) graph and F a subset of $V(G)$. Set

$$N(F) = \{x\in V(X)\backslash F : \exists y\in F, st. \quad xy\in E(X)\};$$

$$C(F) = F\cup N(F);$$

$$R(F) = V(X)\setminus C(F).$$

If $F{=}\{x\}$, then we write $N(x)$ and $C(x)$ instead of $N(F)$ and $C(F)$, respectively. Clearly, for a non-empty subset F of $V(X)$, $N(F)$ is a vertex disconnecting set if $R(F){\neq}\emptyset$. A subset $F{\subset}V(X)$ is said to be a *fragment* if $|N(F)|{=}\kappa(X)$ and $R(F){\neq}\emptyset$. A fragment of minimum cardinality is called an *atom* of X. The notion of atom was introduced by Watkins[4]. On the cardinality of an atom, Hamidoune and Watkins proved following two results, respectively.

Theorem 2.1[6] If $X{=}(V,E)$ is a connected vertex-transitive graph, then the cardinality of an atom of X is at most $\kappa(X)$.

Theorem 2.2[4] Let $X=(V,E)$ be a nontrivial connected graph which is not a complete graph. Then
(i) $\kappa(X) = \delta(X)$ if and only if every atom of X has cardinality 1;
(ii) if $\kappa(X) < \delta(X)$, then each atom has cardinality at most $[(|V| - \kappa(X))/2]$ and induces a connected subgraph of X.

An *imprimitive block* for a group U of permutations on a set T is a proper, nontrivial subset A of T such that if $\sigma \in U$ then either $\sigma(A) = A$ or $\sigma(A) \cap A = \emptyset$. A subset A of $V(X)$ is called an imprimitive block for X if it is an imprimitive block for $\mathrm{Aut}(X)$ on $V(X)$.

Theorem 2.3[3] If $X=(V,E)$ is a nontrivial connected graph which is not a complete graph, then distinct atoms of X are disjoint. Thus if $\kappa(X) < \delta(X)$, the atoms of X are imprimitive blocks of X.

Theorem 2.4[3] Let $X=(V,E)$ be a nontrivial connected graph. If W is a minimum vertex disconnecting set and A an atom of X, then $A \cap W = \emptyset$, or $A \subseteq W$.

3.Connectivity of Bi-Cayley Graphs

Before proceeding, we cite a result proved by Mader.

Theorem 3.1[5] If $X=(V,E)$ is a connected vertex transitive graph which is K_4-free, then $\kappa(X)=\delta(X)$.

Thus, for a connected vertex transitive Bi-Cayley graph, we have the following result:

Corollary 3.2 If $X=BC(G,S)$ is a connected vertex transitive Bi-Cayley graph, then $\kappa(X)=\delta(X)$.

Proof. By Theorem 3.1, the result is obvious since X is a connected vertex transitive bipartite graph which is K_4-free. □

In what follows we always assume that $X=BC(G,S)$ is a connected Bi-Cayley graph, and let $X_i = \{(x,i)|x \in G\}, i = 0,1$.

Lemma 3.3 If $\kappa(X) < \delta(X)$, then $\mathrm{Aut}(X)$ has exactly two orbits X_0, X_1.

Proof. By the definition of Bi-Cayley graph, BR(G) acts transitively on

X_0, X_1, respectively. If $\mathrm{Aut}(X)$ has a unique orbit, then X is vertex transitive, thus $\kappa(X) = \delta(X)$. So, when $\kappa(X) < \delta(X)$, X is not vertex transitive, namely, $\mathrm{Aut}(X)$ has exactly two orbits X_0, X_1. □

Lemma 3.4 Let A be an atom of X. If $\kappa(X) < \delta(X)$, then $A_i{=}A \cap X_i (i = 0, 1)$ are nontrivial.

Proof. By Lemma 3.3, $\mathrm{Aut}(X)$ has exactly two orbits X_0, X_1. By Theorem 2.2, the induced subgraph $Y{=}X[A]$ is a nontrivial connected subgraph of X, which is a bipartite graph, thus $A_i{=}A \cap X_i \neq \emptyset (i = 0, 1)$. Suppose that one of these two vertex subsets is trivial, without loss of generality, we assume that $|A_0|{=}1$. Thus, $|A_1| \leq \delta(X)$ since Y is connected. Set $F = N(A)$. If $|A_1|{=}\delta(X)$, then $|F| \geq \delta(X) - 1$. If $|F|{=}\delta(X) - 1$, then the induced subgraph $Y'{=}X[A \cup F]$ is a connected component of X, which is impossible since X is connected. Thus $|F| > \delta(X) - 1$, namely $|F| \geq \delta(X)$, a contradiction. If $|A_1|{=}m{<}\delta(X)$, then $|N(A_0) \backslash A_1| {=} \delta(X) - m$. Let $n{=}|N(A_1) \backslash A_0|$, then $n \geq \delta(X) - 1$. Since

$$|F| = |N(A)| = |N(A_0) \backslash A_1| + |N(A_1) \backslash A_0| = \delta(X) - m + n \geq 2\delta(X) - m - 1$$

and $|F| = \kappa(X) < \delta(X)$, we have $\delta(X) > 2\delta(X) - m - 1$, namely, $m \geq \delta(X)$, a contradiction. The result follows. □

Lemma 3.5 Let A be an atom of X, and $Y{=}X[A]$. If $\kappa(X) < \delta(X)$, then
(i) $\mathrm{Aut}(Y)$ acts transitively on $A_i = A \cap X_i (i = 0, 1)$.
(ii) If A_i contains $(1_G, i)(i = 0, 1)$, then $A_i = H_i \times \{i\}$, where H_i is a subgroup of G.

Proof. (i) By Lemma 3.4, $A_i{=}A \cap X_i (i{=}0, 1)$ are nontrivial. By the definition of Bi-Cayley graph, for any $g \in G$, $\mathrm{BR}(g)$ is an automorphism of X. For any $(g, i), (h, i) \in A_i (i = 0, 1)$, $\mathrm{BR}(g^{-1}h) \in \mathrm{Aut}(X)$ and $\mathrm{BR}(g^{-1}h)(g, i){=}(h, i)$. Since $\mathrm{BR}(g^{-1}h)(A) \cap A \neq \emptyset$ and A is an atom of X, by Theorem 2.3, $\mathrm{BR}(g^{-1}h)(A) = A$ and A is an imprimitive block of X. Thus the restriction of $\mathrm{BR}(g^{-1}h)$ on A induces an automorphism of Y, which maps (g, i) to (h, i). The result follows.

(ii) By Theorem 2.3, the atom A is a imprimitive block of X containing $(1_G, i)$. By Lemma 3.4, A_i is nontrivial. Then for any $(g, i) \in A_i$, $(1_G, i) \in \mathrm{BR}(g^{-1})(A) = Ag^{-1}$, and hence $Ag^{-1}{=}A$, $A_i g^{-1}{=}A_i$. Thus $(g, i), (h, i) \in A_i$ implies that $(hg^{-1}, i) \in A_i$, from which, the result is obvious. □

Lemma 3.6 Let A be an atom of X. If $\kappa(X) < \delta(X)$, then
(i) Every vertex of X lies in an atom;

(ii) $|A| \leq \kappa(X)$.

Proof. (i) By Lemma 3.3, $\mathrm{Aut}(X)$ has exactly two orbits X_0, X_1. By Lemma 3.4, the induced subgraph $Y{=}X[A]$ is a nontrivial connected subgraph of X, thus at least one vertex of $X_i (i = 0, 1)$, respectively, lies in an atom. By the transitivity of X_i, every vertex of X lies in an atom.

(ii) Let $F{=}N(A)$. Since $A{=}A_0 \cup A_1$ and $A_i (i = 0, 1)$ is nontrivial, $F_i = F{\cap}X_i (i = 0, 1)$ is not empty respectively. For any $(x, i) \in F_i (i = 0, 1)$, by (i), (x, i) lies in an atom A' of X. By Theorem 2.4, $A'{\subseteq}F$, then $|A| = |A'| \leq |F| = \kappa(X)$. □

Theorem 3.7 If $X{=}BC(G, S)$ is connected , then $\kappa(X) = \delta(X)$.

Proof. Suppose to the contrary that $\kappa(X) < \delta(X)$. By Theorem 2.3, distinct atoms are disjoint. Thus, by Lemma 3.6, $V(X)$ is a disjoint union of distinct atoms. Let A be an atom of X, then there exist $\sigma_i \in \mathrm{Aut}(X)(i = 1, \cdots, k)$, such that

$$V(X) = \bigcup_{i=1}^{k} \sigma_i(A).$$

By Lemma 3.4, the induced subgraph $Y{=}X[A]$ is a nontrivial connected subgraph of X, and $A_i{=}A \cap X_i (i = 0, 1)$ is nontrivial. By Lemma 3.3, $\mathrm{Aut}(X)$ has exactly two orbits X_0, X_1, thus for any $1 \leq i, j \leq k$ and $i \neq j$, $\sigma_i(A_0) \cap \sigma_j(A_0) = \emptyset$ and $\sigma_i(A_0), \sigma_j(A_0) \subseteq X_0$. So, we have $X_0 = \bigcup_{i=1}^{k} \sigma_i(A_0)$, and $X_1 = \bigcup_{i=1}^{k} \sigma_i(A_1)$. Since $|X_0| = |X_1|$, we have $|A_0| = |A_1|$ and $|A_i| | |X_i| (i = 0, 1)$.

If G is a prime-order group, then we will deduce a contradiction. Otherwise, by Lemma 3.5, $\mathrm{Aut}(Y)$ acts transitively on $A_i = A{\cap}X_i (i = 0, 1)$, then Y is regular. Let $d = \delta(Y)$ and $F = N(A)$, since $A{=}A_0{\cup}A_1$ and $A_i (i = 0, 1)$ are nontrivial, $F_i = F \cap X_i (i = 0, 1)$ are not empty. Since every vertex of A_i has $\delta(X) - d$ neighbours in F, we have $|F| = \kappa(X) \geq 2(\delta(X) - d)$. By $\kappa(X) < \delta(X)$, we have $d > \delta(X)/2$, and $|A| \geq 2d > \delta(X) > \kappa(X)$. By Lemma 3.6, we deduce a contradiction. □

Acknowledgements

The author thanks the referees for many helpful suggestions and comments.

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