On Uniquely List Colorable Complete Multipartite Graphs

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Abstract

A graph G is called *uniquely k-list colorable*, or $UkLC$ for short, if it admits a *k-list assignment* L such that G has a *unique L-coloring*. A graph G is said to have *the property* $M(k)$ (M for Marshal Hall) if and only if it is not $UkLC$. In 1999, M. Ghebleh and E.S. Mahmoodian characterized the $U3LC$ graphs for complete multipartite graphs except for nine graphs. At the same time, for the nine exempted graphs, they give an open problem: verify the property $M(3)$ for the graphs $K_{2,2,r}$, for $r = 4, 5, \ldots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, and $K_{1*5,4}$. Until now, except for $K_{1*5,4}$, the other eight graphs have been showed to have the property $M(3)$ by W. He et al.. In this paper, we show that graph $K_{1*5,4}$ has the property $M(3)$, and as consequences, $K_{1*4,4}$, $K_{2,2,4}$ have the property $M(3)$. Therefore the $U3LC$ complete multipartite graphs are completely characterized.

Keywords: List coloring, uniquely 3-list colorable graphs, property $M(3)$, complete multipartite graphs

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1 Introduction

List colorings are generalizations of usual colorings that recently attracted considerable attention. Originally, the idea of list colorings of graphs is due independently to V.G. Vizing [13] and to P. Erdős, A.L. Rubin, and H. Taylar [3]. For a survey on list coloring we refer the interested reader to D.R. Woodall [14] and Ch. Eslahchi, M. Ghebleh, and H. Hajiabolhassan [4]. Here we mention some definitions and results about list colorings which are referred throughout the paper.

We consider undirected, finite, simple graphs. For the necessary definitions and notation we refer the reader to standard texts, such as [1].

For each vertex v in a graph $G = (V, E)$, let $L(v)$ denote a list of colors available for v. $L = \{L(u)|u \in V(G)\}$ is said to be a list assignment of G. If $|L(v)| = k$ for all $v \in V(G)$, L is called $k-$ list assignment of G. A list coloring from a given collection of lists is a proper coloring c such that $c(v)$ is chosen from $L(v)$. We will refer to such a coloring as an *L-coloring.*

The concept of *unique list coloring* was introduced by J.H. Dinitz and W.J. Martin [2] and independently by E.S. Mahmoodian and M. Mahdian [11], which can be used to study defining sets of k–coloring [12] and critical sets in Latin squares [9]. Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors $L(v)$, such that there exists a unique L-coloring for G, then G is called *uniquely k-list colorable graph* or a *UkLC graph* for short. For a graph G, it is said to have *the property* $M(k)$ (M for Marshal Hall) if and only if it is not uniquely k-list colorable. So G has the property $M(k)$ if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist two list colorings.

M. Mahdian and E.S. Mahmoodian characterized uniquely 2-list colorable graphs [10]. They showed that

Theorem A ([10]). *A connected graph has the property* $M(2)$ *if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

It seems that characterizing $UkLC$ graphs for any k is not easy. Even the $U3LC$ graphs seem to be difficulty to characterize. In [6] M. Ghebleh and E. S. Mahmoodian showed that there are some complete tripartite graphs which have the property $M(3)$ and there are some complete tripartite graphs which are $UkLC$ for any k (for example, the graph $K_{2,2,3}$ has the property $M(3)$, the graph $K_{3,3,3}$ is a $U3LC$ graph). It is very significative that M. Ghebleh and E. S. Mahmoodian have given some results which are helpful for characterizing $UkLC$ graphs [6]. At the same time, they characterized $U3LC$ graphs for complete multipartite graphs except for finitely many of them. They showed that

Theorem B ([6]). *If G is a complete multipartite graph which has an induced $UkLC$ subgraph, then G is $UkLC$.*

Theorem C ([6]). *The graphs $K_{3,3,3}$, $K_{2,4,4}$, $K_{2,3,5}$, $K_{2,2,9}$, $K_{1,2,2,2}$, $K_{1,1,2,3}$, $K_{1,1,1,2,2}$, $K_{1*4,6}$, $K_{1*5,5}$, and $K_{1*6,4}$ are $U3LC$.*

Where K_{s*r} means a complete r-partite graph in which each part is of size s, and notations such as $K_{s*r,t}$ are used similarly.

Theorem D ([6]). *Let G be a complete multipartite graph that is not $K_{2,2,r}$, for $r = 4, 5, \ldots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, or $K_{1*5,4}$ then G is $U3LC$ if and only if it has one of the graphs in Theorem C as an induced subgraph.*

For perfecting the Theorem D, it is clear that the leaving work is to determine whether the nine graphs above are $U3LC$ or not. So M. Ghebleh and E.S. Mahmoodian give the open problem as follows.

Problem ([6]). *Verify the property $M(3)$ for the graphs exempted in Theorem D, i.e. $K_{2,2,r}$, for $r = 4, 5, \ldots, 8$, $K_{2,3,4}$, $K_{1*4,4}$, $K_{1*4,5}$, and $K_{1*5,4}$.*

Last year, except for $K_{1*5,4}$, the other eight graphs were showed to have the property $M(3)$(namely, they are not $U3LC$) by W. He et al.. They showed the following.

Theorem E ([7]). *Graph $K_{2,2,r}$ has the property $M(3)$, where $r = 4, 5, 6, 7, 8$.*

Theorem F ([8]. *Graphs $K_{1*4,5}$ and $K_{1*4,4}$ have the property $M(3)$.*

Theorem G ([15]). *Graph $K_{2,3,4}$ has the property $M(3)$.*

The fact is that it is difficult to verify the property $M(3)$ for the nine exempted graphs in Theorem D with some common techniques(the proof of Theorem F takes eight pages, and the proof of Theorem G takes twenty pages). In this paper we will show that $K_{1*5,4}$ has the property $M(3)$ (as consequences, $K_{1*4,5}$ and $K_{2,2,4}$ have the property $M(3)$):

Theorem 1.1. *$K_{1*5,4}$ has the property $M(3)$.*

Combine Theorem Theorem E, Theorem F, Theorem G and Theorem 1.1 together, we can give an improved version of Theorem D in the following.

Theorem 1.2. *Let G be a complete multipartite graph, then G is $U3LC$ if and only if it has one of the graphs in Theorem C as an induced subgraph.*

Thus, the $U3LC$ complete multipartite graphs are completely characterized.

We will give the proof of Theorem 1.1 in section 3. In section 2, we state some propositions which are helpful in proving our main results. In section 4, we give some propositions of complete multipartite graphs whose m-number are equal to 4, which are useful in characterization of $U4LC$ complete multipartite graphs.

2 Some Propositions

For $K_{1*5,4}$, denote the six parts by $X_i = \{v_i\}$, for $i = 1,2,3,4,5$, and $X_6 = \{v_6, v_7, v_8, v_9\}$. Let

$$c(v_i) = c_{i1}, \quad for \quad i = 1, 2, \cdots, 9 \qquad (*)$$

be a 3-list coloring with $L(v_i) = \{c_{i1}, c_{i2}, c_{i3}\}$ assigned to the vertices of $K_{1*5,4}$. And denote $\{c_{61}, c_{71}, c_{81}, c_{91}\} = S$. Under the above assumption, the following propositions are preparations to prove our main theorem in section 3.

Proposition 2.1. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by $(*)$, then*

(1) $c_{i1} \neq c_{j1}$, $1 \leq i,j \leq 5, i \neq j$; $c_{i1} \neq c_{j1}$, $1 \leq i \leq 5, 6 \leq j \leq 9$.

(2) All vertices in part X_6 take at least two colors in c, i.e. $|S| \geq 2$.

(3) All colors in $\bigcup_{v \in K_{1*5,4}} L(v)$ are used in c.

Proof. (1) It is obvious.

(2) By contradiction. If $|S| = 1$, we can remove the color which appears in S, from the lists $L(v_i)$, for $i = 1,2,3,4,5$, resulting in the new lists $L'(v_i)$ of size at least 2, for $i = 1,2,3,4,5$. By the property $M(2)$ of $K_{1*5} = K_5$ (Theorem A) we can obtain another L-coloring c' for K_{1*5} which is extendible to vertices of $K_{1*5,4}$. This is a contradiction to c being a unique 3-list coloring.

(3) Otherwise, if there is some unused color in the list of some vertex we can obtain a new $L-$coloring of $K_{1*5,4}$ by simply putting that unused color on that vertex. □

Definition 2.1. *For $K_{1*5,4}$, let c be a 3-list coloring given by $(*)$. If there exist $v_{i_1}, v_{i_2}, \cdots, v_{i_k} \in \{v_1, v_2, v_3, v_4, v_5\}$, such that $c_{i_2 1} \in L(v_{i_1})$, $c_{i_3 1} \in L(v_{i_2})$, $\cdots$, $c_{i_k 1} \in L(v_{i_{(k-1)}})$, $c_{i_1 1} \in L(v_{i_k})$, $2 \leq k \leq 5$, then we say c having a **coloring rotation** of size k in $\{v_1, v_2, v_3, v_4, v_5\}$, denoted $CR(c_{i_1 1}, c_{i_2 1}, \cdots, c_{i_k 1})$.*

Proposition 2.2. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by (*), then there is no coloring rotation in $\{v_1, v_2, v_3, v_4, v_5\}$.*

Proof. By contradiction. Assume that there exists a coloring rotation $CR(c_{i_1 1}, c_{i_2 1}, \cdots, c_{i_k 1})$ in $\{v_1, v_2, v_3, v_4, v_5\}$. Let $c'(v_{i_1}) = c_{i_2 1}$, $c'(v_{i_2}) = c_{i_3 1}$, $\cdots$, $c'(v_{i_{(k-1)}}) = c_{i_k 1}$, $c'(v_{i_k}) = c_{i_1 1}$, $c'(v_j) = c_{j1}, j \neq i_1, i_2, \cdots, i_k$, then c' is a different L-coloring for $K_{1*5,4}$. □

Proposition 2.3. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by (*), then $c_{i1} \notin \{c_{j2}, c_{j3}\}$, for $6 \leq i, j \leq 9$, and $c_{ik} \in \{c_{11}, c_{21}, c_{31}, c_{41}, c_{51}\}$, for $i = 6, 7, 8, 9$, and $k = 2, 3$.*

Proof. If $i = j$, it is obvious. If $i \neq j$, to the contrary, there exist some i_0, j_0, where $6 \leq i_0, j_0 \leq 9$ and $i_0 \neq j_0$, such that $c_{i_0 1} \in \{c_{j_0 2}, c_{j_0 3}\}$, then $c_{i_0 1} \neq c_{j_0 1}$. We can obtain a different L-coloring c' by putting $c'(v_{j_0}) = c_{i_0 1}$, and $c'(v_k) = c(v_k)$, for $k \neq j_0, k = 1, 2, \cdots, 9$. Since $c_{i1} \notin \{c_{j2}, c_{j3}\}$, for $6 \leq i, j \leq 9$, and all colors in $\bigcup_{v \in K_{1*5,4}} L(v)$ are used in c by Proposition 2.1(3), it is clear that $c_{ik} \in \{c_{11}, c_{21}, c_{31}, c_{41}, c_{51}\}$, for $i = 6, 7, 8, 9$, and $k = 2, 3$. □

Proposition 2.4. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by (*), then there exists $i_0 \in \{1, 2, 3, 4, 5\}$, such that $\{c_{i_0 2}, c_{i_0 3}\} \subseteq S$.*

Proof. Otherwise, for any $i \in \{1, 2, 3, 4, 5\}$, there is at most one of c_{i2} and c_{i3} in S. Noting that $K_{1*5} = K_5$, we can remove the color which appears in S, from the lists $L(v_i)$ of parts X_i, resulting in the new lists $L'(v_i)$ of size at least 2, for $i = 1, 2, 3, 4, 5$. By the property $M(2)$ of complete graphs(Theorem A), we can obtain another L-coloring c' for K_{1*5} which is extendible to vertices of $K_{1*5,4}$. This is a contradiction to c being a unique 3-list coloring. □

Proposition 2.5. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by (*), then there are exactly two colors in S, and two vertices of each color in X_6.*

Proof. By Proposition 2.1(2), $|S| = 2$, 3 or 4. If the statement of Proposition 2.5 is not true, depending on the value of $|S|$ we consider three cases. For each case we will show that there exists a different L-coloring of $K_{1*5,4}$. That is a contradiction to c being a unique 3-list coloring.

Case 1. $|S| = 2$, but the last statement of the Proposition 2.5 is not true.

Without loss of generality, say $c_{61} = 6, c_{71} = c_{81} = c_{91} = 7$. Noting that $H = K_{1*5,4}[v_1, v_2, \cdots, v_6] = K_6$ is a induced subgraph of $K_{1*5,4}$, we can remove the color 7 from the lists $L(v_i)$, resulting in the lists $L'(v_i)$ of size at least 2, for $i = 1, 2, 3, 4, 5, 6$. By the property $M(2)$ of H(Theorem

A) we can obtain another L-coloring c' for H which is extendible to $K_{1*5,4}$.

Case 2. $|S| = 3$.

Without loss of generality, say $c_{61} = 6, c_{71} = 7, c_{81} = c_{91} = 8$. Add a new edge between v_6 and v_7, the resulting graph is denoted G. Noting that $H = G[v_1, v_2, \cdots, v_7] = K_7$ is an induced subgraph of G, we can remove the color 8 from the lists of H. Similarly to the proof of case 1, by the property $M(2)$ of H we can obtain another L-coloring c' for H which is extendible to G, and which is a legal L-coloring for $K_{1*5,4}$.

Case 3. $|S| = 4$.

Without loss of generality, say $c_{61} = 6, c_{71} = 7, c_{81} = 8, c_{91} = 9$. Add one edge between any pair of vertices in $\{v_6, v_7, v_8, v_9\}$, the resulting graph is K_9. By the property $M(2)$ of K_9(K_9 has property $M(3)$ naturally) we can obtain another L-coloring c' for K_9 which is a legal L-coloring for $K_{1*5,4}$.

Summarize the three cases above, the Proposition 2.5 holds. □

For clarity, without loss of generality we write $c_{i1} = i$, for $i = 1, 2, 3, 4, 5$, by Proposition 2.1(1), write $S = \{6, 7\}$ by Proposition 2.5.

Proposition 2.6. *For $K_{1*5,4}$, suppose c is a unique 3-list coloring given by $(*)$, then there must be two colors x and y such that they can be used to L-recolor the vertices in X_6, where either $x \in \{1,2,3,4,5\}$, $y \in \{6,7\}$ respectively or $\{x, y\} \subseteq \{1,2,3,4,5\}$ (not considering parts X_i, for $i = 1,2,3,4,5$).*

Proof. Consider the 3×4 array $\begin{pmatrix} c_{61} & c_{71} & c_{81} & c_{91} \\ c_{62} & c_{72} & c_{82} & c_{92} \\ c_{63} & c_{73} & c_{83} & c_{93} \end{pmatrix}$. By Proposition 2.5, without loss of generality, let $c_{61} = c_{71} = 6, c_{81} = c_{91} = 7$. By Proposition 2.3, $c_{ik} \in \{1,2,3,4,5\}$, for $i = 6,7,8,9$ and $k = 2,3$. If $\{c_{62}, c_{63}\} \bigcap \{c_{72}, c_{73}\} \neq \Phi$ or $\{c_{82}, c_{83}\} \bigcap \{c_{92}, c_{93}\} \neq \Phi$, then it is clear that there exist x in $\{1,2,3,4,5\}$ and $y \in \{6,7\}$ satisfying the desired conditions. Otherwise, $|\{c_{62}, c_{63}, c_{72}, c_{73}\}| = |\{c_{82}, c_{83}, c_{92}, c_{93}\}| = 4$ and $|\{c_{62}, c_{63}, c_{72}, c_{73}\} \bigcap \{c_{82}, c_{83}, c_{92}, c_{93}\}| \geq 3$. Without loss of generality, $\{c_{62}, c_{63}, c_{72}\} = \{c_{82}, c_{83}, c_{92}\} = \{1,2,3\}$. It is clear that $\{c_{62}, c_{63}\} \bigcap \{c_{82}, c_{83}\} \neq \Phi$, say $1 \in \{c_{62}, c_{63}\} \bigcap \{c_{82}, c_{83}\}$, then $x = 2$ and $y = 3$ satisfy the desired conditions. □

3 Proof of Theorem 1.1

Now we can give the proof of Theorem 1.1.

Proof. By contradiction. Suppose that c is a unique 3-list coloring of $K_{1*5,4}$ with $L(v_i) = \{c_{i1}, c_{i2}, c_{i3}\}$ and $c(v_i) = c_{i1}$, for $i = 1, 2, \cdots, 9$. Without loss of generality, let $c_{i1} = i$, for $i = 1, 2, 3, 4, 5$, and let $S = \{6, 7\}$ by Proposition 2.5. According to Proposition 2.6, we consider two cases and in each case obtain a different L-coloring c' of $K_{1*5,4}$ from the given lists. That is a contradiction to c being a unique 3-list coloring of $K_{1*5,4}$.

Case 1. There are two colors in $\{1, 2, 3, 4, 5\}$ can be used to L-recolor the vertices in X_6.

Without loss of generality, let colors 1 and 2 can be used to L-recolor the vertices of part X_6. Firstly, we let $c'(v_i) = l_i$, where $l_i \in \{1, 2\} \bigcap L(v_i)$, for $i = 6, 7, 8, 9$. Secondly, by Proposition 2.4 and Proposition 2.5, there exists $i_0 \in \{1, 2, 3, 4, 5\}$, such that $\{c_{i_0 2}, c_{i_0 3}\} = \{6, 7\}$. In order to give $c'(v_i)$ for $i = 1, 2, 3, 4, 5$, according to the value of i_0, we consider two subcases as follows.

Subcase 1.1. $i_0 = 1$ or 2, say $i_0 = 1$, that is $L(v_1) = \{1, 6, 7\}$.

If $\{6, 7\} \bigcap L(v_2) \neq \Phi$, say $7 \in L(v_2)$, let $c'(v_2) = 7, c'(v_1) = 6$, $c'(v_i) = i$, for $i = 3, 4, 5$. Then c' is a different L-coloring for $K_{1*5,4}$. If $\{6, 7\} \bigcap L(v_2) = \Phi$, then $\{3, 4, 5\} \bigcap L(v_2) \neq \Phi$, say $3 \in L(v_2)$. Consider $L(v_3)$, if $\{6, 7\} \bigcap L(v_3) \neq \Phi$, say $7 \in L(v_3)$. Let $c'(v_3) = 7, c'(v_2) = 3, c'(v_1) = 6$, $c'(v_i) = i$, for $i = 4, 5$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{6, 7\} \bigcap L(v_3) = \Phi$, as $2 \notin L(v_3)$ by Proposition 2.2(Otherwise there exists a coloring rotation $CR(2, 3)$), so $\{4, 5\} \bigcap L(v_3) \neq \Phi$, say $4 \in L(v_3)$. Consider $L(v_4)$, if $\{6, 7\} \bigcap L(v_4) \neq \Phi$, say $7 \in L(v_4)$. Let $c'(v_4) = 7, c'(v_3) = 4, c'(v_2) = 3, c'(v_1) = 6$, $c'(v_5) = 5$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{6, 7\} \bigcap L(v_4) = \Phi$, as $\{2, 3\} \bigcap L(v_4) = \Phi$ by Proposition 2.2(Otherwise there exists a coloring rotation $CR(2, 3, 4)$ or $CR(3, 4)$), so $L(v_4) = \{4, 1, 5\}$. Consider $L(v_5)$, as $\{2, 3, 4\} \bigcap L(v_5) = \Phi$ by Proposition 2.2, so $\{6, 7\} \bigcap L(v_5) \neq \Phi$, say $7 \in L(v_5)$. Let $c'(v_5) = 7, c'(v_4) = 5, c'(v_3) = 4, c'(v_2) = 3$, $c'(v_1) = 6$, then c' is a different L-coloring for $K_{1*5,4}$.

Subcase 1.2. $i_0 = 3, 4$ or 5, say $i_0 = 5$, that is $L(v_5) = \{5, 6, 7\}$.

Subcase 1.2.1. $\{5, 6, 7\} \subseteq (L(v_1) \bigcup L(v_2))$.

There must be two colors in $\{5, 6, 7\}$, such that one appears in $L(v_1)$ and the other appears in $L(v_2)$, say $5 \in L(v_1)$ and $6 \in L(v_2)$. Let $c'(v_1) = 5, c'(v_2) = 6, c'(v_5) = 7$, $c'(v_i) = i$, for $i = 3, 4$, then c' is a different L-coloring for $K_{1*5,4}$.

Subcase 1.2.2. $|\{5, 6, 7\} \bigcap (L(v_1) \bigcup L(v_2))| = 2$.

Without loss of generality, assume $\{5, 6, 7\} \bigcap (L(v_1) \bigcup L(v_2)) = \{5, 6\}$. If one of 5 and 6 appears in $L(v_1)$ and the other appears in $L(v_2)$, say $5 \in L(v_1)$ and $6 \in L(v_2)$, we can obtain a different L-coloring of $K_{1*5,4}$ similarly to subcase1.2.1. Otherwise, $\{5, 6\} \subseteq L(v_1)$ and $\{5, 6, 7\} \bigcap L(v_2) = \Phi$, or $\{5,6\} \subseteq L(v_2)$ and $\{5,6,7\} \bigcap L(v_1) = \Phi$, say $\{5,6\} \subseteq L(v_1)$ and $\{5,6,7\} \bigcap L(v_2) = \Phi$, that is $L(v_1) = \{1,5,6\}$. Since $\{5,6,7\} \bigcap L(v_2) = \Phi$, then $\{3,4\} \bigcap L(v_2) \neq \Phi$, say $3 \in L(v_2)$. Consider $L(v_3)$, if $\{5,6,7\} \bigcap L(v_3) \neq \Phi$, write $a \in \{5,6,7\} \bigcap L(v_3)$. Let $c'(v_3) = a$, $c'(v_2) = 3$, $c'(v_1) = b$, $c'(v_5) = c$, where $b \in \{5,6\} \backslash \{a\}$, $\{c\} = \{5,6,7\} \backslash \{a,b\}$, $c'(v_4) = 4$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{5,6,7\} \bigcap L(v_3) = \Phi$, as $2 \notin L(v_3)$ by Proposition 2.2, so $L(v_3) = \{3,1,4\}$. Consider $L(v_4)$, as $\{2,3\} \bigcap L(v_4) = \Phi$ by Proposition 2.2, so $\{5,6,7\} \bigcap L(v_4) \neq \Phi$, write $a \in \{5,6,7\} \bigcap L(v_4)$, let $c'(v_4) = a$, $c'(v_3) = 4$, $c'(v_2) = 3$, $c'(v_1) = b$, $c'(v_5) = c$, where $b \in \{5,6\} \backslash \{a\}$, $\{c\} = \{5,6,7\} \backslash \{a,b\}$, then c' is a different L-coloring for $K_{1*5,4}$.

Subcase 1.2.3. $|\{5,6,7\} \bigcap (L(v_1) \bigcup L(v_2))| = 1$.

Without loss of generality, assume $5 \in (L(v_1) \bigcup L(v_2))$, and $5 \in L(v_1)$. Since $\{6,7\} \bigcap L(v_1) = \Phi$, so $\{2,3,4\} \bigcap L(v_1) \neq \Phi$, say $2 \in L(v_1)$, that is $L(v_1) = \{1,2,5\}$. Consider $L(v_2)$, as $\{6,7\} \bigcap L(v_2) = \Phi$, and $1 \notin L(v_2)$ by Proposition 2.2, so $\{3,4\} \bigcap L(v_2) \neq \Phi$, say $3 \in L(v_2)$. Consider $L(v_3)$, if $\{6,7\} \bigcap L(v_3) \neq \Phi$, say $6 \in L(v_3)$. Let $c'(v_1) = 5$, $c'(v_2) = 3$, $c'(v_3) = 6$, $c'(v_5) = 7$, $c'(v_4) = 4$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{6,7\} \bigcap L(v_3) = \Phi$, as $\{1,2\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $L(v_3) = \{3,4,5\}$. Consider $L(v_4)$, as $\{1,2,3\} \bigcap L(v_4) = \Phi$ by Proposition 2.2, so $\{6,7\} \bigcap L(v_4) \neq \Phi$, say $6 \in L(v_4)$. Let $c'(v_1) = 5$, $c'(v_2) = 3$, $c'(v_3) = 4$, $c'(v_4) = 6$, $c'(v_5) = 7$, then c' is a different L-coloring for $K_{1*5,4}$.

Subcase 1.2.4. $\{5,6,7\} \bigcap (L(v_1) \bigcup L(v_2)) = \Phi$, that is $\{c_{12}, c_{13}\} \subseteq \{2,3,4\}$ and $\{c_{22}, c_{23}\} \subseteq \{1,3,4\}$.

If $2 \in L(v_1)$, or $1 \in L(v_2)$, without loss of generality, say $2 \in L(v_1)$ and $L(v_1) = \{1,2,3\}$. Consider $L(v_2)$, as $\{5,6,7\} \bigcap L(v_2) = \Phi$, and $1 \notin L(v_2)$ by Proposition 2.2, so $L(v_2) = \{2,3,4\}$. Consider $L(v_3)$, as $\{1,2\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $\{5,6,7\} \bigcap L(v_3) \neq \Phi$, say $5 \in L(v_3)$. Consider $L(v_4)$, if $\{6,7\} \bigcap L(v_4) \neq \Phi$, say $6 \in L(v_4)$. Let $c'(v_1) = 3$, $c'(v_2) = 4$, $c'(v_3) = 5$, $c'(v_4) = 6$, $c'(v_5) = 7$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{6,7\} \bigcap L(v_4) = \Phi$, as $\{1,2\} \bigcap L(v_4) = \Phi$ by Proposition 2.2, so $L(v_4) = \{4,3,5\}$. Consider $L(v_3)$ again, as $\{1,2,4\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $\{6,7\} \bigcap L(v_3) \neq \Phi$, say $6 \in L(v_3)$. Let $c'(v_1) = 3$, $c'(v_2) = 4$, $c'(v_3) = 6$, $c'(v_4) = 5$, $c'(v_5) = 7$, then c' is a different L-coloring for $K_{1*5,4}$.

If $2 \notin L(v_1)$ and $1 \notin L(v_2)$, then $L(v_1) = \{1,3,4\}$ and $L(v_2) = \{2,3,4\}$. Consider $L(v_3)$, as $\{1,2\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $\{5,6,7\} \bigcap L(v_3) \neq \Phi$, say $5 \in L(v_3)$. Consider $L(v_4)$, if $\{6,7\} \bigcap L(v_4) \neq \Phi$, say $6 \in L(v_4)$. Let $c'(v_1) = 3$, $c'(v_2) = 4$, $c'(v_3) = 5$, $c'(v_4) = 6$, $c'(v_5) = 7$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{6,7\} \bigcap L(v_4) = \Phi$, as $\{1,2\} \bigcap L(v_4) = \Phi$ by Proposition 2.2, so $L(v_4) = \{4,3,5\}$. Consider $L(v_3)$

again, as $\{1,2,4\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $\{6,7\} \bigcap L(v_3) \neq \Phi$, say $6 \in L(v_3)$. Let $c'(v_1) = 3$, $c'(v_2) = 4$, $c'(v_3) = 6$, $c'(v_4) = 5$, $c'(v_5) = 7$, then c' is a different L-coloring for $K_{1*5,4}$.

Case 2. There are one color in $\{1,2,3,4,5\}$ and one color in $\{6,7\}$ can be used to L-recolor the vertices in X_6.

Without loss of generality, we suppose the colors 1 and 7 can be used to L-recolor the vertices of part X_6. Firstly, let $c'(v_i) = l_i$, where $l_i \in \{1,7\} \bigcap L(v_i)$, for $i = 6,7,8,9$. Secondly, by Proposition 2.4 and Proposition 2.5, there exists $i_0 \in \{1,2,3,4,5\}$, such that $\{c_{i_0 2}, c_{i_0 3}\} = \{6,7\}$. In order to give $c'(v_i)$, for $i = 1,2,3,4,5$, according to the value of i_0, we also consider two subcases as follows.

Subcase 2.1. $i_0 = 1$, that is $L(v_1) = \{1,6,7\}$.

Let $c'(v_1) = 6$, $c'(v_i) = i$, for $i = 2,3,4,5$, then c' is a different L-coloring for $K_{1*5,4}$.

Subcase 2.2. $i_0 \in \{2,3,4,5\}$, say $i_0 = 5$, that is $L(v_5) = \{5,6,7\}$.

If $\{5,6\} \bigcap L(v_1) \neq \Phi$, say $5 \in L(v_1)$. Let $c'(v_1) = 5$, $c'(v_5) = 6$, $c'(v_i) = i$, for $i = 2,3,4$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{5,6\} \bigcap L(v_1) = \Phi$, then $\{2,3,4\} \bigcap L(v_1) \neq \Phi$, say $2 \in L(v_1)$. Consider $L(v_2)$, if $\{5,6\} \bigcap L(v_2) \neq \Phi$, say $5 \in L(v_2)$. Let $c'(v_1) = 2$, $c'(v_2) = 5$, $c'(v_5) = 6$, $c'(v_i) = i$, for $i = 3,4$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{5,6\} \bigcap L(v_2) = \Phi$, as $1 \notin L(v_2)$ by Proposition 2.2, so $\{3,4\} \bigcap L(v_2) \neq \Phi$, say $3 \in L(v_2)$. Consider $L(v_3)$, if $\{5,6\} \bigcap L(v_3) \neq \Phi$, say $5 \in L(v_3)$. Let $c'(v_1) = 2$, $c'(v_2) = 3$, $c'(v_3) = 5$, $c'(v_5) = 6$, $c'(v_4) = 4$, then c' is a different L-coloring for $K_{1*5,4}$. If $\{5,6\} \bigcap L(v_3) = \Phi$, as $\{1,2\} \bigcap L(v_3) = \Phi$ by Proposition 2.2, so $L(v_3) = \{3,4,7\}$. Consider $L(v_4)$, as $\{1,2,3\} \bigcap L(v_4) = \Phi$ by Proposition 2.2, so $\{5,6\} \bigcap L(v_4) \neq \Phi$, say $5 \in L(v_4)$. Let $c'(v_1) = 2$, $c'(v_2) = 3$, $c'(v_3) = 4$, $c'(v_4) = 5$, $c'(v_5) = 6$, then c' is a different L-coloring for $K_{1*5,4}$.

Combine all cases above, Theorem 1.1 holds. □

From Theorem 1.1 and the Lemma given in [5], we give two corollaries in the following, which have been obtained by Wenjie He et al.[7, 8].

Lemma 3.1 ([5]). *If G is a complete tripartite U3LC graph, then all vertices in each part can not take the same color in any unique 3-list coloring of G.*

Corollary 3.1. *$K_{1*4,4}$ has the property $M(3)$.*

Proof. By contradiction. It is clear that $K_{1*4,4}$ is a induced subgraph of $K_{1*5,4}$. If $K_{1*4,4}$ is $U3LC$, then $K_{1*5,4}$ is $U3LC$ by Theorem B. This is a contradiction to the Theorem 1.1. □

Corollary 3.2. *$K_{2,2,4}$ has the property $M(3)$.*

Proof. Let the three parts be $X_1 = \{v_1, v_2\}, X_2 = \{v_3, v_4\}$ and $X_3 = \{v_5, v_6, v_7, v_8\}$. By contradiction. Suppose there are assigned color lists, each of size 3, to the vertices in $K_{2,2,4}$ and c is a unique 3-list coloring from these lists. By Lemma 3.1, $c(v_1), c(v_2), c(v_3), c(v_4)$ are pairwise different. Add new edges between any two vertices in parts X_1 and X_2, the resulting graph is a $K_{1*4,4}$, and c is a proper L-coloring of $K_{1*4,4}$. By corollary 3.1, for the given lists, there exists a different L-coloring c' of $K_{1*4,4}$ which is a legal coloring for $K_{2,2,4}$. Hence $K_{2,2,4}$ has the property $M(3)$. □

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