# Graph designs for nine graphs with six vertices and nine edges*

Qingde Kang, Hongtao Zhao
Institute of Mathematics
Hebei Normal University
Shijiazhuang 050016, P. R. China

Chunping Ma
Department of Applied Mathematics
North China Electric Power University
Baoding 071003, P. R. China

**Abstract**

Let $K_v$ be the complete graph with $v$ vertices. Let $G$ be a finite simple graph. A $G$-decomposition of $K_v$, denoted by $G$-$GD(v)$, is a pair $(X, \mathcal{B})$ where $X$ is the vertex set of $K_v$ and $\mathcal{B}$ is a collection of subgraphs of $K_v$, called blocks, such that each block is isomorphic to $G$ and any two distinct vertices in $K_v$ are joined in exactly one block of $\mathcal{B}$. In this paper, nine graphs $G_i$ with six vertices and nine edges are discussed, and the existence for $G_i$-decomposition are complete solved, $1 \le i \le 9$.

**Key words:** $G$-decomposition; $G$-design; holey $G$-design; quasigroup

## 1 Introduction

A *complete graph* of order $v$, denoted by $K_v$, is a graph with $v$ vertices, where any two distinct vertices $x$ and $y$ are joined by one edge $\{x, y\}$. Let $G$ be a finite simple graph. A *$G$-design* or *$G$-decomposition* of $K_v$, denoted by $G$-$GD(v)$, is a pair $(X, \mathcal{B})$, where $X$ is the vertex set of $K_v$ and $\mathcal{B}$ is a collection of subgraphs of $K_v$, called *blocks*, such that each block is isomorphic to $G$ and any two distinct vertices in $K_v$ are joined in exactly

*Research supported by NSFC Grant 10671055, NSFHB Grant A2007000230 and NCEPU Grant 200511016.

one block of $\mathcal{B}$. The necessary conditions for the existence of a $G$-$GD(v)$ are

$$v \geq |V(G)|,\ v(v-1) \equiv 0 \bmod 2|E(G)|,\ (v-1) \equiv 0 \bmod d,$$

where $V(G)$ and $E(G)$ denote the set of vertices and edges of $G$ respectively, $d$ is the greatest common divisor of the degrees of all vertices in $G$.

Let $K_{n_1,n_2,\cdots,n_t}$ be a complete multipartite graph consisting of $t$ parts with vertex set $X = \bigcup_{i=1}^{t} X_i$, where these $X_i$ are disjoint and $|X_i| = n_i$, $1 \leq i \leq t$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G} = \{X_1, X_2, \cdots, X_t\}$. For any given graph $G$, if the edges of $K_{n_1,n_2,\cdots,n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to $G$ and is called as *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design,* denoted by $G$-$HD(T)$, where $T = \{n_1, n_2, \cdots, n_t\}$ is the type of the holey $G$-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes $i$ occurrences of 1, $r$ occurrences of 2, etc. A $G$-$HD(1^{v-w}w^1)$ is called an *incomplete G-design,* denoted by $G$-$ID(v,w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a $G$-$GD(v)$ is a $G$-$HD(1^v)$ or a $G$-$ID(v,w)$ with $w = 0$ or 1.

For the path $P_k$, the star $K_{1,k}$ and the cycle $C_k$, the existence problem of $P_k$-$GD(v)$, $K_{1,k}$-$GD(v)$ and $C_k$-$GD(v)$ have been solved[1,5,10]. The graph design problem for some of other graphs, e.g., $k$-cube[17], cycle with one chord[4,16] and so on[18], have been already researched. On the other hand, for the graphs with less vertices and less edges, the existence of their graph design has already been solved[2,3,6,11–15,19]. For the graphs with six vertices and nine edges, there are twenty graphs without isolated vertices(see the Appendix I in [9]). In this paper, we will discuss nine graphs with six vertices and nine edges, which are listed as follows. For convenience, as a block in graph design, each graph may be denoted by $(a, b, c, d, e, f)$ according to the following vertex-labels.

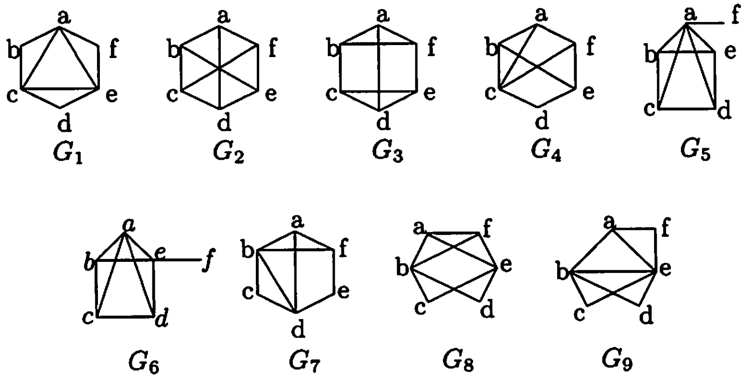


Figure 1: Nine graphs with six vertices and nine edges

In what follows, element $(x,i)$ in $Z_m \times Z_n$ may be denoted by $x_i$ briefly. Thus $x_i + y_j = (x,i) + (y,j) = (x+y, i+j) = (x+y)_{i+j}$, $\infty + x = \infty$, $\infty + x_i = \infty$. For the block $B = (x,y,z,u,v,w)$, $B$ mod $m$ denotes the blocks $(x+t, y+t, z+t, u+t, v+t, w+t)$, $0 \le t \le m-1$. In $Z_m \times Z_n$, a block mod $(m,n)$ denotes that the first coordinate mod $m$ and the second mod $n$, while mod $(m,-)$ denotes that the first coordinate mod $m$ and the second invariant.

In this paper, we shall prove that the necessary conditions for the existence of $G_i$-$GD(v)$, $1 \le i \le 9$, are also sufficient with the exceptions $(v,i) \in \{(9,4),(9,5),(9,6),(9,8),(9,9),(10,2)\}$. The main method to construct these graph designs is the following Lemma.

**Lemma 1.**[12] *For a given graph $G$ and positive integers $h, w, m$, if there exist a $G$-$HD(h^m)$, a $G$-$ID(h+w, w)$ and a $G$-$GD(w)$ (or a $G$-$GD(h+w)$), then a $G$-$GD(mh+w)$ exists, too.*

# 2 Constructions for $HD$

A *pairwise balanced design* $B[K,1;v]$ is a pair $(V, \mathcal{B})$, where $V$ is a $v$-set (*point set*) and $\mathcal{B}$ is a family of subsets (*blocks*) of $V$ with block sizes from $K$ such that every pair of distinct elements of $V$ occurs in exactly one block of $\mathcal{B}$. When $K = \{k\}$, a $B[K,1;v]$=$B[k,1;v]$ is just a balanced incomplete block design.

**Lemma 2.**[12] *Let $K$ be a set of positive integers, $m, v$ be positive integers, and $G$ be a finite simple graph.*

(1) *If there exist a $B[K,1;v]$ and a $G$-$HD(m^k)$ for any $k \in K$, then a $G$-$HD(m^v)$ exists.*

(2) *If there exists a $G$-$HD(m^2)$, then a $G$-$HD((mn)^t)$ exists for any $t \ge 2$ and $n \ge 1$.*

**Lemma 3.** *There exists a $G_2$-$HD(m^t)$ for any integer $t \ge 2$ and $m = 9, 18, 27$.*

**Proof.** First, a $G_2$-$HD(9^2)$ on the set $Z_9 \times Z_2$ can be formed by

$$(4_0, 0_1, 0_0, 6_1, 2_0, 3_1) \bmod (9, -).$$

Then, for positive integer $t \ge 2$, there exist a $G_2$-$HD(9^t)$, a $G_2$-$HD(18^t)$ and a $G_2$-$HD(27^t)$ by Lemma 2(2) (taking $m = 9, n = 1, 2, 3$). ■

## 2.1 Using quasigroups

A *quasigroup* is a set $Q$ with a binary operation "$\cdot$", denoted by $(Q, \cdot)$, such that the equations $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for every pair of elements $a, b \in Q$. It is well known that the multiplication table of a quasigroup defines a Latin square. On the contrary, a quasigroup can be obtained from a Latin square. A quasigroup is said to be *idempotent* (resp. *symmetric*) if the identity $x \cdot x = x$ (resp. $x \cdot y = y \cdot x$) holds for all $x \in Q$ (resp. $x, y \in Q$). Let $S$ be a finite set and $H = \{S_1, S_2, \cdots, S_n\}$ be a partition of $S$. A *holey Latin square* with holes $H$ is a $|S| \times |S|$ array $L$ on $S$ such that:

(1) every cell of $L$ either contains an element of $S$ or is empty;

(2) every element of $S$ occurs at most once in any row or column of $L$;

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$;

(4) element $s \in S$ occurs in row (or column) $t$ if and only if $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^{n} (S_i \times S_i)$.

The type of $L$ is the multiset $T = \{|S_i| : 1 \le i \le n\}$ and will be denoted by exponential notation. A holey symmetric quasigroup corresponding a holey symmetric Latin square with type $T$ is denoted by $HSQ(T) = (S, \mathcal{H}, \cdot)$. Two Latin squares $L_1$ and $L_2$ on a set $S$ said to be *orthogonal* if their superposition yields every ordered pair in $S \times S$. A Latin square is called *self-orthogonal* if it is orthogonal to its transpose. A self-orthogonal quasigroup corresponding to a self-orthogonal Latin Square of order $v$ is denoted by $SOQ(v)$. An idempotent $SOQ(v)$ is denoted by $ISOQ(v)$.

**Lemma 4.**[7,8]

(1) *There exists an idempotent quasigroup of order $v$ if and only if $v \neq 2$;*

(2) *There exists an idempotent symmetric quasigroup of order $v$ if and only if $v$ is odd;*

(3) *There exists an $HSQ(2^n)$ for any $n \ge 3$;*

(4) *There exists an $ISOQ(v)$ for any $v \neq 2, 3, 6$.*

Let $G$ be a finite simple graph, $e = |E(G)|$ and $n \ge 3$. In order to construct a holey graph design $G$-$HD(e^n)$, we may take $Z_e \times I_n$ as the vertex set and $Z_e$ as its automorphism group, where $(I_n, \cdot)$ be an idempotent quasigroup on the set $I_n = \{1, 2, \cdots, n\}$. A $G$-$HD(e^n)$ consists of $\binom{n}{2} e^2/e = n(n-1)e/2$ blocks. For our methods, the range of the subscripts of $n(n-1)/2$ base blocks $A_{i,j}$ is taken as $1 \le i < j \le n$.

On the other hand, in order to construct a holey graph design $G$-$HD((2e)^n)$, we may take $Z_e \times I_{2n}$ as the vertex set and $Z_e$ as its automorphism group, where $I_{2n} = \{1, 2, \cdots, 2n\}$ and $(I_{2n}, \mathcal{H}, \cdot)$ forms an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le n\}$. In fact, for the original $G$-$HD((2e)^n)$, the vertex set $Z_{2e} \times I_n$ contains $n$ holes with size $2e$: $H_i = Z_{2e} \times \{i\}$, $1 \le i \le n$. Now, halve each hole $H_i$ into $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, where each $\overline{H}_j = Z_e \times \{j\}$ has size $e$, $1 \le j \le 2n$. Then, equivalently, the holes of the $G$-$HD((2e)^n)$ can be regarded as $\overline{H}_1, \overline{H}_2, \cdots, \overline{H}_{2n}$ with such restriction that there is no edge between $\overline{H}_{2i-1}$ and $\overline{H}_{2i}$, $1 \le i \le n$.

A $G$-$HD((2e)^n)$ consists of $\binom{n}{2}(2e)^2/e = 2n(n-1)e$ blocks. For our methods, the range of the subscripts of $2n(n-1)$ base blocks $A_{i,j}$ is taken as $1 \le i < j \le 2n$ and $\{i,j\} \notin \mathcal{H}$. Below, it is enough to construct only one base block $A_{i,j}$ for constructing $G$-$HD(e^n)$ and $G$-$HD((2e)^n)$, where $i,j$ are variable in the given range.

Let $x, d \in Z_e$ and $i,j$ be in the given range for $A_{i,j}$ in above-mentioned constructions $G$-$HD(e^n)$ and $G$-$HD((2e)^n)$. Each vertex in the base block may be labelled as one among four forms: $(x,i), (x,j), (x,i\cdot j)$ and $(x,j\cdot i)$, where $(x,i\cdot j)$ and $(x,j\cdot i)$ are same for the symmetric quasigroup. Each unordered edge in the base block may be one among six forms:

$\{(x,i),(x+d,j)\}$, $\{(x,i),(x+d,i\cdot j)\}$, $\{(x,j\cdot i),(x+d,j)\}$,

$\{(x,i),(x+d,j\cdot i)\}$, $\{(x,i\cdot j),(x+d,j)\}$, $\{(x,i\cdot j),(x+d,j\cdot i)\}$.

For a given $d \in Z_e$, $u,v \in \{i,j,i\cdot j,j\cdot i\}$ and $u \ne v$, the edge joining vertices $(x,u)$ and $(x+d,v)$ in base block $A_{i,j}$ is denoted by $d(u,v)$, which represents a mixed difference orbit $\{\{(x,u),(x+d,v)\} : x \in Z_e\}$. And, denote $D(u,v) = \{d : d(u,v) \in A_{i,j}\}$.

**Lemma 5A.**[12] *Let $(I_n,\cdot)$ be an idempotent quasigroup on the set $I_n = \{1,2,\cdots,n\}$ and $G$ be a graph with $e$ edges, then $\mathcal{A} = \{A_{i,j} : 1 \le i < j \le n\}$ can be taken as a base of a $G$-$HD(e^n)$ under the action of automorphism group $Z_e$ if the following conditions hold.*

(1) $D(i,i\cdot j) = D(j,j\cdot i)$, $D(i,j\cdot i) = D(j,i\cdot j)$, $D(i,j) = D(j,i)$;

(2) $D(i\cdot j,j\cdot i) = D(j\cdot i,i\cdot j)$ *when $(I_n,\cdot)$ is self-orthogonal;*

(3) $D(i,j)\cup D(i,i\cdot j)\cup D(j\cdot i,j)\cup D(i,j\cdot i)\cup D(i\cdot j,j)\cup D(i\cdot j,j\cdot i) = Z_e$.

**Lemma 5B.**[12] *Let $(I_{2n},\mathcal{H},\cdot)$ be an $HSQ(2^n)$ with holes $\mathcal{H} = \{\{2r-1,2r\} : 1 \le r \le n\}$ and $G$ be a graph with $e$ edges. Then $\{A_{i,j} : 1 \le i < j \le 2n, \{i,j\} \notin \mathcal{H}\}$ can be taken as a base of a $G$-$HD((2e)^n)$ under the action of automorphism group $Z_e$ if the following conditions hold.*

(1) $D(i,i\cdot j) = D(j,i\cdot j)$;

(2) $D(i,j)\cup D(i,i\cdot j)\cup D(i\cdot j,j) = Z_e$.

**Lemma 6.** *There exist a $G_1$-$HD(9^{2t+1})$, a $G_8$-$HD(9^n)$ and a $G_k$-$HD(18^{t+2})$ for any integer $t \ge 1, n \ge 3, n \ne 6$ and $k = 1,8$.*

**Proof.** By Lemma 4(2), there exists an idempotent symmetric quasigroup $(I_{2t+1},\cdot)$ on the set $I_{2t+1} = \{1,2,\ldots 2t+1\}$ for $t \ge 1$. On the set $X = Z_9 \times I_{2t+1}$ define

$$A_1(i,j) = (0_{i\cdot j}, 4_j, 3_i, 1_{i\cdot j}, 3_j, 4_i);\ A_8(i,j) = (0_i, 2_{i\cdot j}, 5_j, 5_i, 1_{i\cdot j}, 0_j).$$

It is not difficult to verify that both $A_1(i,j)$ and $A_8(i,j)$ satisfy the conditions in Lemma 5A. Thus, for $k = 1,8$, $\mathcal{A}_k = \{A_k(i,j) \bmod (9,-) : 1 \le i < j \le 2t+1\}$ forms a $G_k$-$HD(9^{2t+1})$ indeed. Furthermore, we have

$G_8$-$HD(9^4)$ on the set $Z_9 \times Z_4$:

$$(4_1, 0_0, 8_1, 1_2, 1_3, 5_2) \bmod (9,4),\ (0_0, 2_1, 8_0, 8_2, 2_3, 0_2) + i_j,$$
$$0 \le i \le 8,\ 0 \le j \le 1.$$

$G_8$-$HD(9^8)$ on the set $Z_9 \times Z_8$:

$(6_1, 3_2, 7_3, 5_4, 1_6, 0_7)$, $(0_1, 2_2, 4_6, 6_4, 3_3, 1_5)$,

$(0_2, 8_1, 2_5, 6_6, 2_7, 4_6)$, mod $(9, 8)$,

$(0_2, 0_1, 1_3, 1_7, 0_5, 0_6) + i_j$, $0 \le i \le 8$, $0 \le j \le 3$.

However, there exists a $B[\{3,4,5,8\},1;n]$ for any $n \ge 3, n \ne 6$ by [7]. Thus, there exists a $G_8$-$HD(9^n)$ for any $n \ge 3, n \ne 6$ by Lemma 2(1).

Finally, let's consider the $HSQ(2^{t+2}) = (I_{2t+4}, \mathcal{H}, \cdot)$ with holes $\mathcal{H} = \{\{2r-1, 2r\} : 1 \le r \le t+2\}$, which exists for $t \ge 1$ by Lemma 4(3). Then $\mathcal{A}_k = \{A_k(i,j) \bmod (9,-) : 1 \le i < j \le 2t+4, \{i,j\} \notin \mathcal{H}\}$ will form a $G_k$-$HD(18^{t+2})$ by Lemma 5B, where $k = 1, 8$. ■

**Lemma 7.** *There exists a $G_4$-$HD(18^t)$ for any integer $t \ge 3$.*

**Proof.** By Lemma 4(4), there exists an $ISOQ(t) = (\{1, 2, \cdots t\}, \cdot)$ for $t \ne 2, 3, 6$. It is not difficult to verify that the following base blocks for $G_4$-$HD(18^t)$, on the set $Z_{18} \times I_t$, satisfy the conditions in Lemma 5A:

$(0_{i \cdot j}, 8_j, 1_{j \cdot i}, 17_j, 5_i, 5_j)$, $(0_{j \cdot i}, 8_i, 1_{i \cdot j}, 17_i, 14_j, 5_i)$, mod $(18, -)$,

$1 \le i < j \le t$ .

Furthermore, we can give the direct constructions as follows.

$G_4$-$HD(18^3)$ on the set $Z_{18} \times Z_3$ :

$(0_0, 15_1, 2_2, 9_1, 5_0, 6_1)$, $(2_0, 5_1, 0_2, 1_1, 11_0, 11_1)$, mod $(18, 3)$.

$G_4$-$HD(18^6)$ on the set $Z_{18} \times Z_6$ :

$(0_0, 9_4, 3_1, 16_0, 1_3, 15_2)$, $(0_0, 0_4, 14_1, 13_0, 11_3, 2_2)$,

$(0_0, 8_4, 15_1, 16_0, 8_3, 13_2)$, $(0_0, 17_3, 7_2, 11_4, 12_0, 1_4)$, mod $(18, 6)$,

$(2_0, 0_5, 0_2, 3_4, 11_0, 6_4) + i_j$, $(2_3, 9_2, 0_5, 3_1, 11_3, 6_1) + i_j$,

$0 \le i \le 17$, $0 \le j \le 2$. ■

## 2.2 Using directed product automorphism group

In this section, we construct some $HDs$ by using directed product automorphism group. This method was first used in [16].

**Lemma 8.** *There exist a $G_s$-$HD(9^n)$ and a $G_6$-$HD(9^{2t+1})$ for $s = 4, 5, 7$, any integer $n \ge 3$, $n \ne 6, 8$ and $t \ge 1$.*

**Proof.** For $s = 4, 5, 6, 7$, define $G_s$-block family $A_s$ on the set $Z_9 \times Z_{2t+1}$ as follows.

$A_4 = \{(0_x, 2_0, 0_{-x}, 6_0, 5_x, 4_0) : 1 \le x \le t\}$;

$A_5 = \{(6_0, 8_x, 0_{-x}, 4_x, 3_{-x}, 6_x) : 1 \le x \le t\}$;

$A_6 = \{(6_0, 8_x, 0_{-x}, 4_x, 3_{-x}, 3_0) : 1 \le x \le t\}$;

$A_7 = \{(3_{-x}, 0_0, 0_{2x}, 2_x, 8_0, 4_x) : 1 \le x \le t\}$.

Then $A_s$ mod $(9, 2t+1)$ forms a $G_s$-$HD(9^{2t+1})$ for $s = 4, 5, 6, 7$ and $t \ge 1$. Furthermore, for $s = 4, 5, 7$, we can give direct constructions $G_s$-$HD(9^4) = (Z_9 \times Z_4, \mathcal{B}_s)$ as follows.

$\mathcal{B}_4$ : $(0_3, 3_2, 0_1, 7_0, 5_3, 3_0)$, $(0_2, 3_1, 0_0, 7_1, 5_2, 3_3)$, mod $(9, -)$,

$(0_2, 3_0, 4_3, 0_0, 1_2, 0_1)$ mod $(9, 4)$.

$\mathcal{B}_5 : (0_2,0_0,0_3,8_0,0_1,4_3),\ (7_0,0_1,5_2,2_1,3_3,2_3),\ (5_3,0_1,2_0,8_1,6_2,1_1),$
$(8_1,0_2,6_3,1_2,5_0,3_0),\ (1_0,0_3,7_2,8_3,0_1,3_2),$
$(3_3,0_2,6_0,5_2,1_1,1_0),\ \bmod\ (9,-).$

$\mathcal{B}_7 : (3_0,3_1,0_0,0_3,3_2,0_1),\ (6_1,3_2,3_3,0_0,0_3,6_2),\ \bmod\ (9,-),$
$(0_0,5_1,0_2,1_3,7_2,0_1)\ \bmod\ (9,4).$

Finally, there exists a $B[\{3,4,5\},1;n]$ for any $n \geq 3, n \neq 6,8$ by [7]. So, the conclusions hold by Lemma 2(1). ■

**Lemma 9.** *There exists a $G_k$-$HD(18^n)$ for any integer $n \geq 3$ and $k = 5,6$.*

**Proof.** First, for $k = 5,6$ and $t \geq 1$, a $G_k$-$HD(18^{2t+1}) = (Z_{18} \times Z_{2t+1}, \mathcal{A}_k)$ can be constructed, where $\mathcal{A}_k = \{A_k \bmod (18,2t+1)\}$, and

$A_5 = \{(0_0,3_x,4_{-x},12_x,5_{-x},0_x),\ (0_0,5_x,12_{-x},4_x,3_{-x},9_x) : 1 \leq x \leq t\},$

$A_6 = \{(0_0,3_x,4_{-x},12_x,5_{-x},5_x),\ (0_0,5_x,12_{-x},4_x,3_{-x},12_x) :\ 1 \leq x \leq t\}.$

Furthermore, we have the following direct constructions $G_k$-$HD(18^n)$ on the set $Z_{18} \times Z_n$, where $k = 5,6$ and $n = 4,6,8$.

$G_5$-$HD(18^4)$: $(0_0,9_3,2_2,6_1,8_2,1_2),(0_0,1_3,14_2,3_1,7_2,6_2)$, mod (18,4),
$(0_0,3_3,5_2,13_1,3_2,0_2)+i_j,\ (0_2,3_1,5_0,13_3,3_0,9_0)+i_j,\ i \in Z_{18},\ j = 0,1.$

$G_6$-$HD(18^4)$: $(0_0,9_3,2_2,6_1,8_2,7_0),\ (0_0,1_3,14_2,3_1,7_2,1_0)$, mod (18,4),
$(0_0,3_3,5_2,13_1,3_2,3_0)+i_j,\ (0_2,3_1,5_0,13_3,3_0,12_2)+i_j,\ i \in Z_{18},\ j = 0,1.$

$G_5$-$HD(18^6)$: $(0_0,0_2,1_1,12_4,2_3,5_3),\ (0_0,13_2,0_1,15_4,1_3,6_3),$
$(2_0,0_4,6_2,9_1,7_2,5_4),\ (1_0,0_4,9_2,4_1,8_2,3_4),\ \bmod\ (18,6),$
$(0_0,17_2,8_1,4_4,10_3,0_3)+i_j,\ (0_3,17_5,8_4,4_1,10_0,9_0)+i_j,$
$i \in Z_{18},\ 0 \leq j \leq 2.$

$G_6$-$HD(18^6)$: $(1_0,0_4,9_2,4_1,8_2,6_4),\ (2_0,0_4,6_2,9_1,7_2,4_4),$
$(0_0,0_2,1_1,12_4,2_3,7_0),\ (0_0,13_2,0_1,15_4,1_3,7_0),\ \bmod\ (18,6),$
$(0_0,17_2,8_1,4_4,10_3,10_0)+i_j,\ (0_3,17_5,8_4,4_1,10_0,1_3)+i_j,$
$i \in Z_{18},\ 0 \leq j \leq 2.$

$G_5$-$HD(18^8)$: $(0_0,1_1,13_2,2_3,3_5,14_3),\ (0_0,3_1,12_2,9_3,14_5,13_3),$
$(0_0,14_1,0_2,5_3,1_5,12_3),\ (0_0,8_1,10_2,8_3,12_5,7_3),$
$(0_0,3_2,10_4,0_1,16_3,1_4),\ (0_0,8_2,17_4,17_1,1_3,6_4),\ \bmod\ (18,8),$
$(0_0,15_2,3_4,10_1,3_3,0_4)+i_j,\ (0_4,15_6,3_0,10_5,3_7,9_0)+i_j,$
$i \in Z_{18},\ 0 \leq j \leq 3.$

$G_6$-$HD(18^8)$: $(0_0,14_1,0_2,5_3,1_5,13_0),\ (0_0,3_2,10_4,0_1,16_3,15_5),$
$(0_0,1_1,13_2,2_3,3_5,17_0),\ (0_0,8_2,17_4,17_1,1_3,7_7),$
$(0_0,3_1,12_2,9_3,14_5,9_0),\ (0_0,8_1,10_2,8_3,12_5,1_0),\ \bmod\ (18,8),$
$(0_0,15_2,3_4,10_1,3_3,3_7)+i_j,\ (0_4,15_6,3_0,10_7,3_5,12_3)+i_j,$
$i \in Z_{18},\ 0 \leq j \leq 3.$

Finally, there exists a $B[\{3,4,5,6,8\},1;n]$ for any $n \geq 3$ by [7]. So, the conclusions hold by Lemma 2(1). ■

## 2.3 Other Methods

**Lemma 10.** *There exists a $G_k$-$HD(9^n)$ for any integer $n \geq 3$, $n \neq 6, 8$ and $k = 3, 9$.*

**Proof.** First, give the following direct constructions $G_k$-$HD(9^n)$ on the set $Z_9 \times Z_n$, where $k = 3, 9$ and $n = 3, 4, 5$.

$G_3$-$HD(9^3)$: $(0_0, 0_1, 7_0, 3_2, 8_1, 0_2)$, $(0_0, 3_1, 6_0, 4_2, 1_1, 8_2)$, $(0_0, 7_1, 8_0, 1_2, 4_1, 6_2)$, mod $(9, -)$.

$G_9$-$HD(9^3)$: $(0_2, 3_1, 6_2, 1_2, 0_0, 0_1)$, $(8_0, 4_2, 5_0, 2_0, 0_1, 2_2)$, $(1_1, 2_0, 4_1, 8_1, 0_2, 5_0)$, mod $(9, -)$.

$G_3$-$HD(9^4)$: $(0_0, 0_2, 1_3, 7_1, 5_2, 0_3)$, $(0_1, 1_3, 8_0, 5_2, 7_3, 4_0)$, $(1_2, 0_0, 6_1, 8_3, 4_0, 8_1)$, $(0_3, 0_1, 0_2, 2_0, 5_1, 6_2)$, $(0_0, 1_1, 7_3, 8_2, 0_1, 5_3)$, $(0_1, 0_0, 2_2, 8_3, 7_0, 3_2)$, mod $(9, -)$.

$G_9$-$HD(9^4)$: $(8_2, 3_1, 1_2, 2_2, 0_0, 0_3)$, $(1_3, 2_2, 4_3, 8_3, 0_1, 0_0)$, $(3_1, 2_0, 7_1, 4_1, 0_3, 0_2)$, $(2_0, 7_3, 3_2, 3_0, 0_1, 0_2)$, $(4_2, 7_0, 2_2, 6_1, 0_3, 0_1)$, $(4_0, 3_3, 6_0, 0_0, 0_2, 8_1)$, mod $(9, -)$.

$G_3$-$HD(9^5)$: $(0_0, 0_2, 1_0, 8_4, 3_2, 1_4)$, $(0_0, 0_1, 3_0, 6_2, 1_1, 4_2)$, mod $(9, 5)$.

$G_9$-$HD(9^5)$: $(1_1, 0_2, 4_1, 3_1, 0_0, 8_2)$, $(6_2, 0_4, 7_2, 4_2, 0_0, 7_4)$, mod $(9, 5)$.

Finally, there exists a $B[\{3, 4, 5\}, 1; n]$ for any $n \geq 3, n \neq 6, 8$ by [7]. So, the conclusions hold by Lemma 2(1). ■

**Lemma 11.** *There exist a $G_3$-$HD(18^3)$, a $G_3$-$HD(18^4)$, a $G_9$-$HD(27^3)$ and a $G_9$-$HD(18^n)$ for any integer $n \geq 3$.*

**Proof.** First, give the following direct constructions $G_9$-$HD(18^n)$ on the set $Z_{18} \times Z_n$, where $n = 3, 4, 5, 6, 8$.

$G_9$-$HD(18^3)$: $(3_2, 1_1, 4_2, 5_2, 0_0, 16_1)$, $(7_2, 0_1, 8_2, 9_2, 1_0, 1_1)$, mod $(18, 3)$.

$G_9$-$HD(18^4)$: $(0_0, 6_1, 1_0, 10_0, 6_2, 9_1)$, $(0_0, 4_1, 10_0, 1_0, 3_2, 8_1)$, mod $(18, 4)$, $(0_0, 11_1, 10_0, 1_0, 0_2, 2_1) + i_j$, $(9_2, 11_3, 10_2, 1_2, 0_0, 2_3) + i_j$, $i \in Z_{18}$, $0 \leq j \leq 1$.

$G_9$-$HD(18^5)$: $(14_4, 2_2, 7_4, 8_4, 0_0, 11_2)$, $(0_2, 1_1, 1_2, 4_2, 0_0, 16_1)$, $(5_4, 13_2, 3_4, 4_4, 0_0, 16_2)$, $(17_2, 8_1, 14_2, 15_2, 0_0, 12_1)$, mod $(18, 5)$.

$G_9$-$HD(18^6)$: $(0_4, 8_2, 16_3, 2_1, 6_0, 11_2)$, $(0_3, 2_2, 0_5, 1_5, 4_0, 6_1)$, $(0_4, 3_2, 8_3, 6_1, 13_0, 6_3)$, $(0_0, 10_1, 15_0, 6_5, 3_3, 7_1)$, mod $(18, 6)$, $(0_0, 0_1, 1_5, 1_2, 0_3, 9_1) + i_j$, $(9_3, 1_5, 0_4, 1_4, 0_1, 10_5) + i_j$, $i \in Z_{18}$, $0 \leq j \leq 2$.

$G_9$-$HD(18^8)$: $(0_0, 12_3, 14_1, 6_1, 2_4, 7_3)$, $(0_0, 0_2, 1_1, 2_1, 3_4, 5_1)$, $(0_0, 2_1, 5_2, 6_2, 10_4, 6_1)$, $(0_0, 7_1, 0_2, 1_2, 12_4, 15_1)$, $(0_0, 11_4, 2_5, 8_6, 9_2, 4_4)$, $(0_0, 9_3, 2_1, 13_1, 5_4, 13_3)$, mod $(18, 8)$, $(0_0, 1_2, 0_1, 1_1, 0_4, 10_2) + i_j$, $(9_4, 1_6, 0_7, 1_7, 0_0, 12_6) + i_j$, $i \in Z_{18}$, $0 \leq j \leq 3$.

Furthermore, a $G_9$-$HD(18^n)$ exists by existence of a $B[\{3, 4, 5, 6, 8\}, 1; n]$, $n \geq$

3. Finally, the remaining holey graph designs $HD(m^n)$ on the set $Z_m \times Z_n$ are listed as follows.

$G_9$-$HD(27^3)$: $(17_0, 0_1, 14_0, 12_0, 20_2, 2_1)$, $(16_0, 0_1, 19_0, 18_0, 2_2, 22_1)$, $(26_0, 0_1, 23_0, 22_0, 0_2, 2_1)$, mod $(27, 3)$.

$G_3$-$HD(18^3)$: $(4_1, 0_0, 11_1, 1_2, 12_0, 6_2)$, $(13_1, 0_0, 9_1, 12_2, 8_0, 2_2)$, mod $(18, 3)$.

$G_3$-$HD(18^4)$: $(2_2, 0_0, 11_2, 13_1, 8_0, 0_1)$, $(17_2, 0_0, 13_2, 2_1, 3_0, 9_1)$, mod $(18, 4)$, $(4_2, 0_0, 6_2, 10_1, 6_0, 1_1) + i_j$, $(4_0, 0_2, 6_0, 10_3, 15_2, 1_3) + i_j$, $i \in Z_{18}$, $0 \le j \le 1$. ■

# 3 Nonexistences

In this section, we'll prove the nonexistence for a few graph designs. Let graph $G$ have $m_i$ vertices with degree $d_i, 1 \le i \le r$, and $\sum_{i=1}^{r} m_i = 6$. Suppose there exists a $G$-$GD(v)$ on a $v$-set $V$, with $b$ blocks. If some element $\alpha$ of $V$ appears in $s_i$ blocks as $r_i$-degree vertices, $1 \le i \le t$, we call the element $\alpha$ has the degree-type $r_1^{s_1} r_2^{s_2} \cdots r_t^{s_t}$. The proof consists of by the following steps.

1° Find nonnegative integer solutions for equations

$$\sum_{i=1}^{r} d_i x_i = v - 1 \text{ with restriction } \sum_{i=1}^{r} x_i \le b. \qquad (*)$$

Its one solution $(x_1, x_2, \cdots, x_r) = (a_{1j}, a_{2j}, \cdots, a_{rj})$ means that some element $\alpha$ of $V$ may have the degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}$, $1 \le j \le s$.

2° Solve the further equations

$$\sum_{j=1}^{s} y_j = v \text{ and } \sum_{j=1}^{s} a_{ij} y_j = m_i b, \ 1 \le i \le r. \qquad (**)$$

Each solution $(y_1, y_2, \cdots, y_s)$ means a possible structure of $G$-$GD(v)$: $y_j$ elements of $V$ have degree-type $d_1^{a_{1j}} d_2^{a_{2j}} \cdots d_r^{a_{rj}}, 1 \le j \le s$.

3° For each solution obtained above, discuss the existence of such structure.

**Lemma 12.** *There exist no $G_k$-$GD(9)$ for $k = 4, 5, 6, 8, 9$.*

**Proof.**

(1) $G_4$-$GD(9)$, $v = 9$, $b = 4$, and $(d_1, m_1) = (2, 1)$, $(d_2, m_2) = (3, 4)$, $(d_3, m_3) = (4, 1)$. There are four solutions for $(*)$. And, the equations $\sum_{j=1}^{s} a_{ij} y_j = m_i b$, $1 \le i \le r$, will be in this form.

$$\begin{pmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ 4 \end{pmatrix}.$$

It is not difficult to see that it implies $y_2+2y_4 = -4$, there is no nonnegative integer solution.

(2) $G_5$-$GD(9)$, $v = 9$, $b = 4$, $(d_1, m_1) = (1,1)$, $(d_2, m_2) = (3,4)$, $(d_3, m_3) = (5,1)$. There are three solutions for $(*)$. Furthermore, we have

$$\begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ 4 \end{pmatrix}.$$

It implies $y_2 = -2$, which is impermissible.

(3) $G_6$-$GD(9)$, $v = 9$, $b = 4$, $(d_1, m_1) = (1,1)$, $(d_2, m_2) = (3,3)$, $(d_3, m_3) = (4,2)$. There are three solutions for $(*)$. Furthermore, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 8 \end{pmatrix}.$$

There are two contradictory equations among them. So there exist no solutions for $(**)$.

(4) $G_8$-$GD(9)$, $v = 9$, $b = 4$, $(d_1, m_1) = (2,2)$, $(d_2, m_2) = (3,2)$, $(d_3, m_3) = (4,2)$. There are four solutions for $(*)$. Furthermore, we have

$$\begin{pmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 8 \end{pmatrix}.$$

It has only two solutions $(y_1, y_2, y_3, y_4) = (3,2,4,0)$ and $(4,0,4,1)$. However, since two 4-degree vertices in graph $G_8$ are disjoint, there's at most one element having degree-type $4^2$, which implies $y_1 \leq 1$, it is impossible.

(5) $G_9$-$GD(9)$, $v = 9$, $b = 4$, $(d_1, m_1) = (2,3)$, $(d_2, m_2) = (3,1)$, $(d_3, m_3) = (4,1)$, $(d_4, m_4) = (5,1)$. There are five solutions for $(*)$. Furthermore, we have

$$\begin{pmatrix} 0 & 0 & 2 & 1 & 4 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 4 \\ 4 \end{pmatrix}.$$

It implies $y_1 = 4$, i.e., there are exactly four elements having degree-type $3^1 5^1$. However, there are only one 3-degree vertex and one 5-degree vertex in graph $G_9$, and 3-degree vertex and 5-degree vertex are joint. So there are four edges jointing 3-degree vertex and 5-degree vertex in the four blocks. But there are six edges jointing the four elements which have degree-type $3^1 5^1$. It's a contradiction. ■

**Lemma 13.** *There exists no* $G_2$-$GD(10)$.

**Proof.** $G_2$ is a bipartite graph $K_{3,3}$. If there exists a $G_2$-$GD(10)$ on the set $X = \{0, 1, 2, a, b, c, \cdots\}$, which has five blocks. Without loss of generality, let one of the five blocks be $A$.

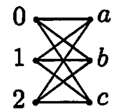


Figure 2: Block $A$

Then the three edges of $K_3$ on the set $\{0, 1, 2\}$ must appear in the other four blocks. The arrangement of the three edges must be one of the following two configurations.

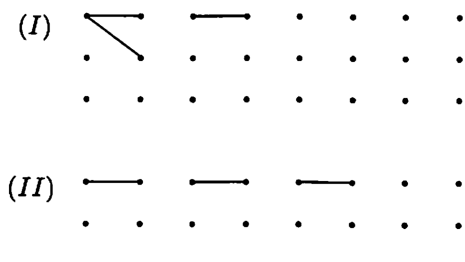


Figure 3: Two configurations

Similarly, the arrangement of the three edges of $K_3$ on the set $\{a, b, c\}$ must also be one of the two configurations above. Since the nine edges jointing the elements between $\{0, 1, 2\}$ and $\{a, b, c\}$ all appear in block $A$, the element in $\{0, 1, 2\}$ and the element in $\{a, b, c\}$ can't appear in any of other blocks simultaneously. So the block configurations for $\{0, 1, 2\}$ and for $\{a, b, c\}$ both have to be $(I)$. Without loss of generality, the arrangement must be.

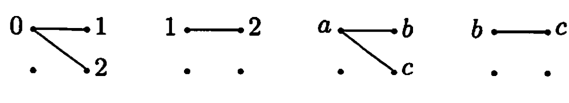


Figure 4: The arrangement

Since every vertex in $G_2$ is 3-degree, each element of $X$ must appear in three blocks of this design. However, in this arrangement, both 0 and $a$ appear only in two blocks. It's a contradiction. ■

# 4 Main constructions

Now, let's list the following table for the desired designs for the nine graphs.

Table 1: $t \geq 1,\ s \neq 2,3,6$

| $v \equiv$ (mod 9) | | 0 | | 1 | |
|---|---|---|---|---|---|
| $G_3$ | $HD$ | | | $9^{2t+1}, 9^4, 18^3, 18^4$ | |
| | $GD$ | | | 10,19 | |
| $G_7$ | $HD$ | $9^{2t+1}, 9^4$ | | $9^{2t+1}, 9^4$ | |
| | $ID$ | (18, 9) | | | |
| | $GD$ | 9 | | 10,19,55,73 | |
| $v \equiv$ (mod 18) | | 0 | 1 | 9 | 10 |
| $G_1$ | $HD$ | | $18^{t+2}$ | $9^{2t+1}$ | |
| | $GD$ | | 19,37 | 9 | |
| $G_2$ | $HD$ | | $18^{t+1}$ | | $27^{t+1}$ |
| | $ID$ | | | | (37,10),(46,19) |
| | $GD$ | | 19 | | 19,28,37 |
| $G_4$ | $HD$ | $9^3, 18^{t+2}$ | $18^{t+2}$ | $9^{2t+1}$ | $9^{2t+1}$ |
| | $ID$ | (18,10),(18,9) | | (27,18) | |
| | $GD$ | 10 | 19,37 | 27 | 10 |
| $G_5$ | $HD$ | $18^{t+2}$ | $18^{t+2}$ | $9^{2t+1}, 9^4$ | $9^{2t+1}$ |
| | $ID$ | | | (18,9) | |
| | $GD$ | 18,36 | 19,37 | 18,27,63,81 | 10 |
| $G_6$ | $HD$ | $18^{t+2}$ | $18^{t+2}$ | $18^{t+2}$ | $9^{2t+1}$ |
| | $ID$ | | | (27,9) | |
| | $GD$ | 18,36 | 19,37 | 27,45 | 10 |
| $G_8$ | $HD$ | $18^{t+2}, 9^3$ | $18^{t+2}$ | $9^{2t+1}, 9^4, 9^8$ | $9^{2t+1}$ |
| | $ID$ | (18,9) | | (18,9) | |
| | $GD$ | 18 | 19,37 | 18,27,63 | 10 |
| $G_9$ | $HD$ | $18^{t+2}$ | $18^{t+2}$ | $9^3, 9^4, 9^5, 27^3$ | $9^3, 9^4, 9^5$ |
| | $ID$ | | | (18,9) | |
| | $GD$ | 18,36 | 19,37 | 18,27,63 | 10 |

In the following sections, we shall construct the desired designs listed in the table.

## 4.1 Constructions for $GD$

In this section, we construct all $G_k$-$GD(v)$ listed in above table, $1 \leq k \leq 9$. The point set is taken as — $Z_v$ for $v = 19, 37, 55, 73$;

$(Z_{\frac{v-1}{2}} \times Z_2) \bigcup \{\infty\}$ for $v = 27, 45, 63, 81$;

$Z_{v-1} \bigcup \{\infty\}$ for $v = 18, 36$;

$Z_5 \times Z_2$ for $v = 10$; and $Z_7 \times Z_4$ for $v = 28$.

$G_1$-$GD(9)$ : $(1,7,0,8,3,5),(2,8,1,6,4,3),(0,6,2,7,5,4),(6,3,7,4,8,5)$;
$G_1$-$GD(19)$: $(0,3,10,6,11,13)$ mod 19;
$G_1$-$GD(37)$: $(0,6,2,5,20,27),(0,12,1,22,9,14)$, mod 37;
$G_2$-$GD(19)$: $(0,9,10,8,13,7)$ mod 19;
$G_2$-$GD(28)$: $(0_0,3_0,1_0,0_1,2_0,3_1),(0_0,0_3,1_0,0_2,2_0,3_2)$,
$(0_0,4_1,0_1,4_2,2_1,2_3),(0_0,1_3,0_1,3_3,5_3,4_3)$,
$(0_1,6_1,0_2,5_3,2_2,6_3),(0_1,0_2,1_2,5_2,0_3,6_2)$, mod $(7,-)$;
$G_3$-$GD(10)$: $(0_0,0_1,3_1,2_0,1_0,4_1)$ mod $(5,-)$;
$G_3$-$GD(19)$: $(0,10,6,14,8,7)$ mod 19;
$G_4$-$GD(10)$: $(4_1,1_0,0_0,2_0,3_1,0_1)$ mod $(5,-)$;
$G_4$-$GD(19)$: $(10,8,0,18,12,7)$ mod 19;
$G_4$-$GD(27)$: $(1_0,0_0,4_0,\infty,1_1,8_1),(2_0,0_0,7_0,2_1,0_1,4_1)$,
$(1_1,2_0,0_1,3_1,11_1,8_0)$, mod $(13,-)$;
$G_4$-$GD(37)$: $(1,18,0,33,30,16),(0,13,2,9,4,10)$, mod 37;
$G_5$-$GD(10)$: $(0_0,4_1,2_1,1_1,1_0,2_0)$ mod $(5,-)$;
$G_5$-$GD(18)$: $(0,8,3,7,6,\infty)$ mod 17;
$G_5$-$GD(19)$: $(9,8,4,7,0,15)$ mod 19;
$G_5$-$GD(27)$: $(0_0,7_0,10_1,7_1,2_0,\infty),(0_1,0_0,4_0,5_1,4_1,\infty)$,
$(0_1,1_0,2_0,5_0,7_1,2_1)$, mod $(13,-)$;
$G_5$-$GD(36)$: $(0,17,7,19,8,\infty),(0,15,1,5,2,6)$, mod 35;
$G_5$-$GD(37)$: $(0,19,22,20,9,8),(0,32,36,13,7,16)$, mod 37;
$G_5$-$GD(63)$: $(0_1,9_0,4_1,9_1,20_0,28_0),(0_1,18_0,6_1,14_1,8_0,17_0)$,
$(0_0,0_1,1_0,3_0,10_1,\infty),(0_0,5_1,4_0,9_0,17_1,12_0)$,
$(0_1,3_0,1_1,3_1,10_0,\infty),(0_0,2_1,6_0,14_0,18_1,13_0)$,
$(0_1,22_0,7_1,18_1,6_0,16_0)$, mod $(31,-)$;
$G_5$-$GD(81)$: $(7_1,15_0,11_1,16_1,31_0,27_1)+i_0,(0_0,5_1,4_0,9_0,19_1,7_1)+i_0$,
$(27_1,35_0,31_1,36_1,11_0,20_0)+i_0$,
$(20_0,25_1,24_0,29_0,39_1,0_0)+i_0$, $0\le i\le 19$;
$(0_0,2_1,11_0,23_0,27_1,20_1),(0_0,9_1,6_0,13_0,28_1,17_1)$,
$(0_1,26_0,8_1,18_1,5_0,16_1),(0_0,0_1,1_0,3_0,11_1,\infty)$,
$(0_1,22_0,6_1,13_1,7_0,12_1),(0_0,29_1,8_0,18_0,12_1,30_1)$,
$(0_1,3_0,1_1,3_1,17_0,\infty)$, mod $(40,-)$.
$G_6$-$GD(10)$: $(0_0,4_1,2_1,1_1,1_0,3_0)$ mod $(5,-)$;
$G_6$-$GD(18)$: $(0,8,3,7,6,\infty)$ mod 17;
$G_6$-$GD(19)$: $(9,8,4,7,0,6)$ mod 19;
$G_6$-$GD(27)$: $(0_0,7_0,10_1,7_1,2_0,\infty),(0_1,0_0,4_0,5_1,4_1,\infty)$,
$(0_1,1_0,2_0,5_0,7_1,9_1)$, mod $(13,-)$;
$G_6$-$GD(36)$: $(0,17,7,19,8,\infty),(0,15,1,5,2,8)$, mod 35;
$G_6$-$GD(37)$: $(0,19,22,20,9,17),(0,32,36,13,7,23)$, mod 37;
$G_6$-$GD(45)$: $(0_1,3_0,17_1,19_1,17_0,\infty),(0_0,3_1,2_0,7_0,17_1,\infty)$,
$(0_0,8_1,1_0,4_0,20_1,14_0),(0_0,9_1,0_1,4_1,13_0,19_0)$, mod $(22,-)$;
$(0_1,11_0,1_1,16_1,1_0,12_0)+i_0$,
$(11_0,12_1,0_0,12_0,5_1,16_1)+i_0$, $0\le i\le 10$.

$G_7$-$GD(9)$ : $(2,a,1,b,x,4),(4,b,y,c,2,3),(3,c,x,a,y,1),(2,x,3,y,4,1)$;
$G_7$-$GD(10)$: $(4_0,1_1,2_0,0_0,0_1,2_1)$ mod 5;
$G_7$-$GD(19)$: $(2,0,7,10,11,6)$ mod 19;
$G_7$-$GD(55)$: $(25,27,22,0,20,1),(23,10,7,0,11,19),$
$(21,15,1,0,17,33)$, mod 55;
$G_7$-$GD(73)$: $(1,2,4,7,3,10),(1,11,22,34,2,26),$
$(1,14,28,45,2,26),(1,28,12,48,27,46)$, mod 73;
$G_8$-$GD(10)$: $(1_0,0_0,2_0,2_1,0_1,1_1)$ mod 5;
$G_8$-$GD(19)$: $(0,2,7,9,6,10)$ mod 19;
$G_8$-$GD(27)$: $(10_1,6_0,\infty,6_1,3_1,0_0,),(9_1,4_0,10_1,12_0,1_1,0_0),$
$(1_0,3_0,11_1,10_1,12_1,0_0)$, mod $(13,-)$;
$G_8$-$GD(37)$: $(0,18,8,9,16,1),(0,6,19,17,5,2)$, mod 37;
$G_8$-$GD(63)$: $(0_1,20_1,2_0,12_0,22_0,7_1),(0_1,22_1,27_0,15_0,18_0,10_1),$
$(0_0,24_0,19_0,5_1,10_1,14_1),(0_0,1_0,\infty,28_0,29_1,0_1),$
$(0_0,17_0,5_1,20_1,4_1,2_0),(0_0,19_0,25_1,3_1,11_1,6_0),$
$(0_0,25_1,22_1,1_0,28_1,8_0)$, mod $(31,-)$.
$G_9$-$GD(10)$: $(1_0,0_0,1_1,3_1,0_1,3_0)$ mod $(5,-)$;
$G_9$-$GD(19)$: $(4,0,6,8,9,11)$ mod 19;
$G_9$-$GD(27)$: $(0_0,1_0,\infty,4_1,6_1,2_1),(0_1,6_0,0_0,10_0,1_1,6_1),$
$(12_1,0_0,5_0,1_1,2_0,2_1)$, mod $(13,-)$;
$G_9$-$GD(37)$: $(18,1,7,13,0,3),(11,0,10,16,2,6)$, mod 37;
$G_9$-$GD(63)$: $(1_0,0_0,1_1,2_1,0_1,\infty),(2_0,0_0,7_0,22_1,5_1,21_1),$
$(0_0,20_1,11_1,8_0,27_0,13_0),(3_0,0_0,10_0,27_1,7_1,26_1),$
$(0_0,6_1,3_1,0_1,20_0,5_0),(26_1,0_0,6_0,8_0,16_1,3_1),$
$(7_1,0_1,26_1,22_0,13_0,3_1)$, mod $(31,-)$.

## 4.2 Constructions for $ID$

In this section, we construct all $G_k$-$ID(v,w)$ listed in above table for $k = 2,4,5,6,7,8,9$.

$G_2$-$ID(27+\omega,\omega)$: $(Z_3\times Z_9)\bigcup\{\infty_1,\infty_2,\cdots,\infty_\omega\}$, $\omega=10,19$

$(\infty_1,1_3,\infty_2,2_4,0_3,1_5),(\infty_3,0_3,\infty_4,2_4,0_2,2_5),$
$(\infty_1,0_6,\infty_2,2_7,0_1,2_8),(\infty_3,0_0,\infty_4,2_1,1_3,2_2),$
$(\infty_i,0_0,\infty_{i+1},0_1,\infty_{i+2},0_2),(\infty_1,1_0,\infty_2,2_1,0_0,1_2),$
$(\infty_3,0_6,\infty_4,2_7,0_2,2_8),(0_0,2_4,1_0,2_5,2_6,1_6),(0_0,1_1,1_0,0_2,0_1,0_3),$
$(0_0,0_7,0_1,1_7,0_2,0_8),(0_4,2_5,0_7,1_6,2_8,0_8),(0_0,2_7,0_3,1_8,0_4,2_8),$
$(0_1,2_2,1_1,0_4,0_2,0_5),(0_1,1_3,0_2,1_4,0_6,1_5),(0_5,2_7,2_6,1_8,0_7,2_8),$
$(0_1,1_6,0_2,2_6,1_3,1_8),(0_3,1_4,1_3,0_5,0_4,1_7),(0_4,1_5,0_5,0_0,2_6,0_7),$
$(\infty_i,0_3,\infty_{i+1},0_4,\infty_{i+2},0_5),(\infty_i,0_6,\infty_{i+1},0_7,\infty_{i+2},0_8)$, mod $(3,-)$.

Here $i=5,8$ when $\omega=10$ and $i=5,8,11,14,17$ when $\omega=19$.

$G_4$-$ID(18,10)$: $Z_8\bigcup\{\infty_1,\infty_2,\cdots,\infty_{10}\}$

$(1,\infty_1,2,\infty_5,3,\infty_2),(4,\infty_1,5,\infty_7,6,\infty_2),(8,\infty_3,3,\infty_6,2,\infty_4),$
$(4,\infty_9,8,6,1,\infty_{10}),(8,\infty_7,7,\infty_{10},2,\infty_8),(5,\infty_5,8,\infty_2,7,\infty_6),$

$(6,\infty_5,4,\infty_3,1,\infty_6),(4,1,7,\infty_1,8,2),(7,\infty_3,5,\infty_8,6,\infty_4),$
$(3,\infty_7,4,\infty_4,1,\infty_8),(6,\infty_9,3,1,5,\infty_{10}),(5,3,2,\infty_9,7,6).$

$G_4$-$ID(18,9)$: $Z_9\bigcup\{\infty_1,\infty_2,\cdots,\infty_9\}$
$(\infty_9,1,4,2,3,7),(1,\infty_7,2,\infty_9,3,\infty_8),(5,\infty_3,7,6,4,\infty_4),$
$(4,\infty_5,9,8,3,\infty_6),(1,\infty_5,7,2,8,\infty_6),(1,\infty_1,5,6,2,\infty_2),$
$(3,\infty_1,4,8,6,\infty_2),(7,\infty_1,9,1,8,\infty_2),(1,\infty_3,6,9,2,\infty_4),$
$(3,\infty_3,9,5,8,\infty_4),(7,\infty_7,8,\infty_9,9,\infty_8),$
$(2,\infty_5,5,3,6,\infty_6),(4,\infty_7,5,\infty_9,6,\infty_8).$

$G_4$-$ID(27,18)$: $Z_9\bigcup\{\infty_1,\infty_2,\cdots,\infty_{18}\}$
$(3,\infty_{18},8,4,7,1),(3,\infty_{11},6,\infty_{18},4,\infty_{12}),(7,\infty_{11},0,\infty_{18},5,\infty_{12}),$
$(2,\infty_1,6,\infty_{13},1,\infty_2),(3,\infty_1,7,\infty_{13},4,\infty_2),(5,\infty_1,0,\infty_{13},8,\infty_2),$
$(2,\infty_3,1,\infty_{14},0,\infty_4),(5,\infty_3,6,\infty_{14},8,\infty_4),(3,\infty_3,4,\infty_{14},7,\infty_4),$
$(0,\infty_5,4,\infty_{15},3,\infty_6),(1,\infty_5,5,\infty_{15},2,\infty_6),(7,\infty_5,8,\infty_{15},6,\infty_6),$
$(7,\infty_7,2,\infty_{16},3,\infty_8),(1,\infty_7,4,\infty_{16},5,\infty_8),(0,\infty_{15},1,6,7,\infty_{16}),$
$(2,\infty_9,3,\infty_{17},1,\infty_{10}),(5,\infty_9,7,\infty_{17},4,\infty_{10}),(2,\infty_{17},5,8,6,4),$
$(6,\infty_9,0,\infty_{17},8,\infty_{10}),(8,\infty_{11},2,\infty_{18},1,\infty_{12}),$
$(0,\infty_7,8,\infty_{16},6,\infty_8),(5,\infty_{13},3,0,2,\infty_{14}).$

$G_5$-$ID(18,9)$: $I_9\bigcup\{\infty_1,\infty_2,\cdots,\infty_9\}$
$(3,\infty_3,1,\infty_4,4,\infty_9),(6,\infty_3,2,\infty_4,8,\infty_9),(7,\infty_3,5,\infty_4,9,\infty_9),$
$(7,\infty_5,3,\infty_6,6,1),(8,\infty_7,1,\infty_8,4,2),(2,\infty_1,1,\infty_2,3,\infty_9),$
$(5,\infty_1,4,\infty_2,6,\infty_9),(4,\infty_5,1,\infty_6,2,7),(8,\infty_1,7,\infty_2,9,\infty_9),$
$(3,\infty_7,5,\infty_8,6,8),(5,\infty_5,8,\infty_6,9,1),(2,\infty_7,7,\infty_8,9,5),$
$(9,\infty_9,1,6,4,3).$

$G_6$-$ID(27,9)$: $Z_9\times Z_3$
$(8_1,0_1,2_1,8_2,4_0,2_0)$ mod$(9,-)$;
$(8_2,4_1,6_0,3_0,0_1,3_2),(0_2,5_1,7_0,4_0,1_1,3_2),(1_2,2_1,6_1,8_0,5_0,0_0),$
$(1_0,4_2,6_0,7_2,2_1,5_2),(2_2,7_1,0_0,6_0,3_1,6_2),(3_2,8_1,1_0,7_0,4_1,7_2),$
$(4_2,0_1,2_0,8_0,5_1,8_2),(2_0,5_2,3_1,8_2,7_0,6_2),(5_2,1_1,3_0,0_0,6_1,0_2),$
$(6_2,2_1,4_0,1_0,7_1,1_2),(7_2,8_1,3_1,5_0,2_0,1_2),(3_0,4_1,0_2,8_0,6_2,5_0),$
$(1_0,3_1,1_2,0_1,0_0,1_1),(2_0,4_1,2_2,1_1,1_0,0_2),(3_0,2_1,2_0,5_1,3_2,0_0),$
$(4_0,7_2,5_1,1_2,0_0,6_2),(4_0,6_1,4_2,3_1,3_0,2_2),(5_0,7_1,5_2,4_1,4_0,3_2),$
$(6_0,8_1,6_2,5_1,5_0,4_2),(5_0,6_1,2_2,1_0,8_2,0_0),(7_0,0_1,7_2,6_1,6_0,5_2),$
$(8_0,7_0,7_1,8_2,1_1,6_2),(0_0,2_1,0_2,8_1,8_0,7_2),(6_0,7_1,0_2,2_0,3_2,5_0),$
$(7_0,8_1,1_2,3_0,4_2,1_1),(8_0,0_1,5_2,4_0,2_2,8_1).$

$G_7$-$ID(18,9)$: $I_9\bigcup\{\infty_1,\cdots,\infty_9\}$
$(2,7,1,\infty_1,8,\infty_2),(4,6,3,\infty_2,9,\infty_1),(\infty_9,3,\infty_1,5,\infty_2,1),$
$(9,4,7,\infty_6,6,\infty_3),(1,9,2,\infty_4,3,\infty_5),(9,7,\infty_3,3,\infty_6,8),$
$(7,5,8,\infty_5,6,\infty_4),(\infty_9,8,\infty_4,4,\infty_5,2),(2,5,9,\infty_7,1,\infty_8),$
$(3,8,6,\infty_8,4,\infty_7),(\infty_9,6,\infty_7,7,\infty_8,9),(5,1,8,\infty_3,2,\infty_6),$
$(1,2,3,4,5,6).$

$G_8$-$ID(18,9)$: $I_9\bigcup\{\infty_1,\cdots,\infty_9\}$
$(2,\infty_1,8,9,\infty_2,3),(\infty_1,6,\infty_2,1,7,5),(\infty_2,1,\infty_1,\infty_9,4,5),$
$(1,\infty_5,7,9,\infty_6,4),(5,\infty_3,1,8,\infty_4,9),(\infty_3,6,\infty_4,9,7,4),$

$(\infty_4, 2, \infty_3, \infty_9, 3, 4), (\infty_5, 2, \infty_6, 5, 3, 8), (\infty_6, 5, \infty_5, \infty_9, 6, 8),$
$(6, \infty_7, 4, 5, \infty_8, 7), (\infty_7, 2, \infty_8, 6, 3, 1), (\infty_8, 8, \infty_7, 4, 9, 1),$
$(\infty_9, 7, 2, 3, 9, 8).$

$G_9$-$ID(18, 9)$: $I_9 \bigcup \{\infty_1, \cdots, \infty_9\}$
$(2, 7, \infty_5, \infty_6, 9, \infty_4), (1, 7, \infty_2, \infty_3, 6, \infty_1), (4, 6, \infty_8, \infty_9, 8, \infty_7),$
$(\infty_7, 5, \infty_8, \infty_9, 2, 6), (\infty_1, 9, \infty_2, \infty_3, 8, 7), (\infty_2, 3, \infty_1, \infty_3, 2, 1),$
$(\infty_3, 5, \infty_2, \infty_1, 4, 1), (\infty_8, 3, \infty_7, \infty_9, 7, 4), (\infty_4, 6, \infty_5, \infty_6, 5, 7),$
$(\infty_5, 3, \infty_4, \infty_6, 4, 2), (\infty_6, 1, \infty_4, \infty_5, 8, 2), (\infty_9, 1, \infty_7, \infty_8, 9, 4),$
$(9, 5, 1, 8, 3, 6).$

## 4.3 Conclusion

**Theorem 1.** *There exists a $G_k$-$GD(v)$ if and only if $v \equiv 0, 1 \pmod 9$ for $4 \le k \le 9$ or $v \equiv 1, 9 \pmod{18}$ for $k = 1$ or $v \equiv 1 \pmod 9$ for $k = 2, 3$ with the exceptions $(v, k) \in \{(9,4), (9,5), (9,6), (9,8), (9,9), (10,2)\}$.*

**Proof.** Obviously, the necessary conditions for the existence of a $G_k$-$GD(v)$ are $v \equiv 0, 1$ (mod 9) for $4 \le k \le 9$; $v \equiv 1, 9$ (mod 18) for $k = 1$; $v \equiv 1$ (mod 9) for $k = 2, 3$. Summarizing Lemma 1, 3, 6–13 and the constructions in §4.1, §4.2, the conclusions hold. ∎

**Acknowledgements.** The authors would like to thank the referees for their careful reading of the paper and helpful comments.

# References


[1] B. Alspach and H. Gavlas, *Cycle decompositions of $K_n$ and $K_n - I$*, Journal of Combinatorial Theory(B), 21(2000), 146-155.

[2] J. C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combinatoria, 10 (1980), 211-254.

[3] J. C. Bermond and J. Schönheim, *G-decomposition of $K_n$, where G has four vertices or less*, Discrete Math. 19(1977), 113-120.

[4] A. Blinco, *On diagonal cycle systems*, Australasian Journal of Combinatorics, 24(2001), 221-230.

[5] J. Bosak, *Decompositions of graphs*, Kluwer Academic Publishers, Boston, 1990.

[6] Y. Chang, *The spectra for two classes of graph designs*, Ars Combinatoria, 65(2002), 237-243.

[7] C. J. Colbourn and J. H. Dinitz(eds.), *The CRC Handbook of Combinatorial Designs,* (*CRC* Press, Boca Raton, 1996).

[8] G. Ge, *Existence of holey LSSOM of type $2^n$ with application to $G_7$-packing of $K_v$*, J. Statist. Plan. Infer. 94(2001), 211-218.

[9] F. Harary, *Graph Theory,* Addison-Wesley, Reading(1969), 274 pp.

[10] K. Heinrich, *Path-decomposition,* le Mathematics (Catania) XLVII (1992), 241-258.

[11] Q. Kang, Y. Du and Z. Tian, Decomposition of $\lambda K_v$ into some graph with six vertices and seven edges, to appear in J. Statist. Plan. Infer..

[12] Q. Kang, Y. Du and Z. Tian, *Decomposition of complete graph into isomorphic subgraphs with six vertices and seven edges,* to appear.

[13] Q. Kang and Z. Wang, *Optimal packings and coverings of $\lambda K_v$ with graph $K_5 - P_5$*, Bull. Inst. Combin. Appl., 41(2004), 22-41.

[14] Q. Kang and Z. Wang, *Maximum $K_{2,3}$-packing and minimum $K_{2,3}$-covering designs of $\lambda K_v$*, to appear in Mathematical Research and Advances.

[15] Q. Kang, L. Yuan and S. Liu, *Graph designs for all graphs with six vertices and eight edges,* submitted to Acta Mathematicae Applicatae Sinica.

[16] Q. Kang, H. Zuo and Y. Zhang, *Decompositions of $\lambda K_v$ into $k$-circuits with one chord,* Australasian Journal of Combinatorics, 30(2004), 229-246.

[17] A. Kotzig, *Decomposition of complete graphs into isomorphic cubes,* J. Combin. Theory B, 31(1981), 292-296.

[18] C. A. Rodger, *Graph-decompositions,* Le Mathematics (Catania) XLV(1990), 119-140.

[19] J. Yin and B. Gong, *Existence of $G$-designs with $|V(G)| = 6$*, in: W. D. Wallis et al.(eds), Combinatorial Designs and Applications (Marcel Dekker, New York, 1990), 201-218.