A Note on the Relationships Between the Generalized Bernoulli and Euler Polynomials*

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Abstract

In this article, we study the generalized Bernoulli and Euler polynomials, and obtain relationships between them, based upon the technique of matrix representation.

Keywords: Generalized Bernoulli polynomials; Generalized Euler polynomials

1. Introduction

In many contexts, matrix representation of a particular counting sequence is considered, and such a representation provides a powerful computational tool for deriving identities and explicit formulas related to the sequence, see, for example [3, 7].

Recently, Cheon [2] rederived several known properties and relationships involving the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, by making use of some fairly standard techniques based upon series rearrangement as well as matrix representation. The main consequence obtained in [2] is the following equation:

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x) . \tag{1}$$

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Srivastava and Pintér [8] followed Cheon's work [2] and showed that the main relationship (1) proven in [2] is essentially the same as some results known before (see [8], p.376, for details). Moreover, they studied the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ as well as the generalized Euler polynomials $E_n^{(\alpha)}(x)$, and established some relationships between them.

The object of the present sequel to their work is to show some relationships between the generalized Bernoulli and Euler polynomials with matrix representation.

This paper will be organized as follows: the Bernoulli and Euler polynomials will be introduced briefly, and the corresponding matrix equations will be constructed in Section 2. And, in Section 3, we will establish two relationships between the generalized Bernoulli and Euler polynomials, which have probably not been realized before. These two relationships are

$$B_n^{(\alpha)}(x) = \frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\sum_{m=0}^{\beta}\binom{\beta}{m}m^l\binom{n-k}{l}B_{n-k-l}^{(\alpha)}(0)]E_k^{(\beta)}(x),$$

$$E_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{l+\beta}}{\prod_{m=1}^{\beta}(l+m)}E_{n-k-l}^{(\alpha)}(0)]B_k^{(\beta)}(x).$$

We will also show that some results presented before are just the special cases of the above equations.

2. Matrix equations of the generalized Bernoulli and Euler polynomials

Let $\mathbb{N}_0 = \{0,1,2,\cdots\}$ and $\mathbb{N} = \{1,2,3,\cdots\}$. The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are defined by the following exponential generating functions:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t|<2\pi), \qquad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t|<\pi).$$

And we know that there are explicit formulas for $B_n(x)$ and $E_n(x)$, respectively,

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}, \tag{2}$$

$$E_n(x) = \frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\binom{n+1}{k}B_k x^{n+1-k}, \tag{3}$$

where $B_k := B_k(0)$ is the Bernoulli number for each $k \in \mathbb{N}_0$. Besides, $B_n(x)$ and $E_n(x)$ satisfy the following equations:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n, \quad (n \in \mathbb{N}_0), \tag{4}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \quad (n \in \mathbb{N}_0). \tag{5}$$

For a real or complex parameter α, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by the following generating functions:

$$(\frac{t}{e^t-1})^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi, 1^\alpha := 1, \tag{6}$$

$$(\frac{2}{e^t+1})^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad |t| < \pi, \ 1^\alpha := 1. \tag{7}$$

It is obvious that

$$B_n^{(1)}(x) = B_n(x), \quad E_n^{(1)}(x) = E_n(x), \quad (n \in \mathbb{N}_0), \tag{8}$$

$$B_n^{(0)}(x) = E_n^{(0)}(x) = x^n, \quad (n \in \mathbb{N}_0). \tag{9}$$

Besides, $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ have the following properties (cf., [8], p.378):

$$B_n^{(\alpha-1)}(x) = \frac{1}{n+1}\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(\alpha)}(x), \quad (n \in \mathbb{N}_0), \tag{10}$$

$$E_n^{(\alpha-1)}(x) = \frac{1}{2}[E_n^{(\alpha)}(x) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)}(x)], \quad (n \in \mathbb{N}_0). \tag{11}$$

We now construct the corresponding matrix representations of (10) and (11).

Consider the case of $\alpha, \beta \in \mathbb{N}_0$ and let $B^{(\alpha)}(x), E^{(\beta)}(x)$ and $X(x)$ be the $(n+1) \times 1$ matrices defined by

$$B^{(\alpha)}(x) = \begin{pmatrix} B_0^{(\alpha)}(x) \\ B_1^{(\alpha)}(x) \\ \vdots \\ B_n^{(\alpha)}(x) \end{pmatrix}, \quad E^{(\beta)}(x) = \begin{pmatrix} E_0^{(\beta)}(x) \\ E_1^{(\beta)}(x) \\ \vdots \\ E_n^{(\beta)}(x) \end{pmatrix}, \quad X(x) = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix},$$

and let P_{n+1}, Q_{n+1} be the $(n+1)\times(n+1)$ lower triangular matrices defined by

$$[P_{n+1}]_{ij} = \begin{cases} \binom{i-1}{j-1}, & if\ i \geq j, \\ 0, & else, \end{cases} \qquad [Q_{n+1}]_{ij} = \begin{cases} \frac{1}{i}\binom{i}{j-1}, & if\ i \geq j, \\ 0, & else. \end{cases}$$

Then (10) and (11) can be expressed as

$$Q_{n+1}B^{(\alpha)}(x) = B^{(\alpha-1)}(x), \quad \frac{1}{2}(P_{n+1} + I_{n+1})E^{(\alpha)}(x) = E^{(\alpha-1)}(x), \quad (12)$$

where I_{n+1} is the identity matrix of order $n+1$.

Since $\alpha \in \mathbb{N}_0$, then

$$B^{(\alpha)}(x) = Q_{n+1}{}^{-1}B^{(\alpha-1)}(x) = (Q_{n+1}{}^{-1})^2 B^{(\alpha-2)}(x) = \cdots$$
$$= (Q_{n+1}{}^{-1})^\alpha B^{(0)}(x) = (Q_{n+1}{}^\alpha)^{-1}X(x), \quad (13)$$
$$Q_{n+1}{}^\alpha B^{(\alpha)}(x) = X(x). \quad (14)$$

Analogously, for the matrix $E^{(\beta)}(x)$, we have

$$E^{(\beta)}(x) = 2^\beta[(P_{n+1} + I_{n+1})^\beta]^{-1}X(x), \quad (15)$$
$$\frac{1}{2^\beta}(P_{n+1} + I_{n+1})^\beta E^{(\beta)}(x) = X(x). \quad (16)$$

By setting $\alpha = 1, \beta = 1$ in (13) and (15), we obtain the matrix equations of the classical Bernoulli and Euler polynomials, respectively:

$$B(x) = Q_{n+1}{}^{-1}X(x), \quad E(x) = 2(P_{n+1} + I_{n+1})^{-1}X(x). \quad (17)$$

In view of (2) and (17), we have

$$[Q_{n+1}{}^{-1}]_{ij} = \binom{i-1}{j-1}B_{i-j}, \quad i \geq j.$$

Moreover, since ([1], p.805, Entry(23.1.20); [6], p.29)

$$E_n(0) = \frac{2(1-2^{n+1})}{n+1}B_{n+1}, \quad n \in \mathbb{N},$$

then from (3), we have

$$E_{i-1}(x) = \sum_{j=1}^{i}\binom{i-1}{j-1}E_{i-j}(0)x^{j-1}.$$

With(17), we immediately deduce that

$$[(P_{n+1}+I_{n+1})^{-1}]_{ij}=\frac{1}{2}\binom{i-1}{j-1}E_{i-j}(0)\,,\quad i\geq j\,.$$

The matrix equations above will lead us to the relationships between the generalized Bernoulli and Euler polynomials.

3. Relationships between the generalized Bernoulli and Euler polynomials

Theorem 1. *The following relationship*

$$B_n^{(\alpha)}(x)=\frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\sum_{m=0}^{\beta}\binom{\beta}{m}m^l\binom{n-k}{l}B_{n-k-l}^{(\alpha)}(0)]E_k^{(\beta)}(x)$$
$$(\alpha\in\mathbb{C},\ \beta\in\mathbb{N}_0\,,\ n\in\mathbb{N}_0) \tag{18}$$

holds between the generalized Bernoulli and Euler polynomials.

Proof. We now determine the (i,j)-entry of

$$\begin{aligned}S_{n+1}:&=({Q_{n+1}}^{\alpha})^{-1}(\frac{1}{2}(P_{n+1}+I_{n+1}))^\beta\\&=\frac{1}{2^\beta}({Q_{n+1}}^{-1})^{\alpha}(P_{n+1}+I_{n+1})^\beta\,,\end{aligned} \tag{19}$$

for integers i,j with $i\geq j$, where $\alpha\,,\,\beta\in\mathbb{N}_0$.

With induction, we can show that

$$[({Q_{n+1}}^{-1})^{\alpha}]_{ij}=\binom{i-1}{j-1}B_{i-j}^{(\alpha)}(0)\,,\quad i\geq j\,, \tag{20}$$

$$[(P_{n+1}+I_{n+1})^\beta]_{ij}=\binom{i-1}{j-1}\sum_{m=1}^{\beta}\binom{\beta}{m}m^{i-j}+(I_{n+1})_{ij}\,,\quad i\geq j\,. \tag{21}$$

Then (19), (20) and (21) will deduce that

$$[S_{n+1}]_{ij}=\frac{1}{2^\beta}\binom{i-1}{j-1}\sum_{l=0}^{i-j}\binom{i-j}{l}B_{i-j-l}^{(\alpha)}(0)\sum_{m=1}^{\beta}\binom{\beta}{m}m^l+\frac{1}{2^\beta}\binom{i-1}{j-1}B_{i-j}^{(\alpha)}(0)\,.$$

Since (19) implies that

$$(\frac{1}{2}(P_{n+1}+I_{n+1}))^\beta={Q_{n+1}}^{\alpha}S_{n+1}\,,$$

then we have $B^{(\alpha)}(x) = S_{n+1}E^{(\beta)}(x)$ from (14) and (16), and we finally gain our ends with some computation:

$$B_n^{(\alpha)}(x) = \sum_{j=1}^{n+1} S_{n+1,j}E_{j-1}^{(\beta)}(x) = \sum_{k=0}^{n} S_{n+1,k+1}E_k^{(\beta)}(x)$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{n-k} \frac{1}{2^\beta}\binom{n}{k}\binom{n-k}{l}B_{n-k-l}^{(\alpha)}(0)\sum_{m=1}^{\beta}\binom{\beta}{m}m^l E_k^{(\beta)}(x)$$

$$+ \sum_{k=0}^{n}\frac{1}{2^\beta}\binom{n}{k}B_{n-k}^{(\alpha)}(0)E_k^{(\beta)}(x)$$

$$= \frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\sum_{m=0}^{\beta}\binom{\beta}{m}m^l\binom{n-k}{l}B_{n-k-l}^{(\alpha)}(0)]E_k^{(\beta)}(x).$$

We have just proved that for $\alpha, \beta \in \mathbb{N}_0$, (18) holds. Since for given $\beta \in \mathbb{N}_0$, both sides of (18) are polynomials in α, it follows that for arbitrary $\alpha \in \mathbb{C}$, (18) still holds.

Remark 1. By setting $\beta = 1$ in Theorem 1, we obtain the following special case:

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}[B_k^{(\alpha)} + \frac{k}{2}B_{k-1}^{(\alpha-1)}]E_{n-k}(x),$$

which yields an equation obtained in [8] (see [8], p.379, equation (35)), where $B_k^{(\alpha)}$ is the generalized Bernoulli number which satisfies $B_k^{(\alpha)} = B_k^{(\alpha)}(0)$. Further special cases of (18) when $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 1$ give us (5) and (1), respectively. Besides, by setting $\beta = 0$ in (18), we obtain the equation

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(\alpha)}x^k.$$

Alternatively, the assertion (18) of Theorem 1 leads us to the following (presumably new) relationships when $\alpha = 0$ and $\alpha = 1$:

$$x^n = \frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}[\sum_{m=0}^{\beta}\binom{\beta}{m}m^{n-k}]E_k^{(\beta)}(x),$$

$$B_n(x) = \frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\sum_{m=0}^{\beta}\binom{\beta}{m}m^l\binom{n-k}{l}B_{n-k-l}]E_k^{(\beta)}(x).$$

Finally, we will show that the generalized Euler polynomials can be expressed by the generalized Bernoulli polynomials too.

Theorem 2. *The following relationship*

$$E_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{l+\beta}}{\prod_{m=1}^{\beta}(l+m)}E_{n-k-l}^{(\alpha)}(0)]B_k^{(\beta)}(x)$$
$$(\alpha\in\mathbb{C},\ \beta\in\mathbb{N}_0\,,\ n\in\mathbb{N}_0) \qquad (22)$$

holds between the generalized Euler and Bernoulli polynomials.

Proof. It is analogous to Theorem 1. Let

$$\begin{aligned} T_{n+1} :&= 2^{\alpha}[(P_{n+1}+I_{n+1})^{\alpha}]^{-1}Q_{n+1}{}^{\beta} \\ &= 2^{\alpha}[(P_{n+1}+I_{n+1})^{-1}]^{\alpha}Q_{n+1}{}^{\beta}. \end{aligned} \qquad (23)$$

The (i,j)-entry of $[(P_{n+1}+I_{n+1})^{-1}]^{\alpha}$ is $\frac{1}{2^{\alpha}}\binom{i-1}{j-1}E_{i-j}^{(\alpha)}(0)$ for integers i and j with $i\geq j$. And, if we notice that ([5], p.2, equation(1.13)):

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j=0,\quad 0\leq j<n,$$

it will be not complex to prove with induction that

$$[Q_{n+1}{}^{\beta}]_{ij}=\frac{\sum_{m=1}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{i-j+\beta}}{\prod_{m=1}^{\beta}(i-j+m)}\binom{i-1}{j-1},\quad i\geq j\,,\beta\in\mathbb{N}. \qquad (24)$$

By using the convention $\prod_{m=1}^{0}(i-j+m)=1$, (24) could be modified as

$$[Q_{n+1}{}^{\beta}]_{ij}=\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{i-j+\beta}}{\prod_{m=1}^{\beta}(i-j+m)}\binom{i-1}{j-1},\quad i\geq j\,,\beta\in\mathbb{N}_0\,.$$

Then, we have

$$[T_{n+1}]_{ij}=\binom{i-1}{j-1}\sum_{k=j}^{i}\binom{i-j}{k-j}\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{k-j+\beta}}{\prod_{m=1}^{\beta}(k-j+m)}E_{i-k}^{(\alpha)}(0),\quad i\geq j\,.$$

Since we can deduce $E^{(\alpha)}(x)=T_{n+1}B^{(\beta)}(x)$ from (14), (16) and (23), then the final consequence can be easily obtained.

Remark 2. By setting $\beta = 1$ in (22) and using the equation $E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x) = 2E_n^{(\alpha-1)}(x)$, we finally have

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}[E_{k+1}^{(\alpha-1)}(0) - E_{k+1}^{(\alpha)}(0)]B_{n-k}(x). \qquad (25)$$

Further special cases of (22) when $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 1$ will return to (4) and the following equation:

$$E_n(x) = -\sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} E_{k+1}(0)B_{n-k}(x), \quad (n \in \mathbb{N}_0).$$

Besides, by setting $\beta = 0$ in (22), we obtain the equation

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(\alpha)}(0)x^k.$$

Alternatively, the special cases of (22) when $\alpha = 0$ and $\alpha = 1$ give us the following relationships, respectively:

$$x^n = \sum_{k=0}^{n} \binom{n}{k} [\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{n-k+\beta}}{\prod_{m=1}^{\beta}(n-k+m)}]B_k^{(\beta)}(x),$$

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}[\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{\sum_{m=0}^{\beta}(-1)^{\beta-m}\binom{\beta}{m}m^{l+\beta}}{\prod_{m=1}^{\beta}(l+m)}E_{n-k-l}(0)]B_k^{(\beta)}(x).$$

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