RECIPROCAL SUMS OF l-TH POWER OF GENERALIZED BINARY SEQUENCES WITH INDICES

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ABSTRACT. Recently in [5], the author considered certain reciprocal sums of general second order recurrence $\{W_n\}$. In this paper, we generalize the results of Xi and we give some new results for the reciprocal sums of l-th power of general second order recurrence $\{W_{kn}\}$ for arbitrary positive integer k.

1. INTRODUCTION

Let a, b, P and Q integers such that $PQ \neq 0$ and $P^2 - 4Q \neq 0$. Define the sequence $\{W_n\}$ as follows: for $n > 1$

$$W_n = PW_{n-1} - QW_{n-2} \tag{1.1}$$

where $W_0 = a, W_1 = b$. The sequence $\{W_n\}$ and its some properties have been studied by several authors. Horadam [3] gave the Binet form of $\{W_n\}$ as shown :

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $\alpha, \beta = \left(P \pm \sqrt{P^2 - 4Q}\right)/2,\ A = b - \beta A, B = b - \alpha a$. We denote W_n by $W_n\,(a, b; P, Q)$. As important special cases, denote $W_n(0, 1, P, Q)$ and $W_n(2, P, P, Q)$ by U_n and V_n, respectively.

Some authors have studied the both finite and infinite reciprocal sums of terms of certain sequences. In [1], the authors derived

$$\sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}} \text{ and } \sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k}\left(\sum_{n=1}^{k} \frac{W_{n+1}}{W_n} - k\alpha\right) \tag{1.2}$$

where $P > 0$ and k, m nonnegative integers. For the case $Q = -1$, the identities in (1.2) are obtained by Good [2]. Regarding taking l−th powers of terms in the sums, the author [5] generalized the results of [1, 2]. For

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example, he derived the following infinitive reciprocal sums:

$$\begin{aligned}&\sum_{n=k}^{\infty}\frac{Q^n}{W_n^l W_{n+m}^l}\sum_{i=0}^{l-1}[(W_{k+1}-W_k\beta)Q^{n-k}\alpha^m-(W_{k+1}-W_k\alpha)\\&\times\beta^{2(n-k)+m}]^{l-1-i}[(W_{k+1}-W_k\beta)Q^{n-k}\beta^m-(W_{k+1}-W_k\alpha)\beta^{2(n-k)+m}]^i\\=\ &\frac{(P^2-4Q)^{l/2}Q^k}{(\alpha^m-\beta^m)(W_{k+1}-W_k\beta)}\sum_{i=0}^{m-1}\frac{\beta^{li}}{W_{k+i}^l}.\end{aligned}$$

In [5], the authors gave the general results including the earlier results by taking l-th powers of terms in the reciprocal sums. In this paper, we generalize the result of [5] regarding reciprocal sums of l-th powers of the terms with indices.

2. The Main Results

In this section we consider both finite and infinite reciprocal sums of products of r-consecutive terms of the sequence $\{W_n\}$. Clearly we will consider the finite and infinite reciprocal sums of $\{W_{rn}\}_{n=0}^{\infty}$ for arbitrary positive integer r. For later use and the readers convenience, we have the following result from [4]:

Lemma 1. *Let W_n be the nth term of sequence $\{W_n\}$. Then for $n, r > 0$,*

$$W_{rn}=V_rW_{r(n-1)}-Q^rW_{r(n-2)} \tag{2.1}$$

where V_n and Q be as before.

For our purpose, we use the generating function of the sequence $\{W_{rn}\}$. We give our first result.

Theorem 1. *Let $P > 0$,*

$$\begin{aligned}&\sum_{n=k}^{t}\frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l}\sum_{i=0}^{l-1}\{[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\alpha^r\\&-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i}\times\\&[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\beta^r-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\}\\=\ &\frac{(\alpha^r-\beta^r)^{l-1}Q^{rk}}{(W_{r(k+1)}-W_{rk}\beta^r)}\left[\frac{1}{W_{rk}^l}-\frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^l}\right].\end{aligned} \tag{2.2}$$

Proof. For ease write and arbitrary $r > 0$, let $f(x)=\sum_{n=k}^{\infty}W_{rn}x^n$. Thus consider

$$f(x)-W_{rk}x^k-W_{r(k+1)}x^{k+1}=V_rx\left(f(x)-W_{rk}x^k\right)-Q^rx^2f(x).$$

Hence

$$f(x)=x^k\frac{W_{rk}+x(W_{r(k+1)}-(\alpha^r+\beta^r)W_{rk})}{1-(\alpha^r+\beta^r)x+(\alpha\beta)^rx^2}.$$

Since $1-(\alpha^r+\beta^r)x+(\alpha\beta)^r x^2=(1-\alpha^r x)(1-\beta^r x)$, $f(x)$ can be decomposed into partial fractions:

$$f(x)=\frac{x^k}{\alpha^r-\beta^r}\left(\frac{W_{r(k+1)}-\beta^r W_{rk}}{1-\alpha^r x}-\frac{W_{r(k+1)}-\alpha^r W_{rk}}{1-\beta^r x}\right).$$

From the coefficients of x^n in both sides above equations, we get

$$W_{rn}=\frac{W_{r(k+1)}-\beta^r W_{rk}}{\alpha^r-\beta^r}\alpha^{r(n-k)}-\frac{W_{r(k+1)}-\alpha^r W_{rk}}{\alpha^r-\beta^r}\beta^{r(n-k)}.$$

Let $T_{rn}=\frac{\beta^{r(n-k)}}{W_{rn}}$. If we compute the difference of two consecutive terms of $\{T_{nk}^l\}$, we obtain

$$\begin{aligned}
&T_{rn}^l-T_{r(n+1)}^l=\frac{(\beta^{r(n-k)}W_{r(n+1)})^l-(\beta^{r(n+1-k)}W_{rn})^l}{W_{rn}^l W_{r(n+1)}^l}\\
=\;&\frac{1}{W_{rn}^l W_{r(n+1)}^l(\alpha^r-\beta^r)^l}\{[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\alpha^r\\
&-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^l\\
&-[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\beta^r-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^l\}\\
=\;&\frac{(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}}{W_{rn}^l W_{r(n+1)}^l(\alpha^r-\beta^r)^{l-1}}\sum_{i=0}^{l-1}\{[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\alpha^r\\
&-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i}[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\beta^r-\\
&(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&T_{rn}^l-T_{r(n+1)}^l\\
=\;&\frac{(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}}{W_{rn}^l W_{r(n+1)}^l(\alpha^r-\beta^r)^{l-1}}\\
&\times\sum_{i=0}^{l-1}\{[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\alpha^r-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i}\\
&\times[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\beta^r-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\}. \qquad (2.3)
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\sum_{n=k}^{t}\frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l}\sum_{i=0}^{l-1}\{[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\alpha^r\\
&-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i}\\
&\times[(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n-k)}\beta^r-(W_{r(k+1)}-W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\}\\
=\;&\frac{(\alpha^r-\beta^r)^{l-1}Q^{rk}}{(W_{r(k+1)}-W_{rk}\beta^r)}\sum_{n=k}^{t}\left(T_{rn}^l-T_{r(n+1)}^l\right)\\
=\;&\frac{(\alpha^r-\beta^r)^{l-1}Q^{rk}}{(W_{r(k+1)}-W_{rk}\beta^r)}\left[\frac{1}{W_{rk}^l}-\frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^l}\right].
\end{aligned}$$

Thus the proof is complete.. □

For example, when $l = 1$ in (2.2), we have the following result for $P > 0$

$$\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+1)}} = \frac{Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)} \left(\frac{1}{W_{rk}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} \right).$$

The case $r = 1$ in the above result can be found in [5]. As a numerical example, if we take $W_n(0, 1, 1, -1) = F_n$, then

$$\sum_{n=1}^{t} \frac{1}{F_{2n}F_{2(n+1)}} = \beta^2 - \frac{\beta^{2t+2}}{F_{2(t+1)}}.$$

Theorem 2. *For $P > 0$,*

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}^{l}W_{r(n+1)}^{l}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\alpha^r \\ &-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i} \\ &\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^r - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{i}\} \\ = \; & \frac{(\alpha^r - \beta^r)^{l-1}Q^{rk}}{W_{rk}^{l}(W_{r(k+1)} - W_{rk}\beta^r)}. \end{aligned}$$

Proof. From Theorem 1,

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}^{l}W_{r(n+1)}^{l}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\alpha^r \\ &-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i} \\ &\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^r - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{i}\} \\ = \; & \frac{(\alpha^r - \beta^r)^{l-1}Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)} \left(\frac{1}{W_{rk}^{l}} - \lim_{t\to\infty} \frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^{l}} \right) \\ = \; & \frac{(\alpha^r - \beta^r)^{l-1}Q^{rk}}{W_{rk}^{l}(W_{r(k+1)} - W_{rk}\beta^r)}. \end{aligned}$$

Since

$$\left|\frac{\alpha}{\beta}\right| > 1 \text{ for } P > 0,$$

and

$$\begin{aligned} &\lim_{t\to\infty} T_{rt}^{l} = \lim_{t\to\infty} \left(\frac{\beta^{r(t-k)}}{W_{rt}} \right)^{l} \\ = \; & \lim_{t\to\infty} \left[\frac{W_{r(k+1)} - \beta^r W_{rk}}{\alpha^r - \beta^r} \left(\frac{\alpha}{\beta} \right)^{r(t-k)} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{\alpha^r - \beta^r} \right]^{-l} = 0. \quad (2.4) \end{aligned}$$

The proof is complete. □

If we take $l = 1$ in Theorem 2, we get

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+1)}} = \frac{Q^{rk}}{W_{rk}(W_{r(k+1)}-W_{rk}\beta^r)}.$$

Theorem 3. *For $P > 0$ and $m > p$,*

$$\sum_{n=k}^{t} \frac{Q^{r(n+p)}}{W^l_{r(n+p)}W^l_{r(n+m)}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i}$$
$$\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)} - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)}-W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left[\frac{\beta^{rli}}{W^l_{r(k+i)}} - \frac{\beta^{rl(t+i+1-k)}}{W^l_{r(t+i+1)}}\right]. \tag{2.5}$$

Proof. By (2.3), we write

$$\begin{aligned}
T^l_{r(n+p)} - T^l_{r(n+m)} &= \left(\frac{\beta^{r(n+p-k)}}{W_{r(n+p)}}\right)^l - \left(\frac{\beta^{r(n+m-k)}}{W_{r(n+m)}}\right)^l \\
&= \frac{(\beta^{r(n+p-k)}W_{r(n+m)})^l-(\beta^{r(n+m-k)}W_{r(n+p)})^l}{W^l_{r(n+p)}W^l_{r(n+m)}} \\
&= \frac{(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n+p-k)}U_{r(m-p)}}{W^l_{r(n+p)}W^l_{r(n+m)}U^l_r} \\
&\quad \times \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\
&\quad -(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\
&\quad \times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^{r(m-p)} \\
&\quad -(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}.
\end{aligned}$$

Thus

$$\begin{aligned}
T^l_{r(n+p)} - T^l_{r(n+m)} &= \frac{(W_{r(k+1)}-W_{rk}\beta^r)Q^{r(n+p-k)}U_{r(m-p)}}{W^l_{r(n+p)}W^l_{r(n+m)}U^l_r} \\
&\quad \times \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\
&\quad -(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\
&\quad [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^{r(m-p)} \\
&\quad -(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}
\end{aligned}$$

Hence we have

$$\sum_{n=k}^{t} \frac{Q^{r(n+p)}}{W^l_{r(n+p)} W^l_{r(n+m)}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i}$$
$$\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{n=k}^{t} (T^l_{r(n+p)} - T^l_{r(n+m)})$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{rli}}{W^l_{r(k+i)}} - \frac{\beta^{rl(t+i+1-k)}}{W^l_{r(t+i+1)}}\right).$$

So we have the conclusion. □

As an example, when $l = 1$ in Theorem 3, one can obtain

$$\sum_{n=k}^{t} \frac{Q^{r(n+p)}}{W_{r(n+p)}W_{r(n+m)}} = \frac{U_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}}\right).$$

Theorem 4. *For $P > 0$ and $m > p$,*

$$\sum_{n=k}^{\infty} \frac{Q^{r(n+p)}}{W^l_{r(n+p)} W^l_{r(n+m)}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i}$$
$$\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \frac{\beta^{rli}}{W^l_{r(k+i)}}.$$

Proof. Considering (2.4) and (2.5), we have

$$\sum_{n=k}^{t} \frac{Q^{r(n+p)}}{W^l_{r(n+p)} W^l_{r(n+m)}} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i}$$
$$\times[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)}$$
$$-(W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i\}$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{rli}}{W^l_{r(k+i)}} - \lim_{t\to\infty} \frac{\beta^{rl(t+i+1-k)}}{W^l_{r(t+i+1)}}\right)$$
$$= \frac{U^l_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \frac{\beta^{rli}}{W^l_{r(k+i)}}.$$

Thus the proof is complete. □

For example, by $W_n(0,1,1,-1) = F_n$, from Theorem 4, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}F_{2n+4}} = \frac{8-3\sqrt{5}}{9}.$$

Also for $W_n(2,1,1,-1) = L_n$, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n+2}^2 L_{2n+6}^2} \left(\frac{(35+21\sqrt{5})}{2} - \left(40 - 18\sqrt{5}\right) \beta^{4n} \right) \right) = \frac{3953}{31\,752} - \frac{5287}{95\,256}\sqrt{5}$$

and

$$\begin{aligned} \sum_{n=1}^{t} \frac{1}{L_{4(n+3)}L_{4(n+6)}} &= \left(\frac{1}{480}\sqrt{5} - \frac{16\,125\,092\,335}{3461\,453\,534\,098} \right) \\ &- \left(\frac{7}{1440}\sqrt{5} - \frac{1}{96} \right) \left(\frac{\beta^{4(t+3)}}{L_{4(t+4)}} + \frac{\beta^{4(t+4)}}{L_{4(t+5)}} + \frac{\beta^{4(t+5)}}{L_{4(t+6)}} \right) \end{aligned}$$

Also choosing by appropriate parameter, many special cases can be obtained.

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