A family of chromatically unique 5-bridge graphs*

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Abstract Let $P(G;\lambda)$ denote the chromatic polynomial of a graph G, expressed in the variable λ. Then G is said to be chromatically unique if G is isomorphic with H for any graph H such that $P(H;\lambda)=P(G;\lambda)$. The graph consisting of s edge-disjoint paths joining two vertices is called an s-bridge graph. In this paper, we provide a new family of chromatically unique 5-bridge graphs.
Keywords: generalized polygon tree; 5-bridge graphs; chromatically equivalent; chromatically unique.

1 Introduction

The graphs that we consider here are finite, undirected and without loops or multiple edges. Let $P(G;\lambda)$, or simply $P(G)$ denote the chromatic polynomial of a graph G. In this paper, $y=\lambda-1$. Two graphs G and H are said to be chromatically equivalent if $P(G)=P(H)$. A graph G is said to be chromatically unique if $P(G)=P(H)$ implies that H is isomorphic with G, denoted by $H\cong G$. Since the notion of chromatic uniqueness was first introduced in 1978 by Chao and Whitehead [1], various classes of chromatically unique graphs have been found successively (see [3], [7]).

A path and a cycle of length l will be denoted by P_l and C_l, respectively. The generalized θ-graph, denoted by $\theta(a,b,c)$, is a 2-connected graph with 3

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edge-disjoint paths P_a, P_b and P_c between a pair of vertices u and v of degree three, where $a \geq 1$, $b \geq 2$ and $c \geq 2$. A graph consisting of s edge-disjoint paths joining two vertices is called an s-bridge graph, which is denoted by $F(k_1, k_2, \cdots, k_s)$, where $k_1, k_2, \cdots, k_s$ are the lengths of s paths.

It was proved by Chao and Whitehead [1] that the cycle C_n is chromatically unique. Loerinc [2] proved that the generalized θ-graph is chromatically unique. That is to say, 2-bridge graphs and 3-bridge graphs are all chromatically unique. Xu et al. [6] gave the sufficient and necessary conditions for a 4-bridge graph to be chromatically unique. In this paper, we give a new family of chromatically unique 5-bridge graphs. For all other notation and terminology not explained here, we can refer to [4].

2 Preliminaries

In this section, we shall give some known results and definitions that will be used to prove our main theorem in section 3.

Definition1[4]. A 2-connected graph G is called a generalized polygon tree if it can be decomposed into cycle class $C = \{C_{i_1}, \cdots, C_{i_r}\}$, and there exist an overlapping process: $H_1 = C_{i_1}$, H_j is obtained from H_{j-1} and C_{i_j} by overlapping in path P_{i_j} where in each step of overlapping, the vertices on P_{i_j}, except end vertices, are with degree 2.

Clearly an s-bridge graph is a generalized polygon tree.

Definition2[4]. Let G be a generalized polygon tree, a pair (u, v) of nonadjacent vertices of G is called an intercourse pair if there are at least three internally disjoint $u-v$ paths in G. The intercourse number of G, $\sigma(G)$ is defined as the number of intercourse pairs of vertices in G.

Theorem1[4] Let G and H be graphs such that $P(G) = P(H)$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $g(G) = g(H)$ and the number of cycles of G and H with the length equal to their girth are equal. Moreover if they are both planar, then the interior regions number $r(G) = r(H)$, and if G is a generalized polygon tree, then H is also a generalized polygon tree and $\sigma(H) = \sigma(G)$.

Theorem2[5] Let H_1, H_2 be two graphs of order n and size m. Suppose H_1,

H_2 have the same girth g and that each has only one cycle of length g, if the lengths of the second shortest cycle of H_1 and H_2 are different, then H_1 is not chromatically equivalent to H_2.

Theorem3[6] Let G be an s-bridge graph, where $j_1, j_2, \cdots, j_s$ are the lengths of s paths, then the chromatic polynomial of G.

$$P(G)=\frac{y}{(y+1)^{s-1}}[\prod(y^{j_i}+(-1)^{j_i+1})+y^{s-1}\prod(y^{j_i-1}+(-1)^{j_i})]=\frac{y}{(y+1)^{s-1}}Q(G)$$

Now suppose that H is obtained from a t-bridge graph and s-t cycles by overlapping on edges, where $k_1, \cdots, k_t$ is the lengths of t paths of t-bridge graph, and $l_1, \cdots, l_{s-t}$ is the lengths of $s-t$ cycles, then the chromatic polynomial of H.

$$P(H)=\frac{y}{(y+1)^{s-1}}[\prod(y^{k_i}+(-1)^{k_i+1})+y^{t-1}\prod(y^{k_i-1}+(-1)^{k_i})]\prod(y^{l_i-1}+(-1)^{l_i})$$

$$=\frac{y}{(y+1)^{s-1}}Q(H)\cdot$$

3 Main Results

Theorem. A 5-bridge graph $F(k_1,k_2,k_3,k_4,k_5)$ is chromatically unique if the positive integers k_1,k_2,k_3,k_4,k_5 assume exactly two distinct values, i.e. $|\{k_1,k_2,k_3,k_4,k_5\}|=2$, ,and $\min\{k_1,k_2,k_3,k_4,k_5\}\geq 2$.

Proof. Our proof is divided into four cases which are considered in the following four lemmas:

Lemma 1. 5-bridge graph $G=F(a,b,b,b,b)$ is chromatically unique for all $a\geq 2,\ b\geq a+1$.

Lemma 2. 5-bridge graph $G=F(a,a,b,b,b)$ is chromatically unique for all $a\geq 2,\ b\geq a+1$.

Lemma 3. 5-bridge graph $G=F(a,a,a,b,b)$ is chromatically unique for all $a\geq 2,\ b\geq a+1$.

Lemma 4. 5-bridge graph $G=F(a,a,a,a,b)$ is chromatically unique for all $a\geq 2,\ b\geq a+1$.

In the following, we will prove all four of these lemmas.

If G is 5-bridge graph $F(k_1,k_2,k_3,k_4,k_5)$, we assume that H is chromatically equivalent to G. By Theorem1, we know that H is also a generalized polygon tree and the interior region number $r(H)=r(G)=4$, the intercourse number $\sigma(H)=\sigma(G)=1$, i.e. H is either a 5-bridge graph or a graph obtained from a 4-bridge graph and a cycle by overlapping on an edge or a graph obtained from a generalized θ-graph and two cycles by overlapping on edges.

By Theorem3, suppose H is obtained from a 4-bridge $F(a_1, a_2, a_3, a_4)$ and a cycle C_{b_1} by overlapping on an edge, then

$$P(H)=\frac{y}{(y+1)^4}[\prod(y^{a_i}+(-1)^{a_i+1})+y^3\prod(y^{a_i-1}+(-1)^{a_i})](y^{b_1-1}+(-1)^{b_1}) \quad (1)$$

Let $P(H)=\frac{y}{(y+1)^4}Q(H)$, then

$$\begin{aligned}Q(H)=[&y^{a_1+a_2+a_3+a_4}+y^{a_1+a_2+a_3+a_4-1}+(y+1)((-1)^{a_1+a_2}y^{a_3+a_4}+(-1)^{a_1+a_3}y^{a_2+a_4}+\\&(-1)^{a_1+a_4}y^{a_2+a_3}+(-1)^{a_2+a_3}y^{a_1+a_4}+(-1)^{a_2+a_4}y^{a_1+a_3}+(-1)^{a_3+a_4}y^{a_1+a_2})+\\&(y^2-1)((-1)^{a_2+a_3+a_4}y^{a_1}+(-1)^{a_1+a_3+a_4}y^{a_2}+(-1)^{a_1+a_2+a_4}y^{a_3}+(-1)^{a_1+a_2+a_3}y^{a_4})\\&+(-1)^{a_1+a_2+a_3+a_4}(y^3+1)](y^{b_1-1}+(-1)^{b_1})\end{aligned}$$

Suppose H is obtained from a generalized θ-graph $\theta(a_1, a_2, a_3)$ and two cycles C_{b_1}, C_{b_2} by overlapping on edges, then

$$P(H)=\frac{y}{(y+1)^4}[\prod(y^{a_i}+(-1)^{a_i+1})+y^2\prod(y^{a_i-1}+(-1)^{a_i})]\prod(y^{b_i-1}+(-1)^{b_i}) \quad (2)$$

We let $P(H)=\frac{y}{(y+1)^4}Q(H)$, then

$$\begin{aligned}Q(H)=&(y+1)y^{a_1+a_2+a_3+b_1+b_2-3}+(-1)^{a_1+a_2}(y+1)y^{a_3+b_1+b_2-2}+(-1)^{a_1+a_3}(y+1)y^{a_2+b_1+b_2-2}+\\&(-1)^{a_2+a_3}(y+1)y^{a_1+b_1+b_2-2}+(-1)^{b_2}(y+1)y^{a_1+a_2+a_3+b_1-2}+(-1)^{b_1}(y+1)y^{a_1+a_2+a_3+b_2-2}\\&+(y+1)y^{b_1-1}[(-1)^{a_1+a_2+b_2}y^{a_3}+(-1)^{a_1+a_3+b_2}y^{a_2}+(-1)^{a_2+a_3+b_2}y^{a_1}]+(y+1)y^{b_2-1}\\&[(-1)^{a_1+a_2+b_1}y^{a_3}+(-1)^{a_1+a_3+b_1}y^{a_2}+(-1)^{a_2+a_3+b_1}y^{a_1}]+(-1)^{a_1+a_2+a_3}(y^2-1)y^{b_1+b_2-2}\\&+(-1)^{a_1+a_2+a_3+b_2}(y^2-1)y^{b_1-1}+(-1)^{a_1+a_2+a_3+b_1}(y^2-1)y^{b_2-1}+(-1)^{b_1+b_2}(y+1)\\&[y^{a_1+a_2+a_3-1}+(-1)^{a_2+a_3}y^{a_1}+(-1)^{a_1+a_3}y^{a_2}+(-1)^{a_1+a_2}y^{a_3}]+(-1)^{a_1+a_2+a_3+b_1+b_2}(y^2-1)\end{aligned}$$

Proof of Lemma 1. Let $G=F(a,b,b,b,b)$, where $a\geq 2$, $b\geq a+1$, by Theorem 3

$$P(G)=\frac{y}{(y+1)^4}[(y^a+(-1)^{a+1})(y^b+(-1)^{b+1})^4+y^4(y^{a-1}+(-1)^a)(y^{b-1}+(-1)^b)^4]$$

We let $P(G)=\frac{y}{(y+1)^4}Q(G)$, then

$$Q(G)=(y+1)y^{a+4b-1}+6y^{a+2b}(y+1)+(-1)^b 4y^{a+b}(y^2-1)+y^a(y^3+1)+$$
$$(-1)^{a+b}4y^{3b}(y+1)+(-1)^a 6y^{2b}(y^2-1)+(-1)^{a+b}4y^b(y^3+1)+(-1)^a(y^4-1)$$

Now suppose H is chromatically equivalent to G, there are the following three cases about H to be considered.

Case1.1: H is 5-bridge graph $F(a_1,a_2,a_3,a_4,a_5)$, where $a_1\ge a_2\ge a_3\ge a_4\ge a_5$.

$$P(H)=\frac{y}{(y+1)^4}[\Pi(y^{a_i}+(-1)^{a_i+1})+y^4\,\Pi(y^{a_i-1}+(-1)^{a_i})]=\frac{y}{(y+1)^4}Q(H)$$

Since $P(H)=P(G)$, we have $Q(H)=Q(G)$. Now we solve the equation $Q(H)=Q(G)$. By $|V(G)|=|V(H)|$, we have $a+4b=a_1+a_2+a_3+a_4+a_5$. After canceling y^4 and constant terms, it is easy to see that the lowest power term in $Q(G)-(-1)^a(y^4-1)$ is y^a, which cannot be canceled with the other terms in $Q(G)-(-1)^a(y^4-1)$. The lowest power term in $Q(H)-(-1)^{\sum a_i}y^4-(-1)^{\sum a_i+1}$ is $(-1)^{\sum_{i\ne 5} a_i}y^{a_5}$, which cannot be canceled by the other terms in $Q(H)-(-1)^{\sum a_i}y^4-(-1)^{\sum a_i+1}$ either. For polynomials to be equal, the coefficients of corresponding power terms must be equal. Hence $a_5=a$. We have known that the girths of both G and H are $g(H)=g(G)=a+b$, the number of cycles whose lengths are equal to the girth is 4, i.e. $C_g(H)=C_g(G)=4$. By $a_5=a$, we know that $g(H)=a+b$ implies that there is at least one among a_1,a_2,a_3,a_4 is b. So we can let $a_4=b$. $C_g(H)=4$ implies either $a_1=a_2=a_3=a$ or $a_1=a_2=a_3=b$. If $a_1=a_2=a_3=a$, we get $g(H)=2a<a+b$, which contradicts $g(H)=a+b$. Therefore $a_1=a_2=a_3=b$, i.e. $H=F(a,b,b,b,b)$.

Case1.2: H is obtained from a 4-bridge graph $F(a_1,a_2,a_3,a_4)$ and a cycle C_{b_1} by overlapping on an edge, where $a_1\ge a_2\ge a_3\ge a_4$. $P(H)$ is given by (1). By $|V(G)|=|V(H)|$, we have $a+4b=\sum a_i+b_1-1$. $g(H)=g(G)=a+b$ implies $b_1\ge a+b$. Obviously the lowest power term in $Q(G)-(-1)^{a+1}$ is y^a or $(-1)^a y^4$, and no cancellation is possible between them. It is also easy to see that the lowest power term occurring in $Q(H)-(-1)^{\sum a_i+b_1}$ is one of $(-1)^{\sum_{i\ne 4} a_i+b_1+1}y^{a_4}$, $(-1)^{\sum a_i+b_1}y^3$ and $(-1)^{\sum a_i}y^{b_1-1}$, which cannot be cancelled by the other terms in $Q(H)-(-1)^{\sum a_i+b_1}$ either, so $\min\{a_4,3,b_1-1\}=\min\{a,4\}$. Since $\min\{a_4,3,b_1-1\}\le 3$, we have $\min\{a,4\}=a\le 3$.

In the following, we will verify that both $b_1 - 1 = a$ and $\min\{a_4, 3, b_1 - 1\} = a_4$ are impossibilities.

If $b_1 - 1 = a$, then $b_1 = a + 1$, which contradicts $b_1 \geq a + b$.

If $\min\{a_4, 3, b_1 - 1\} = a_4$, then $a_4 = a < 3$. As we know $a \geq 2$, so we reach the conclusion $a_4 = a = 2$, $a_1 \geq a_2 \geq a_3 \geq 3$. After canceling the lowest power terms occurring in both $Q(G)$ and $Q(H)$, the lowest power terms occurring in $Q(H)$ are the several y^3 terms, and they cannot be cancelled in $Q(H)$ itself. Otherwise, the lowest power term occurring in $Q(G)$ is $(-1)^{a+b} 4y^b$ or $(-1)^a y^4$. For polynomials to be equal, the coefficients of corresponding powers of y must be equal. Thus we only have $a_1 = a_2 = a_3 = b = 3$. It is noted that we can get $b_1 = 4 < g(G)$ by letting $a = a_4 = 2$, $a_1 = a_2 = a_3 = b = 3$ in $a + 4b = \sum a_i + b_1 - 1$, which is a contradiction.

By the two cases above, we must have $a=3$. Clearly, the coefficient of y^3 in $Q(G)$ is 1, so $a_4 \geq 4$, $b_1 \geq 3 + b \geq 7$. After canceling equal terms in both $Q(G)$ and $Q(H)$, we note that the lowest power term in $Q(G)$ is $(-1)^{a+b} 4y^b$ or $(-1)^a y^4$ which cannot be cancelled by each other, and in $Q(H)$ is $(-1)^{\sum_{i \neq k} a_i + b_1 + 1} y^{a_k}$ or $(-1)^{\sum a_i} y^{b_1 - 1}$ which also cannot be cancelled by each other. Since $b_1 \geq a + b$, $b_1 - 1 \geq b + (a-1) \geq b + 1$, hence the lowest power term in $Q(H)$ is one term or several terms or all terms belonging to $(-1)^{\sum_{i \neq k} a_i + b_1 + 1} y^{a_k}$. On the other hand, we have known $a = 3$, $b \geq a + 1$. Therefore $b \geq 4$.

If $b = 4$, then the lowest power term in $Q(G)$ is $-5y^4$, however for $Q(H)$, even if every a_i is equal to 4, the coefficient of y^4 cannot be more than 4, hence $b \geq 5$. The lowest power term in $Q(G)$ is $(-1)^{a+4b} y^4$, so there is only one in a_1, a_2, a_3 and a_4 equaling to 4, the three others are all more than 4. Without loss of generality we may assume that $a_4 = 4, a_1 \geq a_2 \geq a_3 \geq 5$. By $a + 4b = \sum a_i + b_1 - 1$, $a = 3$ and $a_4 = 4$, we can get $4b = a_1 + a_2 + a_3 + b_1$ easily. After letting $a = 3$, $a_4 = 4$ both in $Q(G)$ and $Q(H)$, and canceling all equal terms, we can find that the lowest power term in $Q(G)$ is $(-1)^{a+b} 4y^b$ or y^6, and in $Q(H)$ is $(-1)^{\sum_{i \neq k} a_i + b_1 + 1} y^{a_k}$ $(k = 1,2,3)$ or $(-1)^{\sum a_i} y^{b_1 - 1}$ or $(-1)^{\sum_{i \neq 4} a_i + b_1} y^{a_4 + 2}$. Note that $b_1 - 1 \geq b + 1$, $a_4 + 2 = 6$, so if $b = 5$, the lowest power term in $Q(G)$ is $4y^5$, however the plus of $(-1)^{\sum_{i \neq k} a_i + b_1 + 1} y^{a_k}$ $(k = 1,2,3)$ in $Q(H)$ is no more than $3y^5$, therefore, we have $b \geq 6$. Since $4b = a_1 + a_2 + a_3 + b_1$, thus we have $y^6 = (-1)^{\sum_{i \neq 4} a_i + b_1} y^{a_4 + 2}$. The lowest power term in $Q(G)$ is $(-1)^{a+b}4y^b$, and in $Q(H)$ is $(-1)^{\sum_{i=1} a_i+b_1+1} y^{a_k}$ $(k=1,2,3)$ or the plus of them. But we can find no matter what b is equal to, the lowest power terms in $Q(G)$ and $Q(H)$ are not equal. Hence $Q(G)\neq Q(H)$, i.e. H is not chromatically equivalent to G.

Case1.3 H is obtained from a generalized θ-graph $\theta(a_1,a_2,a_3)$ and two cycles C_{b_1},C_{b_2} by overlapping on edges, where $a_1\geq a_2\geq a_3\geq 2$. By $g(H)=g(G)=a+b$, we can assume $b_1\geq b_2\geq a+b$. It is noted that $P(H)$ has been presented by (2). By $|V(G)|=|V(H)|$, we have $a+4b=\sum a_i+\sum b_i-2$. At the same time, we can easily find that the lowest power term in $Q(G)-(-1)^{a+1}$ is y^a or $(-1)^a y^4$, which can not be cancelled by each other, and the lowest power term in $Q(H)-(-1)^{\sum a_i+\sum b_i+1}$ is $(-1)^{a_1+a_2+b_1+b_2}y^{a_3}$ or $(-1)^{\sum a_i+\sum b_i}y^2$, which also can not be cancelled by each other. But if $a_3=2$, the coefficient of y^2 occurring in $Q(H)$ is no more than 2, and occurring in $Q(G)$ is no more than 1. So $a_3\geq 3$, and there must be $y^a=(-1)^{\sum a_i+\sum b_i}y^2$, i.e. $a=2$. After canceling equal terms of y^2 occurring in $Q(G)$ and $Q(H)$, the lowest power term in $Q(G)$ is $(-1)^b 4y^b$ or y^4, and in $Q(H)$ is one term or the plus of at least two terms of $(-1)^{\sum_{i=1} a_i+\sum b_i} y^{a_k}$. Clearly, $b>4$. Because if $b\leq 4$, the coefficients of y^b occurring in $Q(G)$ and $Q(H)$ are not equal. So $a_3=4, a_1\geq a_2\geq 5$. Now we let $a_3=4$ in $Q(H)$, and cancel all equal terms in $Q(G)$ and $Q(H)$. We find that the lowest power term in $Q(G)$ is $(-1)^b 4y^b$, which can not be cancelled by the other terms in $Q(G)$. Since $b_i\geq a+b=2+b$, i.e. $b_i-1\geq b+1$, it is easily seen that the absolute value of the coefficient of the lowest power term in $Q(H)$ is no more than 3. Therefore, $Q(G)\neq Q(H)$, i.e. H is not chromatically equivalent with G.

By considering the three cases about H above, we obtain that 5-bridge graph $G=F(a,b,b,b,b)$ is chromatically unique.

Proof of Lemma 2. Let $G=F(a,a,b,b,b)$, where $a\geq 2, b\geq a+1$, similar with the Proof of Lemma 1, by the Theorem3

$$P(G)=\frac{y}{(y+1)^4}[(y^a+(-1)^{a+1})^2(y^b+(-1)^{b+1})^3+y^4(y^{a-1}+(-1)^a)^2(y^{b-1}+(-1)^b)^3]$$

we let $P(G)=\frac{y}{(y+1)^4}Q(G)$, then

$$Q(G)=y^{2a+3b-1}(y+1)+3y^{2a+b}(y+1)+(-1)^{a+b}6y^{a+2b}(y+1)+(-1)^{b}y^{2a}(y^2-1)$$
$$+(-1)^{a}6y^{a+b}(y^2-1)+(-1)^{a+b}2y^{a}(y^3+1)+y^{3b}(y+1)+3y^{b}(y^3+1)$$
$$+(-1)^{b}3y^{2b}(y^2-1)+(-1)^{b}(y^4-1)$$

Now suppose H is chromatically equivalent to G, there are the following three cases about H to be considered.

Case2.1 $H=F(a_1,a_2,a_3,a_4,a_5)$, $a_1\ge a_2\ge a_3\ge a_4\ge a_5$,

$$P(H)=\frac{y}{(y+1)^4}\left[\prod(y^{a_i}+(-1)^{a_i+1})+y^4\prod(y^{a_i-1}+(-1)^{a_i})\right]=\frac{y}{(y+1)^4}Q(H)$$

That H is chromatically equivalent with G implies $|V(G)|=|V(H)|$, i.e. $2a+3b=\sum a_i$. In the following, we analyze $Q(G)$ and $Q(H)$. The lowest power term in $Q(G)-(-1)^b(y^4-1)$ is $(-1)^{a+b}2y^a$, which cannot be cancelled by the other terms in $Q(G)$. The lowest power term in $Q(H)-(-1)^{\sum a_i}y^4-(-1)^{\sum a_i+1}$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i}y^{a_k}$ $(k=1,2,3,4,5)$. For polynomials to be equal, the coefficients of corresponding power terms must be equal. Hence $a_4=a_5=a$. After canceling all equal terms in both $Q(G)$ and $Q(H)$, We note that the lowest power term in $Q(G)$ is $3y^b$ which cannot be cancelled in $Q(G)$, and in $Q(H)$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i}y^{a_k}$ $(k=1,2,3)$ which also cannot be cancelled in $Q(H)$, thus we have $a_1=a_2=a_3=b$, i.e. $H=F(a,a,b,b,b)$.

Case2.2 H is obtained from $F(a_1,a_2,a_3,a_4)$ and a cycle C_{b_1} by overlapping on an edge, where $a_1\ge a_2\ge a_3\ge a_4$.

Because $g(H)=g(G)=2a$, so $b_1\ge 2a$. $P(H)$ is presented by (1). That H is chromatically equivalent with G implies $|V(H)|=|V(G)|$, i.e. $2a+3b+1=\sum a_i+b_1$.

Obviously the lowest power term in $Q(G)-(-1)^{b+1}$ is $(-1)^{a+b}2y^a$ or $(-1)^b y^4$, and no cancellation is possible between them. It is also easy to see that the lowest power term occurring in $Q(H)-(-1)^{\sum a_i+b_1}$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i+b_1+1}y^{a_k}$ $(k=1,2,3,4)$, $(-1)^{\sum a_i+b_1}y^3$ and $(-1)^{\sum a_i}y^{b_1-1}$, which cannot be cancelled by the other terms in $Q(H)-(-1)^{\sum a_i+b_1}$ either, so $\min\{a_k,3,b_1-1\}=\min\{a,4\}$. Since $b_1\ge 2a$, $b_1-1\ge 2a-1\ge a+1$, hence $\min\{a_k,3,\}=\min\{a,4\}=a\le 3$. Since in $Q(G)$ the coefficient of y^a is 2. For polynomials to be equal, the coefficients of corresponding powers of y must be equal. Thus $a=a_4=3$ or $a=a_3=a_4=2$.

If $a=a_4=3$, $a_i \geq 4$ $(i=1,2,3)$. As we know $g(H)=g(G)=2a$, so we have $b_1=2a$. After canceling the terms of y^3 both in $Q(G)$ and $Q(H)$, we can easily find that the lowest power term in $Q(G)$ are the several y^4 terms, and they cannot be cancelled in $Q(G)$ itself. Otherwise, the lowest power term occurring in $Q(H)$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i+b_1+1} y^{a_k}$ $(k=1,2,3)$. If $b=4$, then the lowest power term in $Q(G)$ is $4y^4$, but the coefficient of y^4 in $Q(H)$ is no more than 3, therefore $b\neq 4$, i.e. $b\geq 5$, and the lowest power term occurring in $Q(G)$ is $(-1)^b y^4$, thus $a_3=4$. Because $b\geq 5$, the length of the second shortest cycle of G is $a+b\geq 8$, otherwise, the length of the second shortest cycle of H is $a_3+a_4=7$, for both G and H have only one cycle of the shortest length; by the Theorem2, H is not chromatically equivalent with G. This means $a=a_4=3$ is impossible.

If $a=a_3=a_4=2$, because both G and H has only one cycle of the shortest length, so $b_1\geq 5$. Now letting $a=a_3=a_4=2$ in both $Q(G)$ and $Q(H)$, after canceling all equal terms, i.e. the terms of y^2 and constant terms, the lowest power term in $Q(H)$ is the terms of y^3, but the lowest power term occurring in $Q(G)$ is $3y^b$. Thus $a_1=a_2=b=3$. By $2a+3b+1=\sum a_i+b_1$, we have $b_1=4$, which contradicts $b_1\geq 5$.

Therefore H is not chromatically equivalent with G.

Case2.3 H is obtained from a generalized θ-graph $\theta(a_1,a_2,a_3)$ and two cycles C_{b_1}, C_{b_2} by overlapping on edges, where $a_1\geq a_2\geq a_3\geq 2$. By $g(H)=g(G)=2a$, we can assume $b_1\geq b_2\geq 2a$. It is noted that $P(H)$ has been presented by (2). By $|V(G)|=|V(H)|$, we have $2a+3b=\sum a_i+\sum b_i-2$. At the same time, we can easily find that the lowest power term in $Q(G)-(-1)^{b+1}$ is $(-1)^{a+b}2y^a$ or $(-1)^b y^4$, which can not be cancelled by each other, and the lowest power term in $Q(H)-(-1)^{\sum a_i+\sum b_i+1}$ is $(-1)^{a_1+a_2+b_1+b_2} y^{a_3}$ or $(-1)^{\sum a_i+\sum b_i} y^2$, which also can not be cancelled by each other. For polynomials to be equal, the coefficients of corresponding powers of y must be equal. So the lowest power term in $Q(G)-(-1)^{b+1}$ is $(-1)^{a+b}2y^a$, and we must have $a=a_3=2$, $a_1\geq a_2\geq 3$. With this we have $g(H)=g(G)=4$. Because both G and H have only one cycle of the shortest length, therefore we only have $b_1=4$, $b_2>4$. Now the lowest power term in $Q(H)$ is $(-1)^{a_1+a_2+b_2+1}y^3$, but there are not the terms of y^3 in $Q(G)$. Thus $(-1)^{a_1+a_2+b_2+1}y^3$ must be cancelled in $Q(H)$ itself. Because $a_1 \geq a_2 \geq 3$, then it is only possible that $(-1)^{a_1+a_2+b_2+1}y^3$ can be cancelled by $(-1)^{a_1+a_3+b_1+b_2}y^{a_2}$, i.e. $(-1)^{a_1+b_2}y^{a_2}$. But if $a_2 = 3$, then $(-1)^{a_1+a_2+b_2+1}y^3 = (-1)^{a_1+b_2}y^3$, which is equal to $(-1)^{a_1+b_2}y^{a_2}$. That is to say, $(-1)^{a_1+a_2+b_2+1}y^3$ cannot be cancelled by $(-1)^{a_1+a_3+b_1+b_2}y^{a_2}$. Similarly $(-1)^{a_1+a_2+b_2+1}y^3$ cannot be cancelled by $(-1)^{a_{21}+a_3+b_1+b_2}y^{a_1}$ either. Therefore $Q(G) \neq Q(H)$, i.e. H is not chromatically equivalent with G.

By the three cases of H above, we know that 5-bridge graph $F(a,a,b,b,b)$ is chromatically unique, i.e. the lemma 2 is proven.

Proof of Lemma 3. Assume $G = F(a,a,a,b,b), a \geq 2, b \geq a+1$, ,similar with the Proofs of Lemma 1 and 2, by the Theorem3:

$$P(G) = \frac{y}{(y+1)^4}\left[(y^a+(-1)^{a+1})^3(y^b+(-1)^{b+1})^2 + y^4(y^{a-1}+(-1)^a)^3(y^{b-1}+(-1)^b)^2\right]$$
$$= \frac{y}{(y+1)^4}Q(G)$$

$$Q(G) = y^{3a+2b-1}(y+1) + (-1)^{a+b}6y^{2a+b}(y+1) + 3y^{a+2b}(y+1) + (-1)^a 3y^{2a}(y^2-1) + (-1)^{a+b}2y^b(y^3+1)$$
$$+ y^{3a}(y+1) + (-1)^b 6y^{a+b}(y^2-1) + (-1)^a y^{2b}(y^2-1) + 3y^a(y^3+1) + +(-1)^a(y^4-1)$$

Now assume that H is chromatically equivalent to G. In the following, we will consider all possible cases about H.

Case3.1 $H = F(a_1,a_2,a_3,a_4,a_5)$, $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5$,

$$P(H) = \frac{y}{(y+1)^4}\left[\prod(y^{a_i}+(-1)^{a_i+1}) + y^4\prod(y^{a_i-1}+(-1)^{a_i})\right] = \frac{y}{(y+1)^4}Q(H)$$

That H is chromatically equivalent with G implies $|V(G)| = |V(H)|$, i.e. $3a+2b = \sum a_i$. In the following, we analyze $Q(G)$ and $Q(H)$. By observing $Q(G)$ and $Q(H)$, we can find that the lowest power term in $Q(G)-(-1)^a(y^4-1)$ is $3y^a$, which cannot be cancelled in $Q(G)$, the lowest power term in $Q(H)-(-1)^{\sum a_i}(y^4-1)$ is one term or the plus of at least two terms of $(-1)^{\sum_{i \neq k} a_i} y^{a_k}$ $(k = 1,2,\cdots,5)$. By $Q(G) = Q(H)$, we know that the corresponding power terms are equal in both hands. So $a_3 = a_4 = a_5 = a$. By $3a+2b = \sum a_i$, we get $a_1 + a_2 = 2b$, after canceling all equal terms, the lowest power term in $Q(G)$ is $(-1)^{a+b}2y^b$ and cannot be cancelled in $Q(G)$, the lowest power term in $Q(H)$

is one term or the plus of $(-1)^{\sum_{i\neq k} a_i} y^{a_k}$ $(k=1,2)$ and cannot be cancelled in $Q(H)$ either. So we have $a_1 = a_2 = 2b$. i.e. $H = F(a,a,a,b,b)$.

Case3.2 H is obtained from $F(a_1,a_2,a_3,a_4)$ and a cycle C_{b_1} by overlapping on an edge, where $a_1 \geq a_2 \geq a_3 \geq a_4$.

$P(H)$ is presented by (1). That H is chromatically equivalent with G implies $|V(H)|=|V(G)|$, i.e. $3a+2b+1=\sum a_i + b_1$. Because $g(H)=g(G)=2a$, so $b_1 \geq 2a$.

By observing $Q(G)$ and $Q(H)$, we can find that the lowest power term in $Q(G)-(-1)^{a+1}$ is $3y^a$ or $(-1)^a y^4$, and no cancellation is possible between them; the lowest power term in $Q(H)-(-1)^{\sum a_i+b_1}$ is $(-1)^{\sum_{i\neq k} a_i+b_1+1} y^{a_k}$ or $(-1)^{\sum a_i+b_1} y^3$ or $(-1)^{\sum a_i} y^{b_1-1}$, which also cannot be cancelled by each other. Hence $\min\{a,4\}=\min\{a_k,3,b_1-1\}$. Since $b_1 \geq 2a$, so $b_1-1 \geq 2a-1 \geq a+1$. Therefore $a=\min\{a,4\}=\min\{a_k,3\}$. .The coefficient of y^a in $Q(G)$ is 3. For two polynomials to be equal, the coefficients of corresponding power terms must be equal. So there are only two possible cases: one is $a=a_3=a_4=3$, $a_1 \geq a_2 \geq 4$, and the other is $a=a_2=a_3=a_4=2$.

If $a=a_3=a_4=3$ and $a_1 \geq a_2 \geq 4$, then no matter what the value of b_1 is $C_g(H) \leq 2$. This is a contradiction with $C_g(H)=C_g(G)=3$.

If $a=a_2=a_3=a_4=2$, as $C_g(H)=C_g(G)=3$, so $b_1 \geq 5$. After letting $a=a_2=a_3=a_4=2$ both in $Q(G)$ and $Q(H)$, and canceling the terms of y^2 and constant terms, the lowest power term in $Q(H)$ is $(-1)^{\sum a_i+b_1} y^3$, which cannot be cancelled by the others in $Q(H)$, and at meantime, the lowest power term in $Q(G)$ is $(-1)^{a+b} 2y^b$ or $(-1)^{a+1} 2y^4$. But by $3a+2b+1=\sum a_i + b_1$, we get $2b=a_1+b_1-1 \geq 7$, this means $b \geq 3$. Therefore the lowest power terms both in $Q(G)$ and $Q(H)$ are not equal, i.e. the case $a=a_2=a_3=a_4=2$ is impossible.

From the two cases above, we could get that H is not chromatically equivalent with G.

Case3.3 H is obtained from a generalized θ-graph $\theta(a_1,a_2,a_3)$ and two cycles C_{b_1}, C_{b_2} by overlapping on edges, where $a_1 \geq a_2 \geq a_3 \geq 2$.

$P(H)$ is presented by (2). That H is chromatically equivalent with G implies $|V(H)|=|V(G)|$, i.e. $3a+2b+2=\sum a_i + \sum b_i$. Because $g(H)=g(G)=2a$, so we can assume $b_1 \geq b_2 \geq 2a$.

Similarly the lowest power term in $Q(G)-(-1)^{a+1}$ is $3y^a$ or $(-1)^a y^4$, and no cancellation is possible between them; the lowest power term in $Q(H)-(-1)^{\sum a_i+\sum b_i+1}$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i+b_1+b_2} y^{a_k}$ $(k=1,2,3)$ and $(-1)^{\sum a_i+\sum b_i} y^2$, which also cannot be cancelled by each other. Since $a \ge 2$, thus $a=a_2=a_3=2$ and $a_1>2$. Because $C_g(H)=C_g(G)=3$, therefore we must have $b_1=b_2=4$. By $3a+2b+2=\sum a_i+\sum b_i$, we get $2b=a_1+4$. After canceling the terms of y^2 and constant terms both in $Q(G)$ and $Q(H)$. The lowest power term in $Q(G)$ is $(-1)^{a+b}2y^b$ or $(-1)^a y^4$, i.e. $(-1)^b 2y^b$ or y^4, and no cancellation is possible between them. The lowest power term in $Q(H)$ is $(-1)^{a_2+a_3+b_1+b_2} y^{a_1}$ or $(-1)^{\sum a_i+b_2+1} y^{b_1-1}$ or $(-1)^{\sum a_i+b_1+1} y^{b_2-1}$, i.e. y^{a_1} or $-2y^3$. Thus $(-1)^b 2y^b=-2y^3$, this means $b=3$. Now letting $b=3$ in $2b=a_1+4$, we get $a_1=2$, which contradicts $a_1>2$. So H is not chromatically equivalent with G.

By the three cases of H above, we know that 5-bridge graph $F(a,a,a,b,b)$ is chromatically unique, i.e. the lemma 3 is proven. .

Proof of Lemma 4. Assume $G=F(a,a,a,a,b), a\ge 2, b\ge a+1$, similar with the Proof of the three lemmas above, by the Theorem3:

$$P(G)=\frac{y}{(y+1)^4}\left[(y^a+(-1)^{a+1})^4(y^b+(-1)^{b+1})+y^4(y^{a-1}+(-1)^a)^4(y^{b-1}+(-1)^b)\right]$$
$$=\frac{y}{(y+1)^4}Q(G)$$

$$Q(G)=y^{4a+b-1}(y+1)+6y^{2a+b}(y+1)+(-1)^a 4y^{a+b}(y^2-1)+y^b(y^3+1)$$
$$+(-1)^{a+b}4y^{3a}(y+1)+(-1)^b 6y^{2a}(y^2-1)+(-1)^{a+b}4y^a(y^3+1)+(-1)^b(y^4-1)$$

Now assume that H is chromatically equivalent to G. In the following, we will consider all possible cases about H.

Case4.1 $H=F(a_1,a_2,a_3,a_4,a_5)$, $a_1\ge a_2\ge a_3\ge a_4\ge a_5$,

$$P(H)=\frac{y}{(y+1)^4}\left[\Pi(y^{a_i}+(-1)^{a_i+1})+y^4\,\Pi(y^{a_i-1}+(-1)^{a_i})\right]=\frac{y}{(y+1)^4}Q(H)$$

That H is chromatically equivalent with G implies $|V(G)|=|V(H)|$, i.e. $4a+b=\sum a_i$. In the following, we analyze $Q(G)$ and $Q(H)$. The lowest power term in $Q(G)-(-1)^b y^4$ is $(-1)^{a+b}4y^a$, which cannot be cancelled in $Q(G)$. The lowest power term in $Q(H)-(-1)^{\sum a_i} y^4$ is one term or the plus of at least two terms of $(-1)^{\sum_{i\neq k} a_i} y^{a_k}$. By $Q(G)=Q(H)$, we know that the corresponding power terms are equal in both hands. So $a_2=a_3=a_4=a_5=a$. By $4a+b=\sum a_i$, we get $a_1=b$, i.e. $H=F(a,a,a,a,b)$.

Case4.2 H is obtained from $F(a_1,a_2,a_3,a_4)$ and a cycle C_{b_1} by overlapping on an edge, where $a_1\geq a_2\geq a_3\geq a_4$.

Because $g(H)=g(G)=2a$, so $b_1\geq 2a$. $P(H)$ is presented by (1). That H is chromatically equivalent with G implies $|V(H)|=|V(G)|$, i.e. $4a+b+1=\sum a_i+b_1$.

By observing $Q(G)$ and $Q(H)$, we can find that the lowest power term in $Q(G)-(-1)^{b+1}$ is $(-1)^{a+b}4y^a$ or $(-1)^b y^4$, and no cancellation is possible between them, the lowest power term in $Q(H)-(-1)^{\sum a_i+b_1}$ is $(-1)^{\sum_{i\neq k} a_i+b_1+1} y^{a_k}$ or $(-1)^{\sum a_i+b_1} y^3$ or $(-1)^{\sum a_i} y^{b_1-1}$, which also cannot be cancelled by each other. Hence $\min\{a,4\}=\min\{a_k,3,b_1-1\}$.

As $b_1\geq 2a$, so $b_1-1\geq 2a-1\geq a+1$. Therefore $a=\min\{a,4\}=\min\{a_k,3\}$. The coefficient of y^a in $Q(G)$ is 4, for two polynomials to be equal, the coefficients of corresponding power terms must be equal. So there are only two possible cases: One is $a=a_2=a_3=a_4=3$, $a_1\geq 4$, and the other is $a=a_1=a_2=a_3=a_4=2$.

If $a=a_2=a_3=a_4=3$ and $a_1\geq 4$, then no matter what the value of b_1 is, $C_g(H)\leq 4$. This is a contradiction with $C_g(H)=C_g(G)=6$.

If $a=a_1=a_2=a_3=a_4=2$, as $C_g(H)=C_g(G)=6$, so $b_1\geq 5$. After letting $a=a_1=a_2=a_3=a_4=2$ both in $Q(G)$ and $Q(H)$, and canceling the terms of y^2 and constant terms, the lowest power term in $Q(H)$ is $(-1)^{\sum a_i+b_1} y^3$, which cannot be cancelled by the others in $Q(H)$, and at meantime, the lowest power term in $Q(G)$ is y^b or $(-1)^{b+1}5y^4$. This means that only if $b=3$, we can guarantee $Q(H)=Q(G)$. However, by letting $b=3$ in $4a+b+1=\sum a_i+b_1$, we get $b_1=4$, which contradicts the above $b_1\geq 5$. Hence H is not chromatically equivalent with G.

Case4.3 H is obtained from a generalized θ-graph $\theta(a_1,a_2,a_3)$ and two cycles C_{b_1}, C_{b_2} by overlapping on edges, where $a_1\geq a_2\geq a_3\geq 2$. Since the number of cycles in 5-bridge graph $F(a,a,a,a,b)$ with length g is $C_g(G)=6$. Whereas for H, no matter what a_1,a_2,a_3,b_1 and b_2 are equal to, we always have $C_g(H)\leq 5$. So H is not chromatically equivalent with G.

By the three cases of H above, we know that 5-bridge graph $F(a,a,a,a,b)$ is chromatically unique, i.e. the lemma 4 is proven.

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