# The Graphs $C_{11}^{(t)}$ are Graceful for $t \equiv 0, 1 \pmod 4$ *

Xu Xirong Yang Yuansheng Li Huijun Xi Yue

Department of Computer Science
Dalian University of Technology
Dalian, 116024, P. R. China
yangys@dlut.edu.cn

**Abstract**

Let $C_n$ denote the cycle with $n$ vertices, and $C_n^{(t)}$ denote the graphs consisting of $t$ copies of $C_n$ with a vertex in common. Koh et al. conjectured that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0, 3 \pmod 4$. The conjecture has been shown true for $n = 3, 5, 6, 7, 9, 4k$. In this paper, the conjecture is shown to be true for $n = 11$.
**Keywords:** *graceful graph, vertex labeling, edge labeling*

## 1 Introduction

Let $C_n$ denote the cycle with $n$ vertices, and $C_n^{(t)}$ denote the graphs consisting of $t$ copies of $C_n$ with a vertex in common. Koh et al. [4] conjectured that the graphs $C_n^{(t)}$ are graceful if and only if $nt \equiv 0, 3 \pmod 4$, and proved that the graphs $C_{4k}^{(t)}$ and $C_6^{(2t)}$ are graceful for $t \geqslant 1$. Qian [7] proved that the graphs $C_{2k}^{(2)}$ are graceful. Bermond et al. [1, 2] proved that the graphs $C_3^{(t)}$ (i.e, the friendship graph or Dutch t-windmill) are graceful if and only if $t \equiv 0$ or $1 \pmod 4$. The first author [6, 8, 9] of this paper proved that the graphs $C_5^{(t)}$ and $C_9^{(t)}$ are graceful for $t \equiv 0, 3 \pmod 4$, and $C_7^{(t)}$ are graceful for $t \equiv 0, 1 \pmod 4$. So the conjecture has been shown

*The research is supported by CNSF 60143002, 60373096 and SRFDP 20030141006.

true for $n = 3, 5, 6, 7, 9, 4k$. In this paper, the conjecture is shown to be true for $n = 11$.

For the literature on graceful graphs we refer to [3] and the relevant references given in it.

## 2 The graphs $C_{11}^{(t)}$

Now, we consider the graphs $C_{11}^{(t)}$. Let $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i, v_5^i, v_6^i, v_7^i, v_8^i, v_9^i, v_{10}^i$ be the vertices of the $i$-th cycle, $v_0^i = v$ for all $i$. Then we have

**Theorem 2.1.** The graphs $C_{11}^{(t)}$ are graceful for $t \equiv 0, 1 \pmod 4$.

**Proof.** Case 1. $t \equiv 0 \pmod 4$, say $t = 4k$, i.e. $C_{11}^{(4k)}$

We define a vertex labeling $f$ as follows.

$$
\begin{aligned}
f(v) &= 0, && \\
f(v_1^i) &= 44k + 1 - i, && 1 \le i \le 4k, \\
f(v_2^i) &= 8k + 1 - 2i, && 1 \le i \le 4k, \\
f(v_3^i) &= 40k + 1 - i, && 1 \le i \le 4k, \\
f(v_4^i) &= \begin{cases} 16k + i, & 1 \le i \le 2k, \\ 12k - 1 + i, & 2k + 1 \le i \le 4k, \end{cases} \\
f(v_5^i) &= 36k + 1 - i, && 1 \le i \le 4k, \\
f(v_6^i) &= 8k + 2 - 2i, && 1 \le i \le 4k, \\
f(v_7^i) &= \begin{cases} 22k + 1 - i, & 1 \le i \le 2k, \\ 14k + 1 - i, & 2k + 1 \le i \le 4k, \end{cases} \\
f(v_8^i) &= \begin{cases} 18k + i, & 1 \le i \le 2k, \\ 20k + i, & 2k + 1 \le i \le 4k, \end{cases} \\
f(v_9^i) &= \begin{cases} 32k - i, & 1 \le i \le 2k, \\ 32k - 1 - i, & 2k + 1 \le i \le 3k - 1, \\ 32k - 2 - i, & 3k \le i \le 4k - 2, \\ 16k, & i = 4k - 1, \\ 30k - 1, & i = 4k, \end{cases} \\
f(v_{10}^i) &= \begin{cases} 24k + i, & 1 \le i \le k, \\ 27k - 1, & i = k + 1, \\ 24k - 1 + i, & k + 2 \le i \le 3k - 1, \\ 24k + i, & 3k \le i \le 4k - 2, \\ 32k, & i = 4k - 1, \\ 28k - 1, & i = 4k. \end{cases}
\end{aligned}
$$

Now we prove that $f$ is a graceful labeling of $C_{11}^{(4k)}$ as follows.

Denote by

$$S_j = \{f(v_j^i) \mid 1 \le i \le 4k\}, \quad 0 \le j \le 10.$$

Then

$$\begin{aligned}
S_0 =\ & \{0\},\\
S_1 =\ & \{44k, 44k-1, \ldots, 40k+1\},\\
S_2 =\ & \{8k-1, 8k-3, \ldots, 1\},\\
S_3 =\ & \{40k, 40k-1, \ldots, 36k+1\},\\
S_4 =\ & S_{4.1} \cup S_{4.2}\\
=\ & \{16k+1, 16k+2, \ldots, 18k\} \cup \{14k, 14k+1, \ldots, 16k-1\},\\
S_5 =\ & \{36k, 36k-1, \ldots, 32k+1\},\\
S_6 =\ & \{8k, 8k-2, \ldots, 2\},\\
S_7 =\ & S_{7.1} \cup S_{7.2}\\
=\ & \{22k, 22k-1, \ldots, 20k+1\} \cup \{12k, 12k-1, \ldots, 10k+1\},\\
S_8 =\ & S_{8.1} \cup S_{8.2}\\
=\ & \{18k+1, 18k+2, \ldots, 20k\} \cup \{22k+1, 21k+2, \ldots, 24k\},\\
S_9 =\ & S_{9.1} \cup S_{9.2} \cup S_{9.3} \cup S_{9.4} \cup S_{9.5}\\
=\ & \{32k-1, 32k-2, \ldots, 30k\} \cup \{30k-2, 30k-3, \ldots 29k\}\\
& \cup \{29k-2, 29k-3, \ldots, 28k\} \cup \{16k\} \cup \{30k-1\},\\
S_{10} =\ & S_{10.1} \cup S_{10.2} \cup S_{10.3} \cup S_{10.4} \cup S_{10.5} \cup S_{10.6}\\
=\ & \{24k+1, 24k+2, \ldots, 25k\} \cup \{27k-1\}\\
& \cup \{25k+1, 25k+2, \ldots, 27k-2\} \cup \{27k, 27k+1, \ldots, 28k-2\}\\
& \cup \{32k\} \cup \{28k-1\}.
\end{aligned}$$

Hence

$$\begin{aligned}
& S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9 \cup S_{10}\\
=\ & S_0 \cup S_2 \cup S_6 \cup S_{7.2} \cup S_{4.2} \cup S_{9.4} \cup S_{4.1} \cup S_{8.1} \cup S_{7.1} \cup S_{8.2} \cup S_{10.1}\\
& \cup S_{10.3} \cup S_{10.2} \cup S_{10.4} \cup S_{10.6} \cup S_{9.3} \cup S_{9.2} \cup S_{9.5} \cup S_{9.1} \cup S_{10.5} \cup S_5\\
& \cup S_3 \cup S_1\\
=\ & \{0,\ \ 1, 3, \ldots, 8k-1,\ \ 2, 4, \ldots, 8k,\ \ 10k+1, 10k+2, \ldots, 12k,\\
& 14k, 14k+1, \ldots, 16k-1,\ \ 16k,\ \ 16k+1, 16k+2, \ldots, 18k,\\
& 18k+1, 18k+2, \ldots, 20k,\ \ 20k+1, 20k+2, \ldots, 22k,\\
& 22k+1, 22k+2, \ldots, 24k,\ \ 24k+1, 24k+2, \ldots, 25k,\\
& 25k+1, 25k+2, \ldots, 27k-2,\ \ 27k-1,\ \ 27k, 27k+1, \ldots, 28k-2,\\
& 28k-1,\ \ 28k, 28k+1, \ldots, 29k-2,\ \ 29k, 29k+1, \ldots, 30k-2,\\
& 30k-1,\ \ 30k, 30k+1, \ldots, 32k-1,\ \ 32k,\ \ 32k+1, 32k+2, \ldots, 36k,\\
& 36k+1, 36k+2, \ldots, 40k,\ \ 40k+1, 40k+2, \ldots, 44k\}.
\end{aligned}$$

It is clear that the labels of each vertex are different, and $Max\{f(v_j^i) \mid 1 \le i \le 4k,\ 0 \le j \le 10\} = 44k = |E|$. We thus conclude that $f$ is an injective mapping from the vertex set of $G$ into $\{ 0, 1, \ldots, |E|\}$.

Denote by

$$D_j = \{g(v_j^i, v_{(j+1) \bmod 11}^i) | \ 1 \le i \le 4k\}, \ 0 \le j \le 10,$$

$$g(v_j^i, v_{(j+1) \bmod 11}^i) = |f(v_{(j+1) \bmod 11}^i) - f(v_j^i)|, 1 \le i \le 4k, 0 \le j \le 10.$$

Then

$$
\begin{aligned}
D_0 &= \{|f(v_1^i) - f(v_0^i)| \ 1 \le i \le 4k\} = \{44k + 1 - i | 1 \le i \le 4k\} \\
&= \{44k, 44k - 1, \ldots, 40k + 1\}, \\
D_1 &= \{36k + i | 1 \le i \le 4k\} = \{36k + 1, 36k + 2, \ldots, 40k\}, \\
D_2 &= \{32k + i | 1 \le i \le 4k\} = \{32k + 1, 32k + 2, \ldots, 36k\}, \\
D_3 &= D_{3.1} \cup D_{3.2} \\
&= \{24k + 1 - 2i | 1 \le i \le 2k\} \cup \{28k + 2 - 2i | 2k + 1 \le i \le 4k\} \\
&= \{24k - 1, 24k - 3, \ldots, 20k + 1\} \cup \{24k, 24k - 2, \ldots, 20k + 2\}, \\
D_4 &= D_{4.1} \cup D_{4.2} \\
&= \{20k + 1 - 2i | 1 \le i \le 2k\} \cup \{24k + 2 - 2i | 2k + 1 \le i \le 4k\} \\
&= \{20k - 1, 20k - 3, \ldots, 16k + 1\} \cup \{20k, 20k - 2, \ldots, 16k + 2\}, \\
D_5 &= \{28k - 1 + i | 1 \le i \le 4k\} = \{28k, 28k + 1, \ldots, 32k - 1\}, \\
D_6 &= D_{6.1} \cup D_{6.2} \\
&= \{14k - 1 + i | 1 \le i \le 2k\} \cup \{6k - 1 + i | 2k + 1 \le i \le 4k\} \\
&= \{14k, 14k + 1, \ldots, 16k - 1\} \cup \{8k, 8k + 1, \ldots, 10k - 1\}, \\
D_7 &= D_{7.1} \cup D_{7.2} \cup D_{7.3} \\
&= \{4k + 1 - 2i | 1 \le i \le k\} \cup \{4k + 1 - 2i | k + 1 \le i \le 2k\} \\
&\quad \cup \{6k - 1 + 2i | 2k + 1 \le i \le 4k\} \\
&= \{4k - 1, 4k - 3, \ldots, 2k + 1\} \cup \{2k - 1, 2k - 3, \ldots, 1\} \\
&\quad \cup \{10k + 1, 10k + 3, \ldots, 14k - 1\}, \\
D_8 &= D_{8.1} \cup D_{8.2} \cup D_{8.3} \cup D_{8.4} \cup D_{8.5} \\
&= \{14k - 2i | 1 \le i \le 2k\} \cup \{12k - 1 - 2i | 2k + 1 \le i \le 3k - 1\} \\
&\quad \cup \{12k - 2 - 2i | 3k \le i \le 4k - 2\} \cup \{4k + i | i = 4k - 1\} \\
&\quad \cup \{2k - 1 + i | i = 4k\} \\
&= \{14k - 2, 14k - 4, \ldots, 10k\} \cup \{8k - 3, 8k - 5, \ldots, 6k + 1\} \\
&\quad \cup \{6k - 2, 6k - 4, \ldots, 4k + 2\} \cup \{8k - 1\} \cup \{6k - 1\}, \\
D_9 &= D_{9.1} \cup D_{9.2} \cup D_{9.3} \cup D_{9.4} \cup D_{9.5} \cup D_{9.6} \cup D_{9.7} \\
&= \{8k - 2i | 1 \le i \le k\} \cup \{5k + 1 - i | i = k + 1\} \\
&\quad \cup \{8k + 1 - 2i | k + 2 \le i \le 2k\} \cup \{8k - 2i | 2k + 1 \le i \le 3k - 1\} \\
&\quad \cup \{8k - 2 - 2i | 3k \le i \le 4k - 2\} \cup \{12k + 1 + i | i = 4k - 1\} \\
&\quad \cup \{6k - i | i = 4k\} \\
&= \{8k - 2, 8k - 4, \ldots, 6k\} \cup \{4k\} \cup \{6k - 3, 6k - 5, \ldots, 4k + 1\} \\
&\quad \cup \{4k - 2, 4k - 4, \ldots, 2k + 2\} \cup \{2k - 2, 2k - 4, \ldots, 2\} \\
&\quad \cup \{16k\} \cup \{2k\},
\end{aligned}
$$

$$
\begin{aligned}
D_{10} &= D_{10.1} \cup D_{10.2} \cup D_{10.3} \cup D_{10.4} \cup D_{10.5} \cup D_{10.6} \\
&= \{24k+i \mid 1 \le i \le k\} \cup \{26k-2+i \mid i = k+1\} \\
&\quad \cup \{24k-1+i \mid k+2 \le i \le 3k-1\} \cup \{24k+i \mid 3k \le i \le 4k-2\} \\
&\quad \cup \{28k+1+i \mid i = 4k-1\} \cup \{24k-1+i \mid i = 4k\} \\
&= \{24k+1, 24k+2, \ldots, 25k\} \cup \{27k-1\} \\
&\quad \cup \{25k+1, 25k+2, \ldots, 27k-2\} \cup \{27k, 27k+1, \ldots, 28k-2\} \\
&\quad \cup \{32k\} \cup \{28k-1\}.
\end{aligned}
$$

Let $D$ be the set of labels of all edges, then we have

$$
\begin{aligned}
D &= D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \cup D_7 \cup D_8 \cup D_9 \cup D_{10} \\
&= D_{7.2} \cup D_{9.5} \cup D_{9.7} \cup D_{7.1} \cup D_{9.4} \cup D_{9.2} \cup D_{9.3} \cup D_{8.3} \cup D_{8.5} \\
&\quad \cup D_{9.1} \cup D_{8.2} \cup D_{8.4} \cup D_{6.2} \cup D_{8.1} \cup D_{7.3} \cup D_{6.1} \cup D_{9.6} \cup D_{4.1} \\
&\quad \cup D_{4.2} \cup D_{3.1} \cup D_{3.2} \cup D_{10.1} \cup D_{10.3} \cup D_{10.2} \cup D_{10.4} \cup D_{10.6} \\
&\quad \cup D_5 \cup D_{10.5} \cup D_2 \cup D_1 \cup D_0 \\
&= \{1, 3, \ldots, 2k-1, 2, 4, \ldots, 2k-2, 2k,\ 2k+1, 2k+3, \ldots, 4k-1, \\
&\quad 2k+2, 2k+4, \ldots, 4k-2,\ \ 4k,\ \ 4k+1, 4k+3, \ldots, 6k-3, \\
&\quad 4k+2, 4k+4, \ldots, 6k-2,\ 6k-1,\ 6k+1, 6k+3, \ldots, 8k-3, \\
&\quad 6k, 6k+2, \ldots, 8k-2,\ \ 8k-1,\ \ 8k, 8k+1, \ldots, 10k-1, \\
&\quad 10k, 10k+2, \ldots, 14k-2,\ \ 10k+1, 10k+3, \ldots, 14k-1, \\
&\quad 14k, 14k+1, \ldots, 16k-1,\ 16k,\ 16k+1, 16k+3, \ldots, 20k-1, \\
&\quad 16k+2, 16k+4, \ldots, 20k,\ \ 20k+1, 20k+3, \ldots, 24k-1, \\
&\quad 20k+2, 20k+4, \ldots, 24k,\ \ 24k+1, 24k+2, \ldots, 25k, \\
&\quad 25k+1, 25k+2, \ldots, 27k-2, 27k-1, 27k, 27k+1, \ldots, 28k-2, \\
&\quad 28k-1, 28k, 28k+1, \ldots, 32k-1, 32k, 32k+1, 32k+2, \ldots, 36k, \\
&\quad 36k+1, 36k+2, \ldots, 40k,\ \ 40k+1, 40k+2, \ldots, 44k\} \\
&= \{1, 2, \ldots, 44k\}.
\end{aligned}
$$

It is clear that the labels of each edge are different. So, $g$ maps $E$ onto $\{1, 2, \ldots, |E|\}$. By the definition of graceful graph, we thus conclude that $C_{11}^{(4k)}$ is graceful.

Case 2. $t \equiv 1 \pmod 4$, say $t = 4k+1$, i.e. $C_{11}^{(4k+1)}$

We define a vertex labeling $f$ as follows.

$$
\begin{aligned}
f(v) &= 0, \\
f(v_1^i) &= 44k + 12 - i, && 1 \le i \le 4k+1, \\
f(v_2^i) &= 8k + 3 - 2i, && 1 \le i \le 4k+1, \\
f(v_3^i) &= 40k + 11 - i, && 1 \le i \le 4k+1, \\
f(v_4^i) &= \begin{cases} 20k + 6 + i, & 1 \le i \le 2k, \\ 16k + 5 + i, & 2k+1 \le i \le 4k+1, \end{cases} \\
f(v_5^i) &= 36k + 10 - i, && 1 \le i \le 4k+1,
\end{aligned}
$$

$$f(v_6^i) = \begin{cases} 10k+3+i, & 1 \le i \le 2k, \\ 6k+2+i, & 2k+1 \le i \le 4k+1, \end{cases}$$
$$f(v_7^i) = \begin{cases} 18k+6-i, & 1 \le i \le 2k, \\ 16k+6-i, & 2k+1 \le i \le 4k+1, \end{cases}$$
$$f(v_8^i) = \begin{cases} 14k+5+i, & 1 \le i \le 2k, \\ 20k+6+i, & 2k+1 \le i \le 4k+1, \end{cases}$$
$$f(v_9^i) = \begin{cases} 26k+8-i, & 1 \le i \le 2k, \\ 2i-4k, & 2k+1 \le i \le 4k+1, \end{cases}$$
$$f(v_{10}^i) = \begin{cases} 30k+8, & i=1, \\ 26k+6+i, & 2 \le i \le k+1, \\ 30k+8+i, & k+2 \le i \le 2k, \\ 27k+8+i, & 2k+1 \le i \le 3k-1, \\ 29k+8, & i=3k, \\ 27k+9+i, & 3k+1 \le i \le 4k, \\ 30k+9, & i=4k+1. \end{cases}$$

Similar to the proof in Case 1, it can be shown that this assignment provides a graceful labeling of $C_{11}^{(4k+1)}$. Hence $C_{11}^{(t)}$ is graceful for $t \equiv 0, 1 \pmod 4$. □

In Figure 1, we illustrate our graceful labeling for $C_{11}^{(12)}$ and $C_{11}^{(13)}$.

# References


[1] J. C. Bermond, A. E. Brouwer and A. Germa, Systemes de triplets et differences associèes, *Problems Combinatories et Thèorie des Graphs, Colloq. Intern. du Centre National de la Rech. Scient., 260, Editions du Centre Nationale de la Recherche Scientifique*, Paris(1978), 35-38.

[2] J. C. Bermond, A. Kotzig and J. Turgeon, On a combinatorial problem of antennas in radioastronomy, in Combinatories, A. Hajnal and V. T. Sós. eds., *Colloq. Math. Soc. János Bolyai*, 18, 2 vols. North-Holland. Amsterdam(1978), 135-149.

[3] J. A. Gallian, A Survey: A Dynamic Survey of Graph Labeling, *THE ELECTRONIC JOURNAL OF COMBINATORICS*, 5(2005),#DS6.

[4] K. M. Koh. D. G. Rogers, P. Y. Lee and C. W. Toh, On graceful graphs V:unions of graphs with one vertex in common, *Nanta Math.*, 12(1979), 133-136.

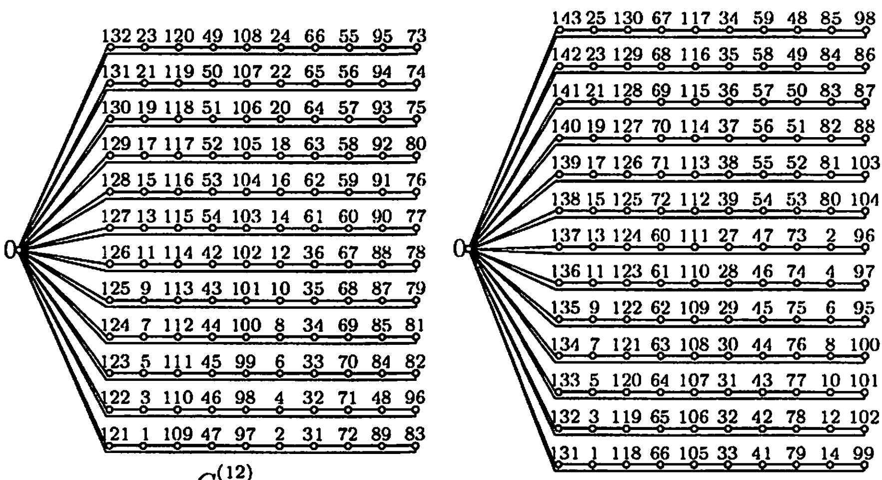
[5] Ma Kejie, Gracefulness of $P(n_1, n_2, ..., n_m)$ and $D_{m,4}$, *Applied Math.*, 4(1989), 95-97.

[6] Yang Yuansheng, Lin Xiaohui, Yu Chunyan, The Graphs $C_5^{(t)}$ are Graceful for $t \equiv 0, 3 \pmod 4$, *ARS COMBINATORIA*, 74(2005), 239-244.

[7] J.Qian, On some conjectures and problems in graceful labelings graphs, *preprint*.

[8] Yang Yuansheng, Xu Xirong, Xi Yue, Li Huijun, Khandoker Mohammed Mominul Haque, The Graphs $C_7^{(t)}$ are Graceful for $t \equiv 0, 1 \pmod 4$, *ARS COMBINATORIA*, to appear.

[9] Yang Yuansheng, Xu Xirong, Xi Yue, Li Huijun, The Graphs $C_9^{(t)}$ are Graceful for $t \equiv 0, 3 \pmod 4$, *ARS COMBINATORIA*, to appear.

$C_{11}^{(12)}$ $C_{11}^{(13)}$

Figure 1: Graceful labelings of $C_{11}^{(12)}$ and $C_{11}^{(13)}$.