Lattices associated with finite vector spaces and finite affine spaces

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Abstract

Let $\mathbb{F}_q^{(n)}$ (resp. $AG(n, \mathbb{F}_q)$) be the n-dimensional vector (resp. affine) space over the finite field $\mathbb{F}_q$. For $1 \le i < i+s \le n-1$ (resp. $0 \le i < i+s \le n-1$), let $\mathcal{L}(i, i+s; n)$ (resp. $\mathcal{L}'(i, i+s; n)$) denote the set of all subspaces (resp. flats) in $\mathbb{F}_q^{(n)}$ (resp. $AG(n, \mathbb{F}_q)$) with dimensions between i and $i+s$ including $\mathbb{F}_q^{(n)}$ and $\{0\}$ (resp. $\emptyset$). By ordering $\mathcal{L}(i, i+s; n)$ (resp. $\mathcal{L}'(i, i+s; n)$) by ordinary inclusion or reverse inclusion, two classes of lattices are obtained. This article discusses their geometricity.

Key words: Vector spaces, Affine spaces, Geometric lattice.

1 Introduction

In this section We recall some terminology and definitions about finite posets and lattices ([1, 2]).

Let P be a poset. For $a, b \in P$, we say a *covers* b, denoted by $b <\cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If P has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b <\cdot a$.

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A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an *atom* in P, we mean an element in P covering 0. We say P is *atomic* if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \forall\, a,\, b \in P.$$

The results on the lattices generated by orbits of subspaces under finite classical groups can be found in Huo, Liu and Wan ([5, 6, 7]), Huo and Wan ([8]), Gao and You ([3]), Orlik and Solomon ([10]), Wang and Feng ([11]).

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^{(n)}$ denote the n-dimensional *row vector space* over $\mathbb{F}_q$. For any integer r with $0 \leq r \leq n$, the cosets of $\mathbb{F}_q^{(n)}$ relative to any r-dimensional vector subspace are called *r-flats.* Define the empty set $\emptyset$ to be the -1-flat. The dimension of an r-flat $U + u$ is defined to be r, denoted by $\dim(U + u) = r$. An r-flat is said to be *incident* with an s- flat, if the r-flat contains or is contained in the s-flat. The point set $\mathbb{F}_q^{(n)}$ with r-flats $(0 \leq r \leq n)$ and the incidence relation among them defined above is said to be the n-dimensional *affine space* and denoted by $AG(n, \mathbb{F}_q)$.

The set of points belonging to both flats $U + u$ and $V + v$ is called the *intersection* of $U + u$ and $V + v$, which is denoted by $(U + u) \cap (V + v)$. It follows that the intersection of all flats containing two given flats $U + u$ and $V + v$ is a flat, which is called the *join* of $U + u$ and $V + v$, denoted by $(U + u) \cup (V + v)$.

Proposition 1.1. *([4]) For any two flats $F_1 = V_1 + x_1$ and $F_2 = V_2 + x_2$, where V_1 and V_2 are vector subspaces, and $x_1, x_2 \in \mathbb{F}_q^{(n)}$, $F_1 \cup F_2 = (V_1 + V_2 + \langle x_2 - x_1 \rangle) + x_1$.*

For $1 \leq i < i + s \leq n - 1$ (resp. $0 \leq i < i + s \leq n - 1$), let $\mathcal{L}(i, i + s; n)$ (resp. $\mathcal{L}'(i, i + s; n)$) denote the set of all subspaces (resp. flats) in $\mathbb{F}_q^{(n)}$ (resp. $AG(n, \mathbb{F}_q)$) with dimensions between i and $i + s$ including $\mathbb{F}_q^{(n)}$ and $\{0\}$ (resp. $\emptyset$). If we define the partial order on $\mathcal{L}(i, i + s; n)$ (resp.

$\mathcal{L}'(i, i+s; n)$) by ordinary inclusion or reverse inclusion, then $\mathcal{L}(i, i+s; n)$ (resp. $\mathcal{L}'(i, i+s; n)$) is a poset, denoted by $\mathcal{L}_O(i, i+s; n)$ or $\mathcal{L}_R(i, i+s; n)$ (resp. $\mathcal{L}'_O(i, i+s; n)$ or $\mathcal{L}'_R(i, i+s; n)$), respectively. When $i = 1$ (resp. $i = 0$), both $\mathcal{L}_O(1, 1+s; n)$ and $\mathcal{L}_R(1, 1+s; n)$ (resp. $\mathcal{L}'_O(0, s; n)$ and $\mathcal{L}'_R(0, s; n)$) are atomic lattices, and the geometricity of these lattices is classified in [8] (resp. [11]). In the present paper we show that both $\mathcal{L}_O(i, i+s; n)$ and $\mathcal{L}_R(i, i+s; n)$ (resp. $\mathcal{L}'_O(i, i+s; n)$ and $\mathcal{L}'_R(i, i+s; n)$) are atomic lattices, and classify their geometricity. Our main results are the following.

Theorem 1.2. *For $1 \le i < i+s \le n-1$, $\mathcal{L}_O(i, i+s; n)$ is a geometric lattice if and only if $i = 1$.*

Theorem 1.3. *For $1 \le i < i+s \le n-1$, $\mathcal{L}_R(i, i+s; n)$ is a geometric lattice if and only if $i+s = n-1$.*

Theorem 1.4. *For $0 \le i < i+s \le n-1$, $\mathcal{L}'_O(i, i+s; n)$ is a geometric lattice if and only if $i = 0$.*

Theorem 1.5. *For $0 \le i < i+s \le n-1$, $\mathcal{L}'_R(i, i+s; n)$ is not a geometric lattice.*

2 Proofs of main results

Proof of Theorem 1.2. Let $M(i; n)$ be the set of all i-dimensional subspaces in $\mathbb{F}_q^{(n)}$. Then $M(i; n)$ is the set of all atoms in $\mathcal{L}_O(i, i+s; n)$. In order to prove $\mathcal{L}_O(i, i+s; n)$ is atomic, it suffices to show that every element of $M(j; n)$ $(i \le j \le i+s)$ is a union of some atoms. The result is trivial for $j = i$. Suppose that the result is true for $j = i+l$. For $P \in M(i+(l+1); n)$, by [9, Corollary 1.8], the number of $i+l$-dimensional subspaces contained in P is

$$\frac{q^{i+l+1}-1}{q-1} \ge 2.$$

It follows that there exist two different $i+l$-dimensional subspaces P_1, $P_2 \subseteq P$ such that $P = P_1 \vee P_2$. Therefore, by induction P is a union of some elements in $M(i; n)$. Therefore, $\mathcal{L}_O(i, i+s; n)$ a finite atomic lattice.

For any $X \in \mathcal{L}_O(i, i+s; n)$, we define

$$r_O(X) = \begin{cases} 0, & \text{if } X = \{0\}, \\ s+2, & \text{if } X = \mathbb{F}_q^{(n)}, \\ \dim(X) - i + 1, & \text{otherwise.} \end{cases}$$

It is routine to check that r_O is the rank function on $\mathcal{L}_O(i, i+s; n)$.

By [8, Theorem 4], $\mathcal{L}_O(1, 1+s; n)$ is a geometric lattice.

Conversely, suppose $i \geq 2$. Then $2 \leq i \leq n-2$. Let $U = \langle v_1, v_2, \ldots, v_i, v_{i+1}, v_{i+2}\rangle$ be the $i+2$-dimensional subspace, and let $V_1 = \langle v_1, \ldots, v_i\rangle$, $V_2 = \langle v_3, \ldots, v_{i+2}\rangle$. Then $\dim(V_1 \cap V_2) = i - 2$. Thus $V_1 \wedge V_2 = \{0\}$, and

$$V_1 \vee V_2 = \begin{cases} U, & \text{if } s \geq 2, \\ \mathbb{F}_q^{(n)}, & \text{if } s = 1. \end{cases}$$

Therefore, $r_O(V_1 \vee V_2) + r_O(V_1 \wedge V_2) = 3 > 2 = r_O(V_1) + r_O(V_2)$ and $\mathcal{L}_O(i, i+s; n)$ is not a geometric lattice. □

Proof of Theorem 1.3. Let $M(i+s; n)$ be the set of all $(i+s)$-dimensional subspaces in $\mathbb{F}_q^{(n)}$. Then $M(i+s; n)$ is the set of all atoms in $\mathcal{L}_R(i, i+s; n)$. By [8, Theorem 5], $\mathcal{L}_R(i, i+s; n)$ is a finite atomic lattice.

For any $X \in \mathcal{L}_R(i, i+s; n)$, we define

$$r_R(X) = \begin{cases} 0, & \text{if } X = \mathbb{F}_q^{(n)}, \\ s+2, & \text{if } X = \{0\}, \\ i+s+1-\dim(X), & \text{otherwise.} \end{cases}$$

It is routine to check that r_R is the rank function on $\mathcal{L}_R(i, i+s; n)$.

For $U, W \in \mathcal{L}_R(i, n-1; n)$, if $\dim(U \cap W) \geq i$, then $U \vee W = U \cap W$. Thus $r_R(U \vee W) + r_R(U \wedge W) = r_R(U) + r_R(W)$. If $\dim(U \cap W) \leq i-1$, then $U \vee W = \{0\}$. We distinguish the following two cases:

Case 1: $U = \{0\}$ or $W = \{0\}$. Clearly, $r_R(U \vee W) + r(U \wedge W) = r_R(U) + r_R(W)$.

Case 2: $U \neq \{0\}$ and $W \neq \{0\}$. Let $\dim U = m_1 \geq i$, $\dim W = m_2 \geq i$, and $\dim(U+W) = d$, then $\dim(U \cap W) = m_1 + m_2 - d$. Thus

$$r_R(U \vee W) + r_R(U \wedge W) = n+1-i+n-d \leq n-m_1+n-m_2 = r_R(U) + r_R(W).$$

Therefore, $\mathcal{L}_R(i, n-1; n)$ is a geometric lattice.

Conversely, suppose $i + s \leq n - 2$. By $1 \leq i < i + s$, $2 \leq i + 1 \leq i + s \leq n - 2$. Let $U = \langle v_1, v_2, \ldots, v_{i+s}, v_{i+s+1}, v_{i+s+2} \rangle$ be the $i + s + 2$-dimensional subspace, and let $V_1 = \langle v_1, \ldots, v_{i+s} \rangle$, $V_2 = \langle v_3, \ldots, v_{i+s+2} \rangle$. Then $\dim(V_1 \cap V_2) = i + s - 2$. Thus

$$V_1 \vee V_2 = \begin{cases} V_1 \cap V_2, & \text{if } s \geq 2, \\ \{0\}, & \text{if } s = 1. \end{cases}$$

Therefore, $r_R(V_1 \vee V_2) + r_R(V_1 \wedge V_2) = 3 > 2 = r_R(V_1) + r_R(V_2)$ and $\mathcal{L}_R(i, i + s; n)$ is not a geometric lattice. $\square$

Proof of Theorem 1.4. Let $M'(i; n)$ be the set of all i-flats in $AG(n, \mathbb{F}_q)$. Then $M'(i; n)$ is the set of all atoms in $\mathcal{L}'_O(i, i + s; n)$. In order to prove $\mathcal{L}'_O(i, i + s; n)$ is atomic, it suffices to show that every element of $M'(j; n)$ $(i \leq j \leq i + s)$ is a union of some atoms. The result is trivial for $j = i$. Suppose that the result is true for $j = i + l$. For $F \in M'(i + (l + 1); n)$, by [9, Theorem 1.18], the number of $i + l$-flats contained in F is

$$\frac{q(q^{i+l+1} - 1)}{q - 1} \geq 2.$$

It follows that there exist two different $i + l$-flats F_1, $F_2 \subsetneq F$ such that $F = F_1 \vee F_2$. Therefore, by induction F is a union of some elements in $M'(i; n)$. Therefore, $\mathcal{L}'_O(i, i + s; n)$ a finite atomic lattice.

For any $X \in \mathcal{L}'_O(i, i + s; n)$, we define

$$r'_O(X) = \begin{cases} 0, & \text{if } X = \emptyset, \\ s + 2, & \text{if } X = \mathbb{F}_q^{(n)}, \\ \dim(X) - i + 1, & \text{otherwise.} \end{cases}$$

It is routine to check that r'_O is the rank function on $\mathcal{L}'_O(i, i + s; n)$.

By [11, Theorem 1.1], $\mathcal{L}'_O(0, s; n)$ is a geometric lattice.

Conversely, suppose $i \geq 1$, then $i + 1 \leq n - 1$. Fix a $i + 1$ dimensional subspace $U = \langle u_1, u_2, \ldots, u_{i+1} \rangle$, then exists a $x \in \mathbb{F}_q^{(n)}$ such that $x \notin U$. Let $U_1 = \langle u_1, u_2, \ldots, u_i \rangle$ and $U_2 = \langle u_2, u_3, \ldots, u_{i+1} \rangle$, then U_1, $U_2 + x \in$

$\mathcal{L}'_O(i, i+s; n)$ and $U_1 \wedge (U_2 + x) = \emptyset$. By Proposition 1.1, $U_1 \cup (U_2 + x) = U + \langle x \rangle$. Thus,

$$U_1 \vee (U_2 + x) = \begin{cases} U + \langle x \rangle, & \text{if } s \geq 2, \\ \mathbb{F}_q^{(n)}, & \text{if } s = 1. \end{cases}$$

Therefore $r'_O(U_1 \vee (U_2+x)) + r'_O(U_1 \wedge (U_2+x)) = 3 > 2 = r'_O(U_1) + r'_O(U_2+x)$ and $\mathcal{L}'_O(i, i+s; n)$ is not a geometric lattice. □

Proof of Theorem 1.5. Let $M'(i+s; n)$ be the set of all $i+s$-flats in $AG(n, \mathbb{F}_q)$. Then $M'(i+s; n)$ is the set of all atoms in $\mathcal{L}'_R(i, i+s; n)$. By [11, Theorem 1.2], $\mathcal{L}'_R(i, i+s; n)$ is a finite atomic lattice.

For any $X \in \mathcal{L}'_R(i, i+s; n)$, we define

$$r'_R(X) = \begin{cases} 0, & \text{if } X = \mathbb{F}_q^{(n)}, \\ s+2, & \text{if } X = \emptyset, \\ i+s+1-\dim(X), & \text{otherwise.} \end{cases}$$

It is routine to check that r'_R is the rank function on $\mathcal{L}'_R(i, i+s; n)$.

Fix a $i+s$ dimensional subspace U, then exists a $x \in \mathbb{F}_q^{(n)}$ such that $x \notin U$. Thus $U \vee (U+x) = \emptyset$ and $U \wedge (U+x) = \mathbb{F}_q^{(n)}$. Therefore $r'_R(U \vee (U+x)) + r'_R(U \wedge (U+x)) = s+2 > 2 = r'_R(U) + r'_R(U+x)$ and $\mathcal{L}'_R(i, i+s; n)$ is not a geometric lattice. □

Acknowledgement

This research is supported by the "LSAZ200702 Program" of Langfang Teachers' College.

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