Hyperbolic Functions with Second Order Recurrence Sequences

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Abstract

In this paper, we introduce an extension of the hyperbolic Fibonacci and Lucas functions which were studied by Stakhov and Rozin. Namely, we define hyperbolic functions by second order recurrence sequences and study their hyperbolic and recurrence properties. We give the corollaries for Fibonacci, Lucas, Pell and Pell-Lucas numbers. We finalize with the introduction some surfaces (the Metallic Shofars) that relate to the hyperbolic functions with the second order recurrence sequences.

Keywords: Fibonacci numbers, Pell numbers, Hyperbolic functions.

1. Introduction

Well known Fibonacci formula is simple case of the second order recurrences. This kind of recurrence relation plays a significant role in many disciplines, like mathematics, physic, biology, economy and so on. Many scholars interested in these numbers, their properties, continuous versions and generalizations [1-19]. Stakhov [4] introduced a new class of recurrence relations generating the generalized Fibonacci $p-$numbers and a new class of mathematical constants named the generalized golden $p-$proportions ($p = 0,1,2,3,\ldots$). Stakhov and Tkachenko [13] defined a new class of hyperbolic functions called hyperbolic Fibonacci and Lucas functions. Stakhov and Rozin [14, 17] introduced symmetrical representation of the hyperbolic Fibonacci and Lucas functions. The function of the "Golden Shofar" [15] follows from this approach. Also

Stakhov and Rozin [16] defined the continuous functions with Fibonacci and Lucas p-numbers which is generalization of the Fibonacci and Lucas numbers. Stakhov [18] gave a wide generalization of the symmetrical hyperbolic Fibonacci and Lucas functions and created a general theory of hyperbolic functions – the *hyperbolic Fibonacci and Lucas m–functions* ($m > 0$ is a given positive real number). It is interesting to note that Falcon and Plaza [19] defined the k-Fibonacci hyperbolic functions similar to Stakhov's hyperbolic Fibonacci and Lucas m-functions [18]. Stakhov's article [18] was available online on December 21, 2006 and Falcon and Plaza [19] was available online on January 2, 2007 that testify the fact that Falcon and Plaza [19] came to a new class of hyperbolic functions independently from Stakhov [18].

The main goal of the present article is to define hyperbolic functions with all second order recurrence sequences $\{U_n\}$ and $\{V_n\}$ and study hyperbolic and recurrence properties of these functions. This article presents the continuous versions of the second order recurrence sequences.

1.1. Second order recurrence sequences, the generalized golden proportions and the generalized Binet formulas

The Argentinean mathematician Vera W. Spinadel introduced [10] a general class of the second order recurrence sequences. Let p and q be nonzero real numbers, such that $p^2+4q \neq 0$. The second order recurrence sequences $\{U_n\}$ and $\{V_n\}$ for all n are defined by

$$U_{n+2} = pU_{n+1} + qU_n, \quad U_0 = 0, U_1 = 1 \tag{1}$$

and

$$V_{n+2} = pV_{n+1} + qV_n, \quad V_0 = 2, V_1 = p \tag{2}$$

It is known that, the characteristic equation for above recurrences is

$$x^2 - px - q = 0. \tag{3}$$

The characteristic equation (3) has two real roots;

$$x_1 = \alpha = \frac{p+\sqrt{p^2+4q}}{2}, \quad x_2 = \frac{-q}{\alpha} = \frac{p-\sqrt{p^2+4q}}{2}. \tag{4}$$

Let us consider the formula for the positive root of the characteristic equation (3):

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}. \tag{5}$$

Note that for the case $p = q = 1$, the formula (5) takes the following form:

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad (6)$$

that is, for the case $p = q = 1$ the formula (5) gives the famous "golden mean" or the "golden proportion" known from the ancient times.

For the case $q = 1$ and $p = m$, the formula (5) is reduced to the formula

$$\alpha = \frac{m+\sqrt{m^2+4}}{2}. \qquad (7)$$

The formula (7) generates an infinite number of new mathematical constants named in [18] the *generalized golden* m*-proportions.*

It is clear that the formula (5), which is a generalization of the formulas (6) and (7), generates an infinite number of new mathematical constants – *the generalized golden* (p,q)*-proportions.* Spinadel [10] named these proportions *metallic means.* Note that Spinadel [10] for the first time introduced the notions of the *silver mean* ($p = 2$, $q = 1$), *bronze mean* ($p = 3$, $q = 1$) and so on.

We can use by the roots x_1 and x_2 (4) for the representation of U_n and V_n sequences in analytical form:

$$U_n = \frac{\alpha^n - (-q)^n \alpha^{-n}}{\alpha + q\alpha^{-1}}, \quad V_n = \alpha^n + (-q)^n \alpha^{-n}, \qquad (8)$$

where $n = 0, \pm1, \pm2, \ldots$. We name the formulas (8) the *generalized Binet formulas,* the generalized Binet formulas (8) may be written as follows

$$U_n = \begin{cases} \dfrac{\alpha^n + q^n\alpha^{-n}}{\alpha + q\alpha^{-1}}, & n \text{ odd} \\[2ex] \dfrac{\alpha^n - q^n\alpha^{-n}}{\alpha + q\alpha^{-1}}, & n \text{ even} \end{cases} \qquad (9)$$

$$V_n = \begin{cases} \alpha^n - q^n\alpha^{-n}, & n \text{ odd} \\ \alpha^n + q^n\alpha^{-n}, & n \text{ even} \end{cases} \qquad (10)$$

where α is a positive root (5) of the characteristic equation (3), $n = 0, \pm1, \pm2, \ldots$ and $\alpha + q\alpha^{-1} = \sqrt{p^2+4q}$. Taking $p = q = 1$ in (1) and (2), we obtain the Fibonacci and Lucas sequences. For this case, the mathematical constant (5) is reduced to the golden mean (6) and the generalized Binet formulas (8)-(10) are reduced to the classical Binet formulas for Fibonacci ($F_n = U_n$) and Lucas ($L_n = V_n$) numbers:

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n}, \qquad (11)$$

$$F_n = \begin{cases} \dfrac{\alpha^n + \alpha^{-n}}{\sqrt{5}}, & n \ odd \\ \dfrac{\alpha^n - \alpha^{-n}}{\sqrt{5}}, & n \ even \end{cases} \tag{12}$$

$$L_n = \begin{cases} \alpha^n - \alpha^{-n}, & n \ odd \\ \alpha^n + \alpha^{-n}, & n \ even \end{cases} \tag{13}$$

For the case $q = 1$ and $p = m$, the formula (5) is reduced to the formula (7) and the generalized Binet formulas (8) are reduced to *Gazale formulas* for the generalized Fibonacci and Lucas m-numbers $F_m(n)$ and $L_m(n)$ [11, 18]:

$$F_m(n) = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{4+m^2}} \qquad L_m(n) = \alpha^n + (-1)^n \alpha^{-n} \tag{14}$$

If we choose $p = 2, q = 1$ in (1) and (2), we get the Pell and Pell-Lucas sequences. Finally, taking $p = 1, q = 2$ in (1) and (2), we obtain the Jacobsthal and Jacobsthal-Lucas sequences [6].

2. Hyperbolic Functions with Second Order Recurrence Sequences

2.1. *Definition of hyperbolic functions with second order recurrence sequences*

The classical hyperbolic functions are defined by

$$sh(x) = \frac{e^x - e^{-x}}{2}, \quad ch(x) = \frac{e^x + e^{-x}}{2}. \tag{15}$$

By using similarity between hyperbolic functions (15) and the Binet formulas (12) and (13), Stakhov and Tkachenko [13] defined *hyperbolic Fibonacci and Lucas functions*. By developing Stakhov and Tkachenko's approach, Stakhov and Rozin [14] defined the so-called *symmetrical hyperbolic Fibonacci and Lucas functions*:

Symmetrical hyperbolic Fibonacci sine

$$sFs(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \tag{16}$$

Symmetrical hyperbolic Fibonacci cosine

$$cFs(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}} \tag{17}$$

Symmetrical hyperbolic Lucas sine

$$sLs(x) = \alpha^x - \alpha^{-x} \tag{18}$$

Symmetrical hyperbolic Lucas cosine

$$cLs(x) = \alpha^x + \alpha^{-x} \tag{19}$$

Also Stakhov [18], Falcon and Plaza [19] defined the Fibonacci and Lucas hyperbolic m-functions ($m > 0$ is a given real number) by using the generalized golden m-proportions (7) as a base of hyperbolic functions. Now we define the hyperbolic functions with second order recurrence sequences (1), (2) based on Stakhov and Tkachenko's definitions [13] as follows.

Definition 1 *Let p and q be nonzero real numbers, such that $p^2 + 4q \neq 0$, and α positive root of characteristic equation (3). The hyperbolic U_n sine and cosine functions defined by*

$$sU(x) = \frac{\alpha^{2x} - q^{2x}\alpha^{-2x}}{\sqrt{p^2 + 4q}}, \tag{20}$$

$$cU(x) = \frac{\alpha^{2x+1} + q^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2 + 4q}}, \tag{21}$$

the hyperbolic V_n sine and cosine functions defined by

$$sV(x) = \alpha^{2x+1} - q^{2x+1}\alpha^{-2x-1}, \tag{22}$$

$$cV(x) = \alpha^{2x} + q^{2x}\alpha^{-2x}. \tag{23}$$

Note that there are the following correlations between U_n and V_n numbers and hyperbolic U_n and V_n functions given by (20)-(23):

$$sU(k) = U_{2k}; \quad cU(k) = U_{2k+1}$$

$$sV(k) = V_{2k}; \quad cV(k) = V_{2k+1}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

The hyperbolic U_n and V_n functions (20)-(23) are not symmetrical with respect to the origin. For this reason, we use Stakhov and Rozin's approach [14] and introduce the *symmetrical representation of the hyperbolic U_n and V_n functions.*

Based on the classical hyperbolic functions (15) and the generalized Binet formulas (8) for U_n and V_n sequences, we can give the definitions of the symmetrical hyperbolic U_n and V_n functions that are different from the definitions (20)-(23):

Symmetrical U_n sine function

$$sUs(x)=\frac{\alpha^x-q^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{24}$$

<u>Symmetrical U_n cosine function</u>

$$cUs(x)=\frac{\alpha^x+q^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{25}$$

<u>Symmetrical V_n sine function</u>

$$sVs(x)=\alpha^x-q^x\alpha^{-x} \tag{26}$$

<u>Symmetrical V_n cosine function</u>

$$cVs(x)=\alpha^x+q^x\alpha^{-x} \tag{27}$$

The symmetrical hyperbolic U_n and V_n functions (24)-(27) are connected between themselves as follows:

$$sVs(x)=\sqrt{p^2+4q}\,sUs(x),\qquad cVs(x)=\sqrt{p^2+4q}\,cUs(x). \tag{28}$$

The U_n and V_n numbers are determined with symmetrical hyperbolic U_n and V_n functions as follows

$$U_n=\begin{cases}cUs(n), & n\ odd\\ sUs(n), & n\ even\end{cases}\qquad V_n=\begin{cases}cVs(n), & n\ odd\\ sVs(n), & n\ even\end{cases} \tag{29}$$

Note that for the case $q=1$ and $p=m$ ($m>0$ is a given positive real number), the symmetrical U_n and V_n functions are reduced to the hyperbolic Fibonacci and Lucas $m-$functions [18]:

<u>Hyperbolic Fibonacci *m*-sine</u>

$$sF_m(x)=\frac{\alpha^x-\alpha^{-x}}{\sqrt{m^2+4}}=\frac{1}{\sqrt{m^2+4}}\left[\left(\frac{m+\sqrt{m^2+4}}{2}\right)^x-\left(\frac{m+\sqrt{m^2+4}}{2}\right)^{-x}\right] \tag{30}$$

<u>Hyperbolic Fibonacci *m*-cosine</u>

$$cF_m(x)=\frac{\alpha^x+\alpha^{-x}}{\sqrt{m^2+4}}=\frac{1}{\sqrt{m^2+4}}\left[\left(\frac{m+\sqrt{m^2+4}}{2}\right)^x+\left(\frac{m+\sqrt{m^2+4}}{2}\right)^{-x}\right] \tag{31}$$

<u>Hyperbolic Lucas *m*-sine</u>

$$sL_m(x)=\alpha^x-\alpha^{-x}=\left(\frac{m+\sqrt{m^2+4}}{2}\right)^x-\left(\frac{m+\sqrt{m^2+4}}{2}\right)^{-x} \tag{32}$$

<u>Hyperbolic Lucas *m*-cosine</u>

$$cL_m(x)=\alpha^x+\alpha^{-x}=\left(\frac{m+\sqrt{m^2+4}}{2}\right)^x+\left(\frac{m+\sqrt{m^2+4}}{2}\right)^{-x} \tag{33}$$

The Fibonacci m-numbers $F_m(n)$ and the Lucas m-numbers $L_m(n)$ are determined identically by the hyperbolic Fibonacci and Lucas m-functions (30)-(33) as follows:

$$F_m(n) = \begin{cases} sF_m(n), & for\ n = 2k \\ cF_m(n), & for\ n = 2k+1 \end{cases}$$
$$L_m(n) = \begin{cases} sL_m(n), & for\ n = 2k \\ cL_m(n), & for\ n = 2k+1 \end{cases} \tag{34}$$

As is proved in [18], for the case $m_e = e - \frac{1}{e} \approx 2.35040238$, the classical hyperbolic functions (15) are connected with the hyperbolic Lucas m-functions (32) and (33) as follows:

$$sh(x) = \frac{sL_m(x)}{2} \text{ and } ch(x) = \frac{cL_m(x)}{2}. \tag{35}$$

2.2. *Recursive properties of the symmetrical hyperbolic U_n and V_n functions*

Now, we can give some properties of the symmetrical hyperbolic U_n and V_n functions.

Proposition 1 *(Recursive relation).*

$$sUs(x+2) = p\,cUs(x+1) + q\,sUs(x),$$
$$cUs(x+2) = p\,sUs(x+1) + q\,cUs(x).$$

Proof. We prove the first identity. From the definitions of symmetrical hyperbolic U_n functions, we have

$$\begin{aligned} p\,cUs(x+1) + q\,sUs(x) &= p\left(\frac{\alpha^{x+1} + q^{x+1}\alpha^{-x-1}}{\sqrt{p^2+4q}}\right) + q\left(\frac{\alpha^x - q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right) \\ &= \frac{\alpha^x(p\alpha+q) - q^{x+1}\alpha^{-x}(1-\frac{p}{\alpha})}{\sqrt{p^2+4q}} \\ &= \frac{\alpha^{x+2} - q^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}} \\ &= sUs(x+2) \end{aligned}$$

Proposition 2 *(Recursive relation).*

$$sVs(x+2) = p\,cVs(x+1) + q\,sVs(x),$$
$$cVs(x+2) = p\,sVs(x+1) + q\,cVs(x).$$

Proposition 3 *(Cassini's identity).*

$$(sUs(x))^2 - cUs(x+1)cUs(x-1) = -q^{-x-1}$$

$$(cUs(x))^2 - sUs(x+1)sUs(x-1) = q^{-x-1}$$

Proof. Noting by (LHS) the left hand side of the first identity, we have

$$(LHS) = \frac{(\alpha^x - q^x\alpha^{-x})^2 - (\alpha^{x+1} + q^{x+1}\alpha^{-x-1})(\alpha^{x-1} + q^{x-1}\alpha^{-x+1})}{\left(\sqrt{p^2+4q}\right)^2}$$

$$= \frac{-q^{x-1}\left(\alpha^2 + q^2\alpha^{-2} + 2q\right)}{p^2+4q}$$

$$= -q^{x-1}$$

The other some properties of symmetrical hyperbolic U_n and V_n functions listed in the following table.

The identities for U_n and V_n numbers	The identities for the symmetrical Hyperbolic U_n and V_n functions
$U_n = -(-q)^n U_{-n}$	$sUs(x) = -q^x sUs(-x)$ $cUs(x) = q^x cUs(-x)$
$V_n = (-q)^n V_{-n}$	$sVs(x) = -q^x sVs(-x)$ $cVs(x) = q^x cVs(-x)$
$V_n = U_{n+1} + qU_{n-1}$	$cVs(x) = cUs(x+1) + q\,cUs(x-1)$ $sVs(x) = sUs(x+1) + q\,sUs(x-1)$
$2U_{n+1} = pU_n + V_n$	$2sUs(x+1) = p\,cUs(x) + sVs(x)$ $2cUs(x+1) = p\,sUs(x) + cVs(x)$
$V_{2n} = V_n^2 - 2(-q)^n$	$cVs(2x) = [sVs(x)]^2 + 2q^x$ $cVs(2x) = [cVs(x)]^2 - 2q^x$
$U_{2n} = U_n V_n$	$sUs(2x) = sUs(x)cVs(x)$ $cUs(2x) = cUs(x)sVs(x)$

2.3. *Hyperbolic properties of the symmetrical hyperbolic* U_n and V_n *functions*

The symmetrical hyperbolic U_n and V_n functions have properties that are similar to the classical hyperbolic functions. Now, we give some hyperbolic properties of the symmetrical hyperbolic U_n and V_n functions.

Proposition 4 *(Pythagorean Theorem). The main property of the symmetrical hyperbolic* U_n and V_n *functions is*

$$[cUs(x)]^2 - [sUs(x)]^2 = 4q^x(p^2+4q)^{-1}$$

$$[cVs(x)]^2 - [sVs(x)]^2 = 4q^x.$$

Proposition 5 *(Sum and Difference).*

$$2\left(\sqrt{p^2+4q}\right)^{-1} cUs(x+y) = cUs\, cUs(y) + sUs(x)\, sUs(y)$$

$$2\left(\sqrt{p^2+4q}\right)^{-1} cUs(x-y) = cUs\, cUs(y) - sUs(x)\, sUs(y)$$

$$2\left(\sqrt{p^2+4q}\right)^{-1} sUs(x+y) = sUs\, cUs(y) + cUs(x)\, sUs(y)$$

$$2\left(\sqrt{p^2+4q}\right)^{-1} sUs(x-y) = sUs\, cUs(y) - cUs(x)\, sUs(y)$$

Proof. We prove the first identity.

$$\begin{aligned} cUs\, cUs(y) + sUs(x)\, sUs(y) &= \left(\frac{\alpha^x + q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right)\left(\frac{\alpha^y + q^y\alpha^{-y}}{\sqrt{p^2+4q}}\right) \\ &\quad + \left(\frac{\alpha^x - q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right)\left(\frac{\alpha^y - q^y\alpha^{-y}}{\sqrt{p^2+4q}}\right) \\ &= \frac{2\alpha^{x+y} + 2q^{x+y}\alpha^{-x-y}}{p^2+4q} \\ &= 2\left(\sqrt{p^2+4q}\right)^{-1} cUs(x+y) \end{aligned}$$

Taking $x = y$ in the first and third identity of the previous formulas, we can give following corollary.

Corollary 1 *(Double argument)*

$$2\left(\sqrt{p^2+4q}\right)^{-1} cUs(2x) = [cUs(x)]^2 + [sUs(x)]^2$$

$$\left(\sqrt{p^2+4q}\right)^{-1} sUs(2x) = sUs(x)\, cUs(x)$$

Proposition 6 *(nth derivatives)*

$$[cUs(x)]^{(n)} = \begin{cases} (\ln\alpha)^n\, sUs(x) + \dfrac{(\ln\frac{q}{\alpha})^n + (\ln\alpha)^n}{\sqrt{p^2+4q}} q^x\alpha^{-x}, & for\ \ n\ \ odd \\[2ex] (\ln\alpha)^n\, cUs(x) + \dfrac{(\ln\frac{q}{\alpha})^n - (\ln\alpha)^n}{\sqrt{p^2+4q}} q^x\alpha^{-x}, & for\ \ n\ \ odd \end{cases}$$

$$[sUs(x)]^{(n)} = \begin{cases} (\ln\alpha)^n cUs(x) - \dfrac{(\ln\frac{q}{\alpha})^n + (\ln\alpha)^n}{\sqrt{p^2+4q}} q^x\alpha^{-x}, & \textit{for } n \textit{ odd} \\ (\ln\alpha)^n sUs(x) - \dfrac{(\ln\frac{q}{\alpha})^n - (\ln\alpha)^n}{\sqrt{p^2+4q}} q^x\alpha^{-x}, & \textit{for } n \textit{ odd} \end{cases}$$

Proposition 7 *(Moivre's equation)*

$$[cUs(x) \pm sUs(x)]^{(n)} = \left(2\left(\sqrt{p^2+4q}\right)^{-1}\right)^{n-1} [cUs(nx) \pm sUs(nx)],$$

$$[cVs(x) \pm sVs(x)]^{(n)} = 2^{n-1}[cVs(nx) \pm sVs(nx)].$$

It should be noted that in the case $p = q = 1$, U_n and V_n numbers would be Fibonacci and Lucas numbers. Therefore these symmetrical hyperbolic U_n and V_n functions generalize the hyperbolic Fibonacci and Lucas functions.

3. The Quasi-sine U_n and V_n Functions

It's possible to insert some continuous functions that takes the values -1 and 1 in the discrete points (x = 0, ±1, ±2,...) that correspondence to the alternating sequence $(-1)^n$ in Binet's formula. The trigonometric function $\cos(\pi x)$ is the simplest. For this reason, we introduce new continuous functions that are associated with the second order recurrence sequences.

Definition 2 *The following continuous functions are called the quasi-sine U_n and V_n functions, respectively.*

$$UU(x) = \frac{\alpha^x - \cos(\pi x) q^x \alpha^{-x}}{\sqrt{p^2+4q}}, \quad VV(x) = \alpha^x + \cos(\pi x) q^x \alpha^{-x}. \tag{36}$$

Note that, taking $x = n$ in (36), we have

$$UU(n) = \frac{\alpha^n - \cos(\pi n) q^n \alpha^{-n}}{\sqrt{p^2+4q}} = U_n,$$

and

$$VV(n) = \alpha^n + \cos(\pi n) q^n \alpha^{-n} = V_n$$

where n = 0, ±1, ±2,....

For $p = q = 1$ in (36), this definition is transformed to the quasi-sine Fibonacci and Lucas functions. The graphs and more information of the quasi-sine Fibonacci and Lucas functions are given in [15]. Taking $p = 2, q = 1$ in (36), this definition is transformed into quasi-sine Pell and

Pell-Lucas functions. The graphs of the quasi-sine Pell and Pell-Lucas functions are given in Figure 1 and 2.

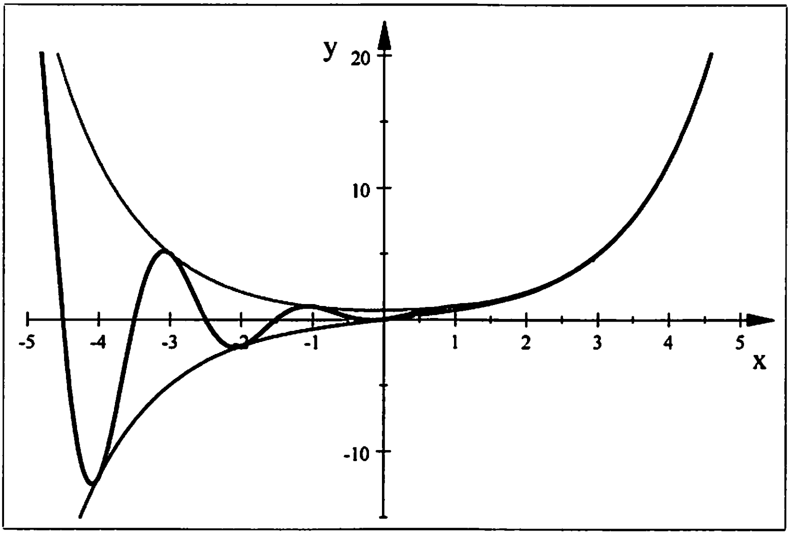

Fig.1. Hyperbolic Pell sine, cosine and quasi-sine Pell functions

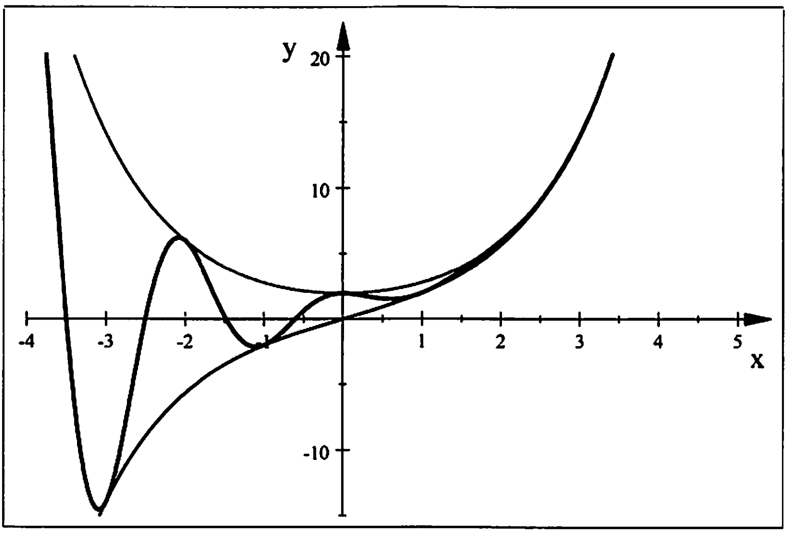

Fig.2. Hyperbolic Pell-Lucas sine, cosine and quasi-sine Pell-Lucas functions

3.1. *Recursive properties of the quasi-sine U_n and V_n functions*

Proposition 8 *(Recursive relation)*

$$UU(x+2)=pUU(x+1)+qUU(x),$$
$$VV(x+2)=pVV(x+1)+qVV(x).$$

Proposition 9 *(Cassini's identity)*

$$[UU(x)]^2-UU(x+1)UU(x-1)=-q^{x-1}\cos(\pi x).$$

We can easily obtain some properties of the quasi-sine U_n *and* V_n functions as following;

The identities for U_n *and* V_n numbers	The identities for the quasi-sine U_n *and* V_n functions
$V_{2n}=V_n^2-2(-q)^n$	$VV(2x)=[VV(x)]^2-2q^x\cos(\pi x)$
$U_{2n}=U_nV_n$	$UU(2x)=UU(x)VV(x)$
$U_{n+1}+qU_{n-1}=V_n$	$UU(x+1)+qUU(x-1)=VV(x)$

Also, the quasi-sine U_n *and* V_n functions have properties that are similar to the classical hyperbolic functions.

It is known that, the three-dimensional Fibonacci spiral is defined in [15] as follows,

$$CFF(x)=\frac{\alpha^x-\cos(\pi x)\alpha^{-x}}{\sqrt{5}}+i\frac{\sin(\pi x)\alpha^{-x}}{\sqrt{5}}.$$

Our purpose is generalized of the three-dimensional Fibonacci spiral for the second order recurrence sequences. $CFF(x)$ function has properties of Fibonacci numbers. Therefore, we can define the three-dimensional spiral for U_n sequence.

Definition 3 *The following complex valued function is called the three-dimensional U_n spiral*

$$CUU(x)=\frac{\alpha^x-\cos(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}+i\frac{\sin(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}, \tag{37}$$

where α is the positive root of the equation (3).

For example, the three-dimensional Pell spiral is

$$CPP(x)=\frac{\alpha^x-\cos(\pi x)\alpha^{-x}}{2\sqrt{2}}+i\frac{\sin(\pi x)\alpha^{-x}}{2\sqrt{2}}.$$

Proposition 10 *(Recursive relation).*

$$CUU(x+2)=pCUU(x+1)+qCUU(x)$$

Proof. Let us note by (RHS) the right hand side of the identity to prove.

$$(RHS) = p\left(\frac{\alpha^{x+1} - \cos(\pi x+\pi)q^{x+1}\alpha^{-x-1} + i\sin(\pi x+\pi)q^{x+1}\alpha^{-x-1}}{\sqrt{p^2+4q}}\right)$$

$$+q\left(\frac{\alpha^x - \cos(\pi x)q^x\alpha^{-x} + i\sin(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right)$$

$$=\frac{\alpha^x\alpha^2 + \cos(\pi x)q^{x+1}\alpha^{-x}(\frac{-q}{\alpha^2}) + i\sin(\pi x)q^{x+1}\alpha^{-x}(\frac{q}{\alpha^2})}{\sqrt{p^2+4q}}$$

$$=\frac{\alpha^{x+2} - \cos(\pi x+2\pi)q^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}} + i\frac{\sin(\pi x+2\pi)q^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}}$$

$$=CUU(x+2)$$

It's clear that, we can easily show that the other recurrence properties of three-dimensional U_n spiral

Selecting the real and imaginary parts in the three-dimensional U_n spiral in (37), we have

$$\mathrm{Re}(CUU(x)) = \frac{\alpha^x - \cos(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}, \tag{38}$$

and

$$\mathrm{Im}(CUU(x)) = \frac{\sin(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}. \tag{39}$$

From (38) and (39), we obtain the following equation systems.

$$y(x) - \frac{\alpha^x}{\sqrt{p^2+4q}} = \frac{-\cos(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}$$

$$z(x) = \frac{\sin(\pi x)q^x\alpha^{-x}}{\sqrt{p^2+4q}}.$$

Let us square both expression of the equation systems and add them. We obtain

$$\left(y - \frac{\alpha^x}{\sqrt{p^2+4q}}\right)^2 + z^2 = \left(\frac{q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right)^2. \tag{40}$$

This formula can be represented in the following form

$$z^2 = [cUs(x) - y][y - sUs(x)],$$

where $cUs(x)$ and $sUs(x)$ are the symmetric hyperbolic U_n cosine and sine, respectively.

Equation (40) corresponds to a surface, which have been called *Metallic Shofar* (see [10]). In the case $p=m, q=1$ the Metallic Shofar is expressed as follows:

$$\left(y-\frac{\alpha^x}{\sqrt{m^2+4}}\right)^2+z^2=\left(\frac{\alpha^{-x}}{\sqrt{m^2+4}}\right)^2,$$

where α is the golden $(m,1)$ – proportion.

Particular cases are:

- If $m=1$, we obtain the classical Golden Shofar with equations

$$\left(y-\frac{\tau^x}{\sqrt{5}}\right)^2+z^2=\left(\frac{\tau^{-x}}{\sqrt{5}}\right)^2,$$

where τ is the golden mean (see [15]).

- If $m=2$, we obtain the Silver Shofar with equation

$$\left(y-\frac{\alpha^x}{\sqrt{8}}\right)^2+z^2=\left(\frac{\alpha^{-x}}{\sqrt{8}}\right)^2,$$

where α is the silver mean (see [10]).

- If $m=3$, we have Bronze Shofar with equation

$$\left(y-\frac{\alpha^x}{\sqrt{13}}\right)^2+z^2=\left(\frac{\alpha^{-x}}{\sqrt{13}}\right)^2,$$

where α is the bronze mean (see [10]).

4. Conclusion

The discovery of Lobachevski's geometry became an epoch-making event in the development not only mathematics, but also of science in general. The Great Russian mathematician and academician Kolmogorov appreciated the role of this discovery in the development of mathematics in following words [20]: "... *It is difficult to overrate the importance of the reorganization of the entire warehouse of mathematical thinking, which happened in the 19th century. In this connection, Lobachevsky's geometry was the most significant mathematical discovery at the start of the 19th century. Based upon this geometric insight the belief in the absolute stability of mathematical axioms was overthrown. This allowed creating essentially new and original abstract mathematical theories and, at last, to demonstrate that similar abstract theories can result in wide and more concrete applications.*" After Lobachevski's discovery, the "hyperbolic ideas" started to penetrate widely into various spheres of science. After the promulgation of the special theory of relativity by Einstein in 1905 and its "hyperbolic interpretation," given by Minkowski in 1908, the "hyperbolic ideas" became universally recognized. Thus, a comprehension of the "hyperbolic character" of the processes in the physical world surrounding us became the main result in the development of science during the 19th and 20th centuries.

The mathematical correlations of Lobachevski's geometry are, of course, based upon the classical hyperbolic functions (15). Why did Lobachevski use these functions, introduced by Vincenzo Riccati in the late 18th century, in his geometry? Apparently, Lobachevski understood that these functions provide the best way to model the "hyperbolic character" of his geometry; on the other hand, he was forced to use these functions because the other hyperbolic functions at that moment simply did not exist. It is necessary to note that Lobachevski's geometry, based on classical hyperbolic functions (15), is historically the first "hyperbolic model" of physical space. Lobachevski's geometry and Minkowski's geometry put forward the hyperbolic functions as the basic plan for modern science.

At the end of the 20th century, the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko [13] broke the monopoly of classical hyperbolic functions in modern science. They introduced a new class of hyperbolic functions based on the golden mean. Later, Alexey Stakhov and Boris Rozin introduced the symmetrical hyperbolic Fibonacci and Lucas functions [14]. The approach of Alexey Stakhov, Ivan Tkachenko and Boris Rozin was based on a similarity between the Binet formulas and hyperbolic functions. This approach resulted in the discovery of a new class of hyperbolic functions, *hyperbolic Fibonacci and Lucas functions.*

The hyperbolic Fibonacci and Lucas functions [13, 14] are an expansion of the Fibonacci and Lucas numbers to the continuous domain. There is a direct analogy between the Fibonacci and Lucas number theory and the theory of hyperbolic Fibonacci and Lucas functions because Fibonacci and Lucas numbers coincide with the hyperbolic Fibonacci and Lucas functions at discrete values of the variable x ($x = 0, \pm 1, \pm 2, \pm 3, \ldots$). Besides, every identity for the Fibonacci and Lucas numbers has its continuous analogy in the form of the corresponding identity for the hyperbolic Fibonacci and Lucas functions, and conversely. This outcome is of great significance for the Fibonacci number theory [1, 3, 7, 9] because this theory as if is transformed into the theory of hyperbolic Fibonacci and Lucas functions [13, 14]. Thanks to this approach, the Fibonacci and Lucas numbers became one of the most important numerical sequences of new hyperbolic geometry.

However, perhaps one of the most important steps in the development of the new "hyperbolic models" of nature was made by Alexey Stakhov [18] and Falcon & Plaza [19]. The hyperbolic Fibonacci and Lucas m-functions are a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions introduced by Stakhov and Rozin [14]. They are based on Gazale formulas [11] and extend ad infinitum a number of new hyperbolic models of nature given by (30)-(33). It is difficult to imagine that the set of new hyperbolic functions given by (30)-(33) is infinite! The hyperbolic Fibonacci and Lucas m-functions complete a general theory of hyperbolic functions, started by Johann Heinrich Lambert and Vincenzo Riccati, and open new perspectives for the development of new "hyperbolic ideas" in modern science.

The main importance of the present article for modern science consists of the fact that it generalizes a general theory of hyperbolic functions, which include the classical hyperbolic functions (15), Stakhov and Tkachenko's hyperbolic Fibonacci and Lucas functions [13], Stakhov and Rozin's symmetrical hyperbolic Fibonacci and Lucas functions [14], Stakhov and Falcon & Plaza's hyperbolic Fibonacci and Lucas m – functions [18, 19] as partial cases. Thus, this article extends infinitely a number of new hyperbolic models of nature. The development of a general theory of hyperbolic functions, stated in the present article, gives us the opportunity to put forward the following unusual hypothesis. Apparently, we can assume that theoretically there are an infinite number of "hyperbolic models of Nature," which correspond to a general class of the hyperbolic functions given by (20)-(23) and (24)-(27). By studying the models of physical phenomenon, researcher may select from the hyperbolic functions (20)-(23) and (24)-(27) some concrete kind of hyperbolic functions, which are adequate to this physical phenomenon.

A new geometric theory of phyllotaxis developed by the Ukrainian researcher Oleg Bodnar [8] demonstrates that the "golden" hyperbolic world exists objectively and independently of our consciousness. This "golden" hyperbolic world is based on the hyperbolic Fibonacci and Lucas functions and persistently appears in nature, in particular, in pine cones, pineapples, cacti, and heads of sunflowers and baskets of various flowers in the form of Fibonacci and Lucas spirals on the surface of these botanical objects. However, the promulgation of the new geometrical theory of phyllotaxis, made by Oleg Bodnar [8], demonstrated that in addition to "Lobachevski's geometry," nature also uses other variants of the so-called "hyperbolic models of nature." The use of hyperbolic Fibonacci and Lucas functions in Bodnar's geometry [8] allowed to solve the "riddle of phyllotaxis," and to explain, how Fibonacci and Lucas spirals appear on the surface of phyllotaxis objects. Bodnar's geometry [8] gives a hope that one can be created other variants of hyperbolic geometries based on the hyperbolic functions developed in the present article.

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