Some Davenport Constants with Weights and Adhikari & Rath's Conjecture

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Abstract

Let $n \in \mathbb{N}$ and let $A \subseteq \mathbb{Z}_n$ be such that A does not contain 0 and it is non-empty. $E_A(n)$ is defined to be the least $t \in \mathbb{N}$ such that for all sequences $(x_1, \ldots, x_t) \in \mathbb{Z}^t$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}$, $1 \leq j_1 < \cdots < j_n \leq t$ and $(\vartheta_1, \cdots, \vartheta_n) \in A^n$ with $\sum_{i=1}^{n} \vartheta_i x_{j_i} \equiv 0 \pmod{n}$. Similarly, for any such set A, we define the Davenport Constant of $\mathbb{Z}_n$ with weight A denoted by $D_A(n)$ to be the least natural number k such that for any sequence $(x_1, \cdots, x_k) \in \mathbb{Z}^k$, there exist a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $(a_1, \cdots, a_l) \in A^l$ such that $\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}$. Das Adhikari and Rath conjectured that for any set $A \subseteq \mathbb{Z}_n \setminus \{0\}$, the equality $E_A(n) = D_A(n) + n - 1$ holds. In this note, we determine some Davenport constants with weights and also prove that the conjecture holds in some special cases.

1. Introduction

Let G be an additive finite abelian group, a finite sequence

$$S = (g_1, g_2, \cdots, g_l) = g_1 g_2 \cdots g_l$$

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of elements of S, where the repetition of elements is allowed and their order is disregarded, is simply called a zero-sum sequence if $g_1 + g_2 + \cdots + g_l = 0$. For any integer n such that $1 \le n \le l$, we denote

$$\sum_n(S) = \{g_{i_1} + g_{i_2} + \cdots + g_{i_n} \mid 1 \le i_1 < i_2 < \cdots < i_n \le l\}.$$

We are interested here in certain generalizations of two important combinatorial invariants related to zero-sum problems (for detailed accounts one may see [9, 3, 15, 8]) in finite abelian groups.

For G a finite abelian group of cardinality n, the Davenport constant $D(G)$ is the smallest natural number t such that any sequence of t elements in G has a non-empty zero-sum subsequence; another interesting constant $E(G)$ is the smallest natural number k such that any sequence of k elements in G has a zero-sum subsequence of length n.

The following result due to Gao [7] (see also [9], Proposition 5.7.9) connects these two invariants.

Theorem 1. *If G is a finite abelian group of order n, then $E(G) = D(G) + n - 1$.*

For the particular group $\mathbb{Z}_n$, the following generalization of $E(G)$ has been considered in [2] and [1] recently. Let $n \in \mathbb{N}$ and assume $A \subseteq \mathbb{Z}_n$. Then $E_A(n)$ is the least $t \in \mathbb{N}$ such that for all sequences $(x_1, \ldots, x_t) \in \mathbb{Z}^t$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}$, $1 \le j_1 < \cdots < j_n \le t$ and $(\vartheta_1, \cdots, \vartheta_n) \in A^n$ with

$$\sum_{i=1}^{n} \vartheta_i x_{j_i} \equiv 0 \pmod{n}.$$

To avoid trivial cases, one assumes that the weight set A does not contain 0 and it is non–empty.

Similarly, for any such set $A \subseteq \mathbb{Z}_n \setminus \{0\}$ of weights, we define the Davenport Constant of $\mathbb{Z}_n$ with weight A denoted by $D_A(n)$ to be the least natural number k such that for any sequence $(x_1, \cdots, x_k) \in \mathbb{Z}^k$, there exist a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $(a_1, \cdots, a_l) \in A^l$ such that

$$\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}.$$

In this case we say that the sequence $(x_1, \cdots, x_k)$ has a zero-sum subsequence $(x_{j_1}, \cdots, x_{j_l})$ with weight A.

Thus, for the group $G = \mathbb{Z}_n$, if we take $A = \{1\}$, then $E_A(n)$ and $D_A(n)$ are respectively $E(G)$ and $D(G)$ as defined earlier.

$E_A(n)$ and $D_A(n)$ were studied in [2], [1] and [12]. They got the following results.

Theorem 2.

(i) *If* $A = \{1, -1\}$, *then* $D_A(n) = [\log_2 n] + 1$ *and* $E_A(n) = n + [\log_2 n]$, *where for a real number* x, $[x]$ *denotes the largest integer which is less than or equals to* x.

(ii) *If* $A = \{1, 2 \cdots n-1\}$, *then* $D_A(n) = 2$ *and* $E_A(n) = n + 1$.

(iii) $A = (\mathbb{Z}_n)^* = \{\ a \mid (a, n) = 1\}$, *then* $D_A(n) = 1 + \Omega(n)$, $E_A(n) = n + \Omega(n)$, *where* $\Omega(n)$ *denotes the number of prime factors of* n, *multiplicity included.*

(iv) *Let* p *be a prime and* $A = \{1, 2, \cdots, r\}$, *where* r *is an integer such that* $1 < r < p$, *we have* $D_A(p) = \lceil \frac{p}{r} \rceil$, *where for a real number* x, $\lceil x \rceil$ *denotes the smallest integer which is greater than or equals to* x, $E_A(p) = p - 1 + D_A(p)$.

(v) *Let* p *be a prime and* A *the set of quadratic residues modulo* p. *Then we have* $D_A(p) = 3$, $E_A(p) = p + 2$.

(vi) $D_A(n) + n - 1 \leq E_A(n) \leq 2n - 1$ *for any* $A \subseteq \mathbb{Z}_n \setminus \{0\}$.

In all these above cases, one has $E_A(n) = D_A(n) + n - 1$. It is natural that Adhikari and Rath [2] suggested the following conjecture.

Conjecture. For any set $A \subseteq \mathbb{Z}_n \setminus \{0\}$ of weights, the equality $E_A(n) = D_A(n) + n - 1$ holds.

In this paper, we obtain the following main results.

Theorem 3.

(i) *Let* $n \in \mathbb{N}$ *and* $A = \{1, 2, \cdots, r\}$, *then* $D_A(n) = \lceil \frac{n}{r} \rceil$.

(ii) *Let p be a prime. Then we have $E_A(p) = D_A(p) + p - 1$ for any $A \subseteq \mathbb{Z}_p \setminus \{0\}$.*

(iii) *If $A = \{1, 2\} \subseteq \mathbb{Z}_n \setminus \{0\}$ and n is even, then $D_A(n) = \frac{n}{2}$ and $E_A(n) = n + \frac{n}{2} - 1$.*

(iv) (1) *If $A = \{1, 2, \cdots, \lceil\frac{n}{2}\rceil\} \subseteq \mathbb{Z}_n \setminus \{0\}$, then $D_A(n) = 2$ and $E_A(n) = n + 1$.*

(2) *Let p be a prime. For any $A \subseteq \mathbb{Z}_p \setminus \{0\}$ with $|A| = \lceil\frac{p}{2}\rceil$, we have $D_A(p) = 2$ and $E_A(p) = p + 1$.*

(3) (a) *If n is even, $2^k \parallel n$ and $A_1 = \{\, a \mid a \text{ is odd}, 1 \leq a < n\}$, $A_2 = \{\, a \mid a \text{ is even}, 1 < a < n\}$, then $D_{A_1}(n) = k+1$ and $E_{A_1}(n) = n+k$; $D_{A_2}(n) = 2$ and $E_{A_2}(n) = n + 1$.*

(b) *If n is odd, $A_1 = \{\, a \mid a \text{ is odd}, 1 \leq a < n\}$, $A_2 = \{\, a \mid a \text{ is even}, 1 < a < n\}$, then $D_{A_1}(n) = 3$ and $E_{A_1}(n) = n + 2$; $D_{A_2}(n) = 3$ and $E_{A_2}(n) = n + 2$.*

2. Proof of Theorem 3.

The proof of (i) or (ii) is almost identical to that of Theorem 2 [2].

We need the following theorems.

In 1961, P. Erdős, A. Ginzburg and A. Ziv [6] proved the following classical theorem.

The EGZ Theorem. If S is a sequence of elements from $\mathbb{Z}_n$ of length $2n - 1$, then $0 \in \sum_n(S)$.

The following famous theorem was originally proved by Cauchy [4], and later independently re-derived by Davenport [5].

The Cauchy-Davenport Theorem. If A_1, A_2, $\cdots$, A_n are a collection of nonempty subsets of $\mathbb{Z}_p$ with p prime, then

$$|\sum_{i=1}^{n} A_i| \geq \min\{p, \sum_{i=1}^{n} |A_i| - n + 1\}.$$

For any subset A of an abelian group G and let $H(A)$ denote the maximal subgroup of G such that $A + H(A) = A$. The following theorem is a classical theorem of Kneser [11].

Kneser's Theorem. Let G be a finite abelian group. Suppose that $A_1, A_2, \cdots, A_n$ are a collection of nonempty subsets of G. Then

$$|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i + H| - (n-1)|H|,$$

where $H = H(\sum_{i=1}^{n} A_i)$.

Grynkiewicz [10] gave several statements which are equivalent to Kneser's Theorem. What we state below is one of them and we will use it afterwards.

Grynkiewicz's Theorem. Let G be a finite abelian group. Let A_1, A_2, A_3 be a collection of nonempty subsets of G and

$$\sum_{i=1}^{3} |A_i| \geq |G| + |H(\sum_{i=1}^{3} A_i)| + 1.$$

Then $\sum_{i=1}^{3} A_i = G$.

Proof of Theorem 3.

(i) Consider any sequence $S = (s_1, \cdots, s_{\lceil \frac{n}{r} \rceil})$ of elements of $\mathbb{Z}_n$ of length $\lceil \frac{n}{r} \rceil$. Considering the sequence

$$S' = (\overbrace{s_1, s_1, \cdots, s_1}^{r \text{ times}}, \overbrace{s_2, s_2, \cdots, s_2}^{r \text{ times}}, \cdots, \overbrace{s_{\lceil \frac{n}{r} \rceil}, \cdots, s_{\lceil \frac{n}{r} \rceil}}^{r \text{ times}}),$$

obtained from S by repeating each element r times, and observing that the length of this sequence is $\geq n$, it follows that

$$D_A(n) \leq \left\lceil \frac{n}{r} \right\rceil. \tag{1}$$

On the other hand, considering the sequence

$$(\overbrace{1, 1, \cdots, 1}^{(\lceil \frac{n}{r} \rceil - 1) \text{ times}}),$$

for any non-empty subsequence $(s_{j_1}, \cdots, s_{j_l})$ of this sequence and $(a_1, \cdots a_l) \in A^l$, we have

$$0 < \sum_{i=1}^{l} a_i s_{j_i} \leq rl = r(\left\lceil \frac{n}{r} \right\rceil - 1) \leq n - 1.$$

Therefore,

$$D_A(n) \geq \left\lceil \frac{n}{r} \right\rceil. \tag{2}$$

From equations (1) and (2), the result follows.

(ii) Now, consider any sequence $S = (s_1, \cdots, s_N)$ of elements of $\mathbb{Z}_p$ of length $N = p - 1 + D_A(p)$. $A = \{a_1, a_2, \cdots, a_r\} \subseteq \mathbb{Z}_p \setminus \{0\}$ for $r \geq 2$.

Case 1. (The sequence S has at least p non-zero elements in it).

Let $(s_{i_1}, s_{i_2}, \cdots, s_{i_p})$ be a subsequence of S of p non-zero elements and let

$$A_k = \{a_1 s_{i_k}, a_2 s_{i_k}, \cdots, a_r s_{i_k}\}$$

for $k = 1, \cdots, r$. Since $|A_k| \geq 2$ for all k, by the Cauchy-Davenport Theorem it follows that

$$|A_1 + A_2 + \cdots + A_p| \geq \min\{p, \sum_{k=1}^{p} |A_k| - p + 1\} = p,$$

and hence

$$\sum_{k=1}^{p} a'_k s_{i_k} = 0, \text{ where } a'_k \in \{a_1, a_2, \cdots, a_r\} \subseteq A.$$

Case 2. (The sequence S has less than p non-zero elements in it).

In this case, at least $D_A(p)$ elements of the sequence are equal to zero. We reorder the sequence in such a way that $s_1 = s_2 = \cdots = s_t = 0$ and the remaining elements are non-zero. We have $N - t < p$. Let

$$B = \{r_1, \ldots, r_l\} \subseteq \{t + 1, t + 2, \cdots, N\}$$

be maximal with respect to the property that there exist $a_1, \cdots, a_l \in A = \{a_1, a_2, \cdots, a_r\}$ with

$$\sum_{j=1}^{l} a_j s_{r_j} = 0.$$

Now we claim that $l+t \geq p$. Indeed, if this were not the case then the set $C = \{t+1, \cdots, N\} \setminus \{r_1, \cdots, r_l\}$ would contain $N-t-l \geq D_A(p)$ elements. Hence by the definition of $D_A(p)$, there would exist a non-empty $B' \subseteq C$ and $a_j \in A = \{a_1, a_2, \cdots, a_r\}$ for each $j \in B'$ such that

$$\sum_{j \in B'} a_j s_j = 0.$$

Now, $B \cup B'$ would contradict the maximality of B. Hence $l+t \geq p$. Therefore, appending the sequence B to $(s_1, s_2, \cdots, s_{p-l}) = (0, 0, \cdots, 0)$, we get a sequence of length p with desired property.

(iii) From (i), we know that $D_A(n) = 2$. So we will prove that $E_A(n) = n + \frac{n}{2} - 1$.

Let $S = (s_1, \cdots, s_N)$ be a sequence of elements in $\mathbb{Z}_n$ of length $N = n + \frac{n}{2} - 1$. It suffices to prove that S has a zero-sum subsequence of length n with weight A. Considering S in $\mathbb{Z}_{\frac{n}{2}}$, by the EGZ Theorem we know that S has a zero-sum subsequence of length $\frac{n}{2}$. Without loss of generality(w.l.o.g.), assume that $T_1 = \prod_{i=1}^{\frac{n}{2}} s_i$ is a zero-sum subsequence of length $\frac{n}{2}$. Similarly, ST^{-1} has a zero-sum subsequence of length $\frac{n}{2}$. W.l.o.g., assume that $T_2 = \prod_{i=\frac{n}{2}+1}^{n} s_i$ is a zero-sum subsequence of length $\frac{n}{2}$. Clearly, $\sum_{i=1}^{n} 2s_i \equiv 0 \pmod{n}$. It follows that (iii) of Theorem 3 holds.

(iv) (1) From (i), we know that $D_A(n) = 2$. Let $S = (s_1, \cdots, s_{n+1})$ be a sequence of elements in $\mathbb{Z}_n$. To prove $E_A(n) = n+1$, it suffices to prove that S has a zero-sum subsequence of length n with weight A. Set

$$\Gamma = \{s_i \mid as_i = 0,\ a \in A,\ s_i \in S\}.$$

If $\Gamma \neq \emptyset$, then we have $|\Gamma|$ zero-sum subsequences of S of length 1 with weight A. Since $D_A(n) = 2$, arguing as in the proof of case 2 in (ii) we can produce a zero-sum subsequence of S of length n with weight A.

If $\Gamma = \emptyset$, which means $\gcd(s_i, n) = 1$, then $|s_i A| = |A| = \lceil \frac{n}{2} \rceil$ for $i = 1, 2, \cdots, n+1$.

If n is even, then s_i is odd for $i = 1, 2, \cdots, n+1$ and $\frac{n}{2} \in A$. So

$$\sum_{i=1}^{n} s_i \frac{n}{2} \equiv 0 \pmod{n}.$$

If n is odd, then we have $|s_iA| = |A| = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ for $i = 1, 2, \cdots, n+1$. So

$$\sum_{i=1}^{3} |s_iA| = \frac{3(n+1)}{2} \geq n + \frac{n}{3} + 1.$$

By Grynkiewicz's Theorem, we get $\sum_{i=1}^{3} s_iA_i = \mathbb{Z}_n$. Hence

$$\sum_{i=1}^{n} s_iA_i = \mathbb{Z}_n.$$

It follows that $0 \in \sum_{i=1}^{n} s_iA_i$.

(2) One has only to consider the case where p is an odd prime. Let $A = \{a_1, \cdots, a_r\}$, where $0 < a_i \leq p-1$, for $i = 1, \cdots, r$, $r = \lceil \frac{p}{2} \rceil = \frac{p+1}{2}$.

Let $S = (s_1, s_2)$ be a sequence of elements in $\mathbb{Z}_p$. We will show that S has a zero-sum subsequence with weight A. If $s_1 = 0$ or $s_2 = 0$, the result is trivial. Assume that $s_i \neq 0$ for every $i = 1, 2$. Set

$$A_i = s_iA = \{s_ia_1, s_ia_2, \cdots, s_ia_r\}$$

for $i = 1, 2$. By the Cauchy-Davenport Theorem, we have

$$|A_1 + A_2| \geq \min\{p, |A_1| + |A_2| - 1\} = 2\left\lceil \frac{p}{2} \right\rceil - 1 = p,$$

so $0 \in A_1 + A_2$. Hence, we have $D_A(p) \leq 2$. Obviously, $D_A(p) > 1$. Thus $D_A(p) = 2$. And from (ii), we obtain $E_A(p) = p + 1$.

(3) (a) If n is even and $2^k \parallel n$, then $\frac{n}{2^k} \in A_1$, where

$$A_1 = \{1, 3, \cdots, \frac{n}{2^k}, \cdots, n-1\}.$$

First, we prove that $D_{A_1}(n) > k$. Note that

$$\begin{aligned} A_1 =& \{1, 3, \cdots, \frac{n}{2^k}, \cdots, n-1\}, \\ \frac{n}{2}A_1 =& \{\frac{n}{2}\}, \\ \frac{n}{2^2}A_1 =& \{\frac{n}{2^2}, \frac{3n}{2^2}\}, \\ &\cdots\cdots \\ \frac{n}{2^k}A_1 =& \{\frac{n}{2^k}, \frac{3n}{2^k}, \cdots, \frac{(2^k-1)n}{2^k}\}. \end{aligned}$$

We assert that $0 \notin \sum_{i\in I} \frac{n}{2^i} A_1$ with $I \subseteq \{1, 2, \cdots, k\}$. We proceed by induction on the cardinality of I. Note that for $|I| = 1$, the result follows trivially. Inductively, assume that the result holds true for $1 \le |I| < k$. Now consider $|I| = k$. If $0 \in \sum_{i=1}^{k} \frac{n}{2^i} A_1$, then there must exist $a_i \in A_1$ for $i = 1, 2, \cdots, k$ such that

$$\frac{n}{2}a_1 + \frac{n}{2^2}a_2 + \cdots + \frac{n}{2^k}a_k \equiv 0 \pmod{n}.$$

Multiplying both sides of the above equation by 2, we get

$$\frac{n}{2}a_2 + \frac{n}{2^2}a_3 + \cdots + \frac{n}{2^{k-1}}a_k \equiv 0 \pmod{n}.$$

Hence, $0 \in \frac{n}{2}A_1 + \frac{n}{2^2}A_1 + \cdots + \frac{n}{2^{k-1}}A_1$, a contradiction to the induction hypothesis.

Next, we prove that $D_{A_1}(n) \le k+1$. Let $S = (s_1, \cdots, s_N)$ be a sequence of elements in $\mathbb{Z}_n$ of length $N = k+1$. We will prove that S has a zero-sum subsequence with weight A_1. Consider the sequence of $2^{k+1} - 1$ integers

$$\left(\sum_{i\in I} s_i\right)_{\emptyset \ne I \subseteq \{1,2,\cdots,k+1\}}$$

that cannot contain distinct integers modulo 2^k. Therefore, there exist $I_1, I_2 \subseteq \{1, 2, \cdots, k+1\}$ with $I_1 \ne I_2$ such that

$$\sum_{i\in I_1} s_i \equiv \sum_{i\in I_2} s_i \pmod{2^k}.$$

We distinguish three cases.

Case 1. If $I_1 \cap I_2 = \emptyset$, then we have

$$\sum_{i\in I_1} s_i(n - \frac{n}{2^k}) + \sum_{i\in I_2} s_i \frac{n}{2^k} \equiv 0 \pmod{n},$$

where $n - \frac{n}{2^k}, \frac{n}{2^k} \in A_1$.

Case 2. If $I_1 \subsetneq I_2$ or $I_2 \subsetneq I_1$, set $I = I_2 \backslash I_1$ or $I = I_1 \backslash I_2$, then $\sum_{i\in I} s_i \equiv 0 \pmod{2^k}$. It follows that

$$\sum_{i\in I} s_i \frac{n}{2^k} \equiv 0 \pmod{n},$$

where $\frac{n}{2^k} \in A_1$.

Case 3. If $I_1 \cap I_2 \neq \emptyset$, $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$, then

$$\sum_{i \in I_1 \setminus I_1 \cap I_2} s_i \equiv \sum_{i \in I_2 \setminus I_1 \cap I_2} s_i \pmod{2^k}.$$

It reduces to Case 1. Thus, we prove that $D_{A_1}(n) = k+1$.

Last, we will prove that $E_{A_1}(n) = n+k$. Assume that $S = (s_1, \cdots, s_{N'})$ is a sequence of elements in $\mathbb{Z}_n$ of length $N' = n+k$. To prove $E_{A_1}(n) = n+k$, it suffices to prove that S has a zero-sum subsequence of length n with weight A_1 because of Theorem 2(vi). We partition S into the following multisets (sets with repetitions allowed):

$$M_i = \{s_j \mid 2^i \parallel s_j,\ s_j \in S\}, \text{ for } i = 0, 1, 2, \cdots, k.$$

Note that every pair of elements $s_i^{(1)}$, $s_i^{(2)}$ in M_i constitutes a zero-sum subsequence of S with weight A_1 since

$$s_i^{(1)} \cdot \frac{s_i^{(2)}}{2^i} + s_i^{(2)}\left(n - \frac{s_i^{(1)}}{2^i}\right) \equiv 0 \pmod{n},$$

where $\frac{s_i^{(2)}}{2^i}$, $n - \frac{s_i^{(1)}}{2^i} \in A_1$, for $i = 0, 1, \cdots, k-1$.

While every element s_k' in M_k produces a zero-sum subsequence of S of length 1 with weight A_1 since $s_k' \frac{n}{2^k} \equiv 0 \pmod{n}$, $\frac{n}{2^k} \in A_1$. So we can choose n elements from M_i for $i = 0, 1, \cdots, k$, say, $s_1, s_2, \cdots, s_n$ and n elements $a_i \in A_1$ such that

$$\sum_{i=1}^{n} s_i a_i \equiv 0 \pmod{n}.$$

This completes the proof of $E_{A_1}(n) = n+k$.

Now we consider the case of A_2.

If $n \equiv 0 \pmod 4$, then $\frac{n}{2} \in A_2$. Note that $s \cdot \frac{n}{2} \equiv 0 \pmod{n}$ when s is even and $(s_1 + s_2) \cdot \frac{n}{2} \equiv 0 \pmod{n}$ when s_1 and s_2 are both odd. So from the definition of $D_A(n)$, $E_A(n)$ and Theorem 2(vi), it is easy to deduce that $D_{A_2}(n) = 2$ and $E_{A_2}(n) = n+1$.

If $n \equiv 2 \pmod 4$, then $\frac{n}{2} \notin A_2$. For arbitrary sequence $S = s_1 s_2$ of elements in $\mathbb{Z}_n$, assume that $\gcd(s_1, n) = 2^i d_1$ and $\gcd(s_2, n) = 2^j d_2$ where $0 \le i, j \le 1$, $2 \nmid d_1$, $2 \nmid d_2$.

If $d_1 > 1$ or $d_2 > 1$, then it is easy to see that $0 \in s_1 A_2$ or $0 \in s_2 A_2$.

If $d_1 = d_2 = 1$, then $s_1 A_2 = s_2 A_2 = A_2$. Thus $0 \in s_1 A_2 + s_2 A_2 = A_2 + A_2$.

So arguing as above, it is easy to deduce that $D_{A_2}(n) = 2$ and $E_{A_2}(n) = n + 1$.

(b) If n is odd .

First, we consider the cases of A_1. For the sequence $(1, 1)$, note that $0 \notin A_1 + A_1$, it follows that $D_{A_1}(n) > 2$.

For arbitrary sequence $S = s_1 s_2 s_3$ of elements in $\mathbb{Z}_n$, if there exists one of elements, say s_1, such that $\gcd(s_1, n) \neq 1$, then we have $0 \in s_1 A_1$ because of $\frac{n}{\gcd(s_1, n)} \in A_1$. In this case S has a zero-sum subsequence of length 1 with weight A_1.

If $\gcd(s_i, n) = 1$, for $i = 1, 2, 3$, then

$$|s_1 A_1| + |s_2 A_1| + |s_3 A_1| = \frac{3(n-1)}{2} \ge n + \frac{n}{3} + 1$$

when $n \ge 15$. Hence, by Grynkiewicz's Theorem, we have

$$s_1 A_1 + s_2 A_1 + s_3 A_1 = \mathbb{Z}_n$$

when $n \ge 15$. So S has a zero-sum subsequence of length 3 with weight A_1. Clearly, $D_{A_1}(n) = 3$ when $n \ge 15$.

When $n \le 13$ and $n = p$ is prime, by the Cauchy-Davenport Theorem we have

$$|s_1 A_1 + s_2 A_1| \ge \min\{p, |s_1 A_1| + |s_2 A_1| - 1\} = p - 2.$$

Thus

$$s_1 A_1 + s_2 A_1 + s_3 A_1 = \mathbb{Z}_n.$$

Hence, $D_{A_1}(n) = 3$ when $n \le 13$ and n is prime.

When $n = 9$, we have $A_1 = \{1,3,5,7\}$ and $s_i \in \{1,2,4,5,7,8\}$ for $i = 1,2,3$. Then, it is not difficult to check case by case that $s_1A_1 + s_2A_1 + s_3A_1 = \mathbb{Z}_9$. Thus, we have $D_{A_1}(9) = 3$. In sum, we get $D_{A_1}(n) = 3$ when $n > 1$.

We will prove that $E_{A_1}(n) = n + 2$. Assume that $S = (s_1, \cdots, s_N)$ is a sequence of elements in $\mathbb{Z}_n$ of length $N = n + 2$. It suffices to prove S has a zero-sum subsequence of length n with weight A_1.

If(w.l.o.g.) $\gcd(\mathrm{s_i}, \mathrm{n}) \neq 1$ for $i = 1, 2, \cdots, t$ $(t \geq 2)$ and $\gcd(\mathrm{s_i}, \mathrm{n}) = 1$ for $i = t+1, t+2, \cdots, N$. Arguing as in the proof of case 2 in (ii), it is easy to see that we can choose $I \subseteq \{1, 2, \cdots, N\}$ with $|I| = n$ such that

$$\sum_{i\in I} s_i a_i \equiv 0 \pmod{n},$$

where $a_i \in A_1$ for $i \in I$.

If (w.l.o.g.) $\gcd(\mathrm{s_i}, \mathrm{n}) = 1$ for $i = 1, 2, \cdots, N-1$, arguing as above, we can prove that $0 \in s_1A_1 + s_2A_1 + \cdots + s_nA_1 = \mathbb{Z}_n$. It follows that $E_{A_1}(n) = n + 2$.

Next, we consider the cases of A_2. Arguing as in the proof of case A_1, we can prove that $D_{A_2}(n) = 3$ and $E_{A_2}(n) = n + 2$. This completes the proof. □

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References

[1] S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin and F. Pappalardi, *Contributions to zero-sum problems*, Discrete Math. **306**, 1–10 (2006).

[2] S. D. Adhikari and Purusottam Rath, *Davenport constant with weights and some related questions*,Integers: 6(2006),#A30.

[3] Y. Caro, *Zero-sum problems - A survey*, Discrete Math. **152**, 93–113 (1996).

[4] A. L. Cauchy, *Recherches sur les nombres*, J. Ecôle Polytech. **9**, 99–123 (1813).

[5] H. Davenport, *On the addition of residue classes*, J. London Math. Soc. **22**, 100–101 (1947).

[6] P. Erdős, A. Ginzburg and A. Ziv, *Theorem in the additive number theory*, Bull. Research Council Israel, **10F**, 41–43 (1961).

[7] W. D. Gao, *A combinatorial problem on finite abelian groups*, J. Number Theory, **58**, 100–103 (1996).

[8] W. D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: A survey*, to appear.

[9] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations*, Chapman & Hall, CRC (2006).

[10] David J.Grynkiewicz, *Sumsets, Zero-Sums and Extremal Combinatorics*,Ph.D.Thesis,2005.

[11] M.Kneser, *Ein satz über abelsche gruppen mit anwendungen auf die geometric der zahlen*, Math. Z. **61**, 429–434 .

[12] Florian Luca, *A generalization of a classical zero-sum problem*, Preprint.

[13] Melvyn B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[14] J. -P. Serre, *A course in Arithmetic*, Springer, 1973.

[15] R. Thangadurai, *Interplay between four conjectures on certain zero-sum problems*, Expo. Math. **20**, no. 3, 215–228 (2002).