The bounds of spectral radius of graphs with a given size of independent set *

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Abstract

Let n, k be integers and $k < n$. Denote by $\mathcal{G}_{n,k}$ and $\mathcal{G}'_{n,k}$ the set of graphs of order n with k independent vertices and the set of graphs of order n with k independent edges, respectively. The bounds of the spectral radius of graphs in $\mathcal{G}_{n,k}$ and $\mathcal{G}'_{n,k}$ are obtained.
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1 Introduction

In this paper, we consider connected simple graphs only. Let $A(G)$ be the adjacent matrix of graph G. The spectral radius, $\rho(G)$, of a graph G is the largest eigenvalue of $A(G)$. For the results on the spectral radii of general graphs, the reader is referred to [1–3].When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem(see [4]), the spectral radius of $A(G)$ is simple and has a unique positive eigenvector. We will refer to such an eigenvector as

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the Perron vector of $A(G)$. The following Proposition is a well-known result.

Proposition 1 *Let $G = (V, E)$ be a connected graph, for $x, y \in V(G)$, $G^* = G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, then*

$$\rho(G) \leq \rho(G^*)$$

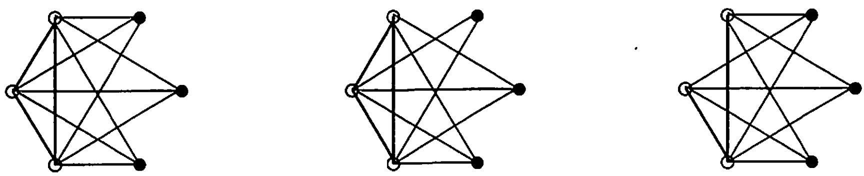
A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. The number of vertices in a maximum independent set of G is called the *vertex independence number* of G. The independent edge set is a set of edges no two of which are adjacent, i.e. a matching. The number of edges in a maximum matching is called the *edge independence number* of G. These definitions can be found in [5].

In [6], Brualdi and Solheid proposed the following problem: *Given a set of graphs, $\mathcal{G}$, find a bound for the spectral radii of graphs in $\mathcal{G}$ and characterize the graph in which the maximal spectral radius is attained.* Some special kinds of graphs have been studied in [8, 9]. In this paper, we study this kind of question for $\mathcal{G}_{n,k}(k < n)$ and $\mathcal{G}'_{n,k}(k < n)$, the set of graphs of order n with k independent vertices and the set of graphs of order n with k independent edges, respectively.

We denote by K_n the complete graph with n vertices, and denote by lK_r the graph of l copies of K_r. Let $H_1 = (V_1, E_1), H_2 = (V_2, E_2)$, the *direct sum* $H_1 \cup H_2$, is the graph $H = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The *complete product* $H_1 \nabla H_2$ of graphs H_1 and H_2 is the graph obtained from $H_1 \cup H_2$ by joining every vertex of H_1 with every vertex of H_2.

2 Notations and Lemmas

First, we introduce some notations. Let $m = n - k$, $G_1 = K_m^k = K_m \nabla (kK_1)$. Obviously, $G_1 \in \mathcal{G}_{n,k}$. We give a partition of $V(G_1)$ and $E(G_1)$. Let $V(G_1) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \cdots, v_k\}$ is the independent vertex set of K_m^k and $V_2 = \{v_{k+1}, \cdots, v_n\}$ is the vertex set of K_m. $E(G_1) = E_1 \cup E_2$, where E_1 denotes the set of edges between V_1 and V_2, and E_2 denotes the set of edges whose ends both inside V_2. We color the edges in E_1 with color red, the edges in E_2 with color blue. Denote by e_r, e_b the red edges and blue edges in G_1, respectively. when $2 \leq k \leq n-2$, we define $G_2 = G_1 - e_r, G_3 = G_1 - e_b$. For $n = 6$, the graphs defined above are shown in Fig.1(in which the blue edges are represented by thick lines and the independent vertices are represented by black dots).



G1 G2 G3

Fig.1

Without loss of generality, in the following discussion, we suppose that the vertices of $G_i (1 \leq i \leq 3)$ are put on a cycle in a counter-clockwise order. $G_2 = G_1 - v_k v_{k+1}$, $G_3 = G_1 - v_{k+1} v_{k+2}$.

Next we give some important results which we use later in this paper.

Lemma 1 *([1]P57) The characteristic polynomial of the complete product of regular graphs H_1 and H_2 is given by the relation:*

$$P_{H_1 \triangledown H_2}(x) = \frac{P_{H_1}(x) P_{H_2}(x)}{(x - r_1)(x - r_2)} [(x - r_1)(x - r_2) - n_1 n_2],$$

where n_i is the order of H_i and r_i is the degree of vertices in $V(H_i)$.

Lemma 2 *[7] Let G be a connected graph and $\rho(G)$ the spectral radius of $A(G)$. Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \cdots, v_s \in N(v) \setminus N(u) (1 \leq s \leq d_v)$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i (1 \leq i \leq n)$. Let G^* be the graph obtained from G by* *deleting the edges* (v, v_i) *and adding the edges* $(u, v_i)(1 \leq i \leq s)$. *If* $x_u \geq x_v$, *then*

$$\rho(G) < \rho(G^*).$$

Lemma 3 *Let* $\rho(G)$ *be the spectral radius of* $A(G)$, $(x_1, x_2, \cdots, x_n)^T$ *be the Perron vector of* $A(G)$, *where* x_i *corresponds to the vertex* $v_i (1 \leq i \leq n)$.
(i) If $d(v_i) = d(v_j) = n - 1$, *then* $x_i = x_j$;
(ii) If $N(v_i) = N(v_j)$, *then* $x_i = x_j$.

Proof. (i) Let $s = \sum_{i=1}^{n} x_i$. By the definition of eigenvalue,

$$\rho\, x_i = \sum_{v_j \ adj \ v_i} x_j \qquad (1 \leq i \leq n) \tag{1}$$

Since $d(v_i) = d(v_j) = n - 1$,

$$\rho\, x_i = s - x_i,$$
$$\rho\, x_j = s - x_j.$$

Hence

$$x_i = x_j = \frac{s}{\rho + 1}.$$

(ii) It's a direct result from (1). □

3 The bound of spectral radius of graphs in $\mathcal{G}_{n,k}$

Theorem 1 *Let* $G \in \mathcal{G}_{n,k}$ *and* $m = n - k$, *then*

$$\rho(G) \leq \frac{m - 1 + \sqrt{(m-1)^2 + 4km}}{2}, \tag{2}$$

the equality holds if and only if $G \cong G_1$.

Proof. Choose $G \in \mathcal{G}_{n,k}$ such that the spectral radius of G is as large as possible. Let $V_1 = \{v_1, v_2, \cdots, v_k\}$ be the set of independent vertices of G. We claim that the induced subgraph $G - V_1$ must be a complete graph K_m, Otherwise, there exists a pair of nonadjacent vertices, say $v_i, v_j (k+1 \le i, j \le n)$. Then we add an edge $v_i v_j$ in G. By Proposition 1, $\rho(G) < \rho(G + v_i v_j)$, a contradiction to the choice of G. By similar argument, each vertex in V_1 must be adjacent to each vertex in $V(K_m)$. Thus, we have shown that for any $G \in \mathcal{G}_{n,k}$, $\rho(G) \le \rho(G_1)$, the equality holds uniquely at G_1.

Next, we will calculate $\rho(G_1)$.

Let $H_1 = K_m, H_2 = kK_1$. Since $P_{H_1}(x) = (x-m+1)(x+1)^{m-1}$, $P_{H_2}(x) = x^k$, from Lemma 1,

$$P_{H_1 \nabla H_2}(x) = x^{k-1}(x+1)^{m-1}[(x-m+1)x - mk].$$

Therefore, the spectral radius ρ of the graph $H_1 \nabla H_2$ satisfies the equation

$$(x-m+1)x - mk = 0.$$

The result follows immediately by solving the above equation. □

If $k = n-1$, G_1 is a star of order n. So Theorem 1 implies the following corollary.

Corollary 1 *Let $K_{1,n-1}$ be a star of order n. then*

$$\rho(K_{1,n-1}) = \sqrt{n-1}.$$

Theorem 2 *Let $2 \le k \le n-2, m = n-k$. For any $G \in \mathcal{G}_{n,k} \setminus \{G_1\}$,*

$$\rho(G) \le \rho(G_2),$$

the equality holds if and only if $G \cong G_2$.

Proof. First, we will show the following facts:

Fact 1. In view of isomorphism, by deleting an edge in G_1, we only get two graph: G_2 and G_3. This is an obvious result.

Fact 2. $\rho(G_3) < \rho(G_2)$.

Proof of Fact 2. Let $V(G_2) = V(G_3) = V(G_1) = \{v_1, v_2, \cdots, v_n\}$. $G_2 = G_1 - v_k v_{k+1}, G_3 = G_1 - v_{k+1} v_{k+2}$. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ is the Perron vector of $A(G_3)$, where x_i corresponds to the vertex $v_i (1 \le i \le n)$.

From Lemma 3, if $N(v_i) = N(v_j)$ or $d(v_i) = d(v_j) = n - 1$, then $x_i = x_j$. Therefore,

$$x_1 = x_2 = \cdots = x_k = a$$
$$x_{k+1} = x_{k+2} = b$$
$$x_{k+3} = x_{k+4} \cdots = x_n = c$$

Using (1), we get

$$\rho a = (m-2)c + 2b, \quad (3)$$
$$\rho b = (m-2)c + ka. \quad (4)$$

By (3),(4),we have

$$\rho a - 2b = \rho b - ka,$$

Hence,

$$\frac{x_{k+2}}{x_k} = \frac{b}{a} = \frac{\rho + k}{\rho + 2} \geq 1.$$

Since $x_i > 0(0 \leq i \leq n)$, we see that $x_{k+2} \geq x_k$. By deleting $v_{k+1}v_k$ and adding $v_{k+1}v_{k+2}$, we obtain G_2, then from Lemma 2

$$\rho(G_3) < \rho(G_2).$$

For any $G \in \mathcal{G}_{n,k} \setminus \{G_1\}$, G is either a subgraph of G_2 or a subgraph of G_3. In both cases, $\rho(G) \leq \rho(G_2)$ is still valid, and if $G \neq G_2$, by Proposition 1, $\rho(G) < \rho(G_2)$. $\square$

Theorem 3 *The spectral radius of the graph G_2 satisfies the equation*

$$\rho^4 - (m-2)\rho^3 - [m(k+1) - 2]\rho^2 - m(k-1)\rho + (k-1)(m-1) = 0$$

Proof. Let $G_2 = G_1 - v_kv_{k+1}$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ is the Perron vector of $A(G_2)$, where x_i corresponds to the vertex $v_i(1 \leq i \leq n)$. Let $s = \sum_{i=1}^{n} x_i$, then From Lemma 3 we have

$$x_1 = \cdots = x_{k-1}$$
$$x_{k+2} = \cdots = x_n = \frac{s}{\rho + 1}$$
$$\rho x_1 = (m-1)x_{k+2} + x_{k+1}$$
$$\rho x_k = (m-1)x_{k+2}$$
$$\rho x_{k+1} = (m-1)x_{k+2} + (k-1)x_1$$

.

Simplify the array of these equations, we get

$$x_1 = x_2 = \cdots = x_{k-1} = \frac{(m-1)s}{\rho^2 - k + 1}$$
$$x_k = \frac{(m-1)s}{\rho(\rho+1)}$$
$$x_{k+1} = \frac{\rho(m-1)s}{\rho^2 - k + 1} - \frac{(m-1)s}{\rho+1}$$
$$x_{k+2} = x_{k+3} = \cdots = x_n = \frac{s}{\rho+1}$$

Recall in mind that $s = \sum_{i=1}^{n} x_i$, the result follows after a simple calculating. □

Corollary 2 *Let ρ be the spectral radius of the graph G_2, then*

$$\rho < \frac{m-2+\sqrt{(m-2)^2+4(m+1)(k+1)+4m-18}}{2}.$$

Proof. By Theorem 3, ρ satisfies the following equation

$$\rho^4 - (m-2)\rho^3 - [m(k+1)-2]\rho^2 - m(k-1)\rho + (k-1)(m-1) = 0.$$

Since $\rho > 0$,

$$\rho^2 - (m-2)\rho - [m(k+1)-2] = \frac{m(k-1)\rho - (k-1)(m-1)}{\rho^2}. \quad (5)$$

It's easy to see that G_2 contains a complete bipartite graph $K_{m,k-1}$ as a subgraph, which implies $\rho > \sqrt{m(k-1)}$. G_2 is not a complete graph, so $\rho < m+k-1$. Recall these facts in mind, and combine the equation (5) , we have

$$\begin{aligned}\rho^2 - (m-2)\rho - [m(k+1)-2] &< \frac{m(k-1)(m+k-1)-(k-1)(m-1)}{m(k-1)} \\ &= m+k-2+\frac{1}{m} \\ &< m+k-\frac{3}{2}.\end{aligned}$$

Hence,

$$\rho^2 - (m-2)\rho - (mk+2m+k-\frac{7}{2}) < 0.$$

Solving this inequality , we obtain the result. □

4 The bound of spectral radius of graphs in $\mathcal{G}'_{n,k}$

The graph $\mathcal{K}_n^l$ is a graph obtained by joining l independent vertices to one vertex of K_{n-l}.

Now we introduce some graphical concepts involving matching. Let M be a matching in $G = (E, V)$. A matching M *saturats* a vertex v, and v is said to be $M-$ *saturated* , if some edge of M is incident with v; otherwise, v is $M-$*unsaturated.* An *M-alternating path* in G is a path whose edges are alternately in $E \setminus M$ and M. An *M-augmenting path* is an M-alternating path whose origin and terminus are M-unsaturated.

Lemma 4 *([5]P70) A matching M in G is a maximum matching if and only if G contains no M-augmenting path.*

Lemma 5 *([8]) Let ρ be the spectral radius of the graph $\mathcal{K}_n^l$, then*

$$\rho < \begin{cases} n-l-1+\dfrac{l}{(n-l)^2-n}, & 1 \le l \le n-1-\sqrt{n-1}; \\ \sqrt{n-1}+\dfrac{n-l-2}{2(\sqrt{n-1}-(n-l-2))}, & n-1-\sqrt{n-1} < l \le n-3. \end{cases} \tag{6}$$

Theorem 4 *Let $G \in \mathcal{G}'_{n,k}, l = n-2k (l \le n-2)$, then*

$$\rho \le \begin{cases} \rho(K_n), & l = 0, 1; \\ \rho(\mathcal{K}_n^l), & l > 1. \end{cases}$$

Proof. If $l = 0, 1$, then $\rho(G) \le \rho(K_n)$ is a clear result. So we assume that $l \ge 2$ next.

Let $G = (V, E) \in \mathcal{G}'_{n,k}$ be a graph with as large spectral radius as possible. $E_1 = \{e_1 = v_1v_2, e_2 = v_3v_4, \cdots, e_k = v_{2k-1}v_{2k}\}$ be a maximum matching of G, $V_1 = \{v_1, v_2, \cdots, v_{2k}\}, V_2 = V \setminus V_1 = \{v_{2k+1}, \cdots, v_n\}$.

First, we will show the following facts:

Fact 1. For any $G \in \mathcal{G}'_{n,k}$, $V_2 = \{v_{2k+1}, \cdots, v_n\}$ is an independent set in G.

Proof of Fact 1. Otherwise, Let $e_{k+1} = v_iv_j$, $2k+1 \le i < j \le n$ is an edge of G, then $E_2 = E_1 \cup \{e_{k+1}\}$ is a matching of G satisfying $|E_2| > |E_1|$, a contradiction.

Fact 2. $G[V_1] = K_{2k}$.

Proof of Fact 2. Otherwise, there exists a pair of nonadjacent vertices, say $v_i, v_j (1 \le i < j \le 2k)$. Then we add an edge v_iv_j in G. By Proposition 1, $\rho(G) < \rho(G + v_iv_j)$, a contradiction to the choice of G.

Fact 3. We denote by $[V_1, V_2]$ the set of edges with one end in V_1 and the other in V_2. Then there is no independent edges in $[V_1, V_2]$.

Proof of Fact 3. Otherwise, suppose uv_i, wv_j are such edges with $v_i, v_j \in V_1, u, w \in V_2$, then $uv_iv_{i-1}v_{j-1}v_jw$ is an E_1-augmenting path in G, from Lemma 4 , E_1 is not a maximum matching. This is contrary to the hypothesis.

Fact 4. For each vertex $u \in V_2$, $d(u) = 1$.

Proof of Fact 4. Otherwise, suppose there's a vertex $u \in V_2$ and $d(u) \ge 2$. From Fact 1, $N(u) \subset V_1$, so we can find two neighbors of u, say, $v_i, v_j \in V_1$. Since $|V_2| = l \ge 2$, there exist another vertex $w \in V_2 \setminus \{u\}$. Let $v_t \in V_1$ be adjacent to w, then wv_t and uv_i(or uv_j) are two independent edges in $[V_1, V_2]$, which is contrary to Fact 3.

Fact 5. Distinct vertices in V_2 must be joined to an identical vertex v_i in V_1.

Proof of Fact 5. If this is not true, without loss of generality, we suppose that there exist $u, w \in V_2$, and u is joined to v_i , w is joined to $v_j (j \ne i)$. Then uv_i and wv_j are two independent edges in $[V_1, V_2]$, which is contrary to Fact 3.

Combining the facts above, we have $\rho(G) \le \rho(\mathcal{K}_n^l)(l > 1)$, with the equality holding if and only if $G = \mathcal{K}_n^l$. □

The following theorem was obtained in [8]. Here we give another proof.

Theorem 5 *([8]) Let ρ be the spectral radius of $\mathcal{K}_n^l (1 \le l \le n-2)$,* *then ρ satisfies the following equation*

$$\rho^3 - (n-l-2)\rho^2 - (n-1)\rho + (n-l-2)l = 0 \tag{7}$$

Proof. Without loss of generality, Let $V_1 = \{v_1, v_2, \cdots, v_l\}$ be the set of vertices of degree one, their common neighbor is v_{l+1}. The left vertices are $v_{l+2}, \cdots, v_n$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be the Perron vector of $\mathcal{K}_n^l$.

$$\sum_{i=1}^{n} x_i = s. \tag{8}$$

Then from Lemma 3,

$$x_1 = x_2 = \cdots = \frac{s}{\rho(\rho+1)},$$
$$x_{l+1} = \frac{s}{\rho+1},$$
$$x_{l+2} = x_{l+3} = \cdots = x_n = \frac{[\rho(\rho+1)-l]s}{\rho(\rho+1)^2}.$$

Substituting these equations to (8), we have

$$\frac{l}{\rho(\rho+1)} + \frac{1}{\rho+1} + (n-l-1)\cdot\frac{\rho(\rho+1)-l}{\rho(\rho+1)^2} = 1 \tag{9}$$

The result follows after Simplifying the equation above . □

Using the Cardano's formula (see [10],PP120-121), we obtain the following estimation.

Corollary 3 *Let $1 \le l \le n-2$, then*

$$\rho(\mathcal{K}_n^l) < \frac{2\sqrt{(n-l-2)^2 + 3(n-1)}}{3}. \tag{10}$$

Remark. For $\mathcal{K}_5^2$, $\rho(\mathcal{K}_5^2) = 2.3429$, from (10), $\rho < 2.4037$. The use of (6) lead to $\rho < 2.5$.

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