# Certain classes of groups with commutativity degree $d(G) < \frac{1}{2}$

H. Doostie[1] and M. Maghasedi[2]

[1]Mathematics Department, Teacher Training University, 49 Mofateh Ave., Tehran 15614, Iran.
Email: doostih@saba.tmu.ac.ir
[2]Mathematics Department, Islamic Azad University, Karaj Branch, Iran.
E-mail: mmaghasedi@gmail.com

**Abstract**

For a finite group $G$ the commutativity degree,

$$d(G) = \frac{|\{(x,y)|x,y \in G, xy = yx\}|}{|G|^2}$$

is defined and studied by several authors and when $d(G) \geq \frac{1}{2}$ it is proved by P. Lescot in 1995 that $G$ is abelian , or $\frac{G}{Z(G)}$ is elementary abelian with $|G'| = 2$, or $G$ is isoclinic with $S_3$ and $d(G) = 1$. The case when $d(G) < \frac{1}{2}$ is of interest to study. In this paper we study certain infinite classes of finite groups and give explicit formulas for $d(G)$. In some cases the groups satisfy $\frac{1}{4} < d(G) < \frac{1}{2}$. Some of the groups under study are nilpotent of high nilpotency classes.

**AMS Subject Classification:** 11b39, 20K05

**Key words:** Commutativity degree, Finite groups.

## 1. Introduction

The notion of commutativity degree of a finite group $G$,

$$d(G) = \frac{|\{(x,y)|x,y \in G, xy = yx\}|}{|G|^2}$$

or $d(G) = \frac{k(G)}{|G|}$, where $k(G)$ is the number of conjugacy classes of $G$, defined in 1973 by Gallagher [2] and studied during the years for certain properties (one may refer to [3,6,7] ). In obtaining the properties of $d(G)$, Gustafson [3] proved that for a non-abelian finite group $G$, $d(G) \leq \frac{5}{8}$, and P. Lescot [6] studied the groups where $d(G) \geq \frac{1}{2}$ and classified these groups. Moghaddam and etal in [7] studies the isoclinism of groups and the n-nilpotency degree of finite groups where n-nilpotency degree of a finite group $G$ is defined by:

$$d_G^n = \frac{1}{|G|^{n+1}} |\{(x_1, \ldots, x_{n+1})|x_i \in G, [x_1, \ldots, x_{n+1}] = 1\}|,$$

where the notation $[x_1, \ldots, x_{n+1}]$ is used for the commutator

$$[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}].$$

In fact they prove the equality $d^n(N \times H) = d^n(N) \times d^n(H)$ for every $n \geq 1$, where $N$ and $H$ are finite CN-groups (a CN-group is a finite group where the centralizer of every element is a normal subgroup).

In this paper we study certain infinite classes of finite groups which are not CN-groups and we give explicit formulas for $d(G)$. We use the notation $N :_\varphi H$ for the semidirect product of a group $N$ by a group $H$ with respect to a homorphism $\varphi : H \longrightarrow Aut(N)$ where $h\varphi = \varphi_h$, for every $h \in H$. Certainly $N$ is a normal subgroup of $N :_\varphi H$ and $\frac{N :_\varphi H}{N} \cong H$. Our considered classes of groups are as follows :

$G_1(m,n) = D_{2n} : Z_{2m}$, $\quad m,n \geq 3$,
$G_2(m,n) = Q_{2^n} : Z_{2m}$, $\quad m,n \geq 3$,
$G_3(n) = Z_{2^n} \wr Z_2$, $\quad n \geq 2$, (the wreath product of $Z_{2^n}$ by $Z_2$,)
$G_4(n) = S_n$, $\quad n \geq 5$, (the symmetric group of degree $n$).

## 2. The Computation of $d(G)$

The main results of this section are:

**Proposition 2.1.** For every integers $m, n \geq 3$,

(i) if $G = G_1(m,n)$ then

$$d(G) = \begin{cases} \frac{n+3}{4n} & \text{, if } n \text{ is odd,} \\ \frac{n+6}{4n} & \text{, if } n \text{ is even,} \end{cases}$$

which is independent of $m$;

(ii) if $G = G_2(m,n)$ then $d(G) = \frac{2^{n-3}+3}{2^n}$, which is also independent of $m$;

(iii) if $G = G_3(n)$ then $d(G) = \frac{2^n+3}{2^{n+2}}$;

(iv) if $G = S_n$ then $d(G) = \frac{P(n)}{n!}$, where $P(n)$ is the number of partitions of the integer $n$.

**Proof.** Let $d'(G) = |\{(x,y) | x, y \in G, xy = yx\}|$.

To prove (i), let $G = G_1(m,n)$ and we get the following presentation for $G$,

$$G = \langle a, b, c | a^2 = b^n = c^{2m} = 1, c^{-1}aca = 1, c^{-1}bcb = 1 \rangle.$$

Every element $x$ of $G$ may be represented as $x = a^i b^j c^k$, where $i \in \{0,1\}$, $j \in \{0, 1, \ldots, n-1\}$ and $k \in \{0, 1, \ldots, 2m-1\}$. For every $x = a^i b^j c^k$ and $y = a^s b^t c^l$ of $G$, where $i, s \in \{0,1\}$, $j, t \in \{0, 1, \ldots, n-1\}$ and $k, l \in \{0, 1, \ldots, 2m-1\}$, if $xy = yx$ then

$$(a^i b^j c^k)(a^s b^t c^l) = (a^s b^t c^l)(a^i b^j c^k),$$

so

$$a^{i+s} b^{(-1)^s j + (-1)^k t} c^{k+l} = a^{s+i} b^{(-1)^i t + (-1)^l j} c^{l+k}.$$

Hence we obtain

$$b^{(-1)^s j + (-1)^k t} = b^{(-1)^i t + (-1)^l j},$$

or

$$(-1)^s j + (-1)^k t \equiv (-1)^i t + (-1)^l j \pmod{n}. \qquad (\dagger)$$

Let $A_{i,s} = \{(i,j,k,s,t,l) | \ j,t \in \{0,1,\ldots,n-1\},\ i,s \in \{0,1\},\ k,l \in \{0,1,\ldots,2m-1\}\}$, where $(i,j,k,s,t,l)$ satisfies the condition $(\dagger)$. Then we deduce that

$$|\bigcup_{i=0}^{1}\bigcup_{s=0}^{1} A_{i,s}| = \sum_{i=0}^{1}\sum_{s=0}^{1} |A_{i,s}| = d'(G).$$

To compute $|A_{0,0}|$, $|A_{0,1}|$, $|A_{1,0}|$ and $|A_{1,1}|$ we consider two cases for $n$:
*Case 1: $n$ is odd.* First we suppose that $i = s = 0$ and show that $|A_{0,0}| = m^2(n^2+3n)$. The values of $|A_{0,1}|$, $|A_{1,0}|$ and $|A_{1,1}|$ may be determined in a similar way. By the assumption $i = s = 0$ the relation $(\dagger)$ will be reduced to:

$$(-1)^k t - t \equiv (-1)^l j - j \pmod{n}, \qquad (\ddagger)$$

and there are four possible cases to consider the solutions of $(\ddagger)$, as follows:

(*a*). if $k$ and $l$ are even then $(\ddagger)$ holds for every values of $t$ and $j$,
(*b*). if $k$ and $l$ are odd then $(\ddagger)$ holds for $t = j$,
(*c*). if $k$ is odd and $l$ is even then $(\ddagger)$ holds for $t = 0$,
(*d*). if $l$ is odd and $k$ is even then $(\ddagger)$ holds for $j = 0$.

Since each of the integers $k$ and $l$ take $m$ possible values, there are $m^2(n^2 + n + n + n)$ solutions $(i,j,k,s,t,l)$ for $(\ddagger)$ when $i = s = 0$; i.e., $|A_{0,0}| = m^2(n^2+3n)$.

In a similar way we obtain $|A_{1,0}| = |A_{0,1}| = |A_{1,1}| = m^2(n^2+3n)$, and hence $d'(G) = 4m^2(n^2+3n)$. Since $|G| = 4mn$ one obtains $d(G) = \frac{n+3}{4n}$, as desired.
*Case 2: $n$ is even.* In this case we show that

$$|A_{0,0}| = |A_{1,0}| = |A_{0,1}| = |A_{1,1}| = m^2(n^2+6n).$$

A similar proof to that of case 1 may be used for this calculations. For simplicity we give the possible cases for the solutions of $(\ddagger)$ when $i = s = 0$:

(e). if $k$ and $l$ are even, then (‡) holds for every values of $t$ and $j$;

(f). if $k$ and $l$ are odd, then (‡) holds for $t \equiv j (\text{mod } \frac{n}{2})$;

(g). if $k$ is odd and $l$ is even, then (‡) holds for $t \equiv 0 (\text{mod } \frac{n}{2})$ and $j$ is arbitrary;

(h). if $l$ is odd and $k$ is even, then (‡) holds for $j \equiv 0 (\text{mod } \frac{n}{2})$ and $t$ is arbitrary.

We note that in the case (f), for every value of $t$ there are two different values for $j$. Consequently, there are $m^2(n^2 + 2n + 2n + 2n)$ solutions for (‡), when $i = s = 0$ and the result follows immediately.

To prove (ii) we may consider the following presentation for $G = G_2(m, n)$:

$$G = \langle a, b, c | a^{2^{n-1}} = c^{2m} = 1, b^2 = a^{2^{n-2}}, b^{-1}aba = c^{-1}aca = c^{-1}bcb = 1 \rangle.$$

Then every $x \in G$ may be presented as $x = a^i b^j c^k$, where $i \in \{0, 1, \dots, 2^{n-1} - 1\}$, $j \in \{0, 1\}$ and $k \in \{0, 1, \dots, 2m - 1\}$. Now two elements $x = a^i b^j c^k$ and $y = a^s b^t c^l$ of $G$ commute if and only if

$$a^{i(1-(-1)^{l+t})+s((-1)^{k+j}-1)} = b^{t(1-(-1)^k)+j(-1+(-1)^l)}.$$

Equivalently, the equations

$$(*) \qquad \begin{cases} i(1 - (-1)^{l+t}) + s((-1)^{k+j} - 1) \equiv 0 & (\text{mod } 2^{n-1}), \\ t(1 - (-1)^k) + j((-1)^l - 1) \equiv 0 & (\text{mod } 4), \end{cases}$$

hold. Now, computing $d(G)$ is reduced to determining the number of elements of the set

$$A = \{(i, j, k, s, t, l) | i, s \in \{0, 1, \dots, 2^{n-1} - 1\}, j, t \in \{0, 1\}, k, l \in \{0, 1, \dots, 2m - 1\}\},$$

where $(i, j, k, s, t, l)$ satisfies the above system of equations. We observe that $|A| = d'(G)$, then we try to calculate $|A|$ by considering four cases for $k$ and $l$.

If $k$ and $l$ are even then we must only consider the possible cases for $t$ and $j$. For the values $t = j = 0$, each of the integers $i$ and $s$ admit $m$ values and there are $m \times m \times 2 \times 2^{n-1} = 2^m m^2$ solutions of the system (*)

in $A$. Using a similar manner as above for each case, when $(t = 1,\ j = 0)$ and $(t = 0,\ j = 1)$ we get $2^m m^2$ elements of $A$ satisfying $(*)$. In the final case, $t = j = 1$, the system $(*)$ holds for every values of $i$ and $s$, and there are $m^2 \times 2^{2n-2}$ solutions. So, for the even velues of $k$ and $l$ there are $m^2 2^n(2^{n-2} + 3)$ solutions of $(*)$ in $A$.

If at least one of $k$ or $l$ is odd, we consider three cases and in each case as the above we get $m^2 2^n(2^{n-2} + 3)$ solutions. Consequently, $|A| = 4m^2 2^n(2^{n-2} + 3)$ and then $d(G) = \frac{2^{n-3}+3}{2^n}$.

The proof of (iii) is similar to those of (i) and (ii). Indeed, the group $G = G_3(n)$ may be presented as

$$G = \langle a, b, c | a^{2^n} = b^{2^n} = c^2 = 1, c^{-1}ac = b, c^{-1}bc = a \rangle,$$

and hence we immediately obtain $d(G) = \frac{2^n+3}{2^{n+2}}$.

The assertion (iv) may be proved by considering the permutations $\theta$ and $\psi$ of $S_n$. We now that $\theta$ and $\psi$ are conjugate if and only if $\theta$ and $\psi$ have the same cycle structures. Let $n = n_1 + n_2 + \ldots + n_k$ be a partition of $n$ where $n_1 \leq n_2 \leq \ldots \leq n_k$, we denote this partition by $n = (n_1, n_2, \ldots, n_k)$. Define the cycles

$$\begin{cases} \theta_1 = (1, 2, \ldots, n_1), \\ \theta_2 = (n_1 + 1, n_1 + 2, \ldots, n_1 + n_2), \\ \vdots \\ \theta_k = (n_1 + n_2 + \ldots + n_{k-1} + 1, \ldots, n_1 + n_2 + \ldots + n_{k-1} + n_k), \end{cases}$$

and let $\psi_{(n_1, n_2, \ldots, n_k)} = \theta_1\theta_2 \ldots \theta_k$. Now, if $P(\mathbf{n})$ is the set of all partitions of n and $C(\mathbf{n})$ is the set of all disjoint conjugacy classes of $S_n$ then we may define

$$f : P(\mathbf{n}) \longrightarrow C(\mathbf{n})$$

given by

$$f(n_1, n_2, \ldots, n_k) = [\psi_{(n_1, n_2, \ldots, n_k)}],$$

where $[\psi_{(n_1, n_2, \ldots, n_k)}]$ is the conjugacy class of $\psi_{(n_1, n_2, \ldots, n_k)}$. Since $f$ is a bijection, it follows that $|P(\mathbf{n})| = |C(\mathbf{n})|$. This implies that $P(n) = k(S_n)$

and by the definition of $d(G)$ we get the required result as $d(S_n) = \frac{P(n)}{n!}$. $\square$

**Corollary 2.2.**

(i) For the groups $G = G_1(m,n)$, $G = G_2(m,n)$ and $G = G_3(n)$, $\frac{1}{4} < d(G) < \frac{1}{2}$.

(ii) If $G = G_4(n)$, then $d(G) < \frac{1}{2}$ and $\lim_{n\to\infty} d(G) = 0$.

**Proof.** (i) is a straightforward consequence of Proposition 2.1. For (ii) we observe that

$$\begin{cases} P(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \\ n \to \infty. \end{cases}$$

(To prove one may refer to [1]). Then $\lim_{n\to\infty} \frac{P(n)}{n!} = \lim_{n\to\infty} \frac{\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}}{n!} = 0$. $\square$

# References


[1] S. Ahlgren, The partition function modulo composite integers $m$, *Math. Ann.* **318** (2000), 795-803.

[2] P. X. Gallagher, The number of conjugacy classes in finite groups, *Math. Z.* **118** (1970), 175-179.

[3] W. H. Gustafson, What is the propability that two group elements commute? *Amer. Math. Monthly.* **80** (1973), 1031-1034.

[4] M. R. Jones, Muliplcators of $p$-groups, *Math. Z.* **127** (1972), 165-166.

[5] M. R. Jones, Some inequalities for the muliplcator of a finite group, *Proc. Amer. Math. Soc.* **39** (1973), 450-456.

[6] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177** (1995), 847-869.

[7] M. R. R. Moghaddam, A. R. Salemkar and K. Chiti, $n$-isoclinism classes and $n$-nilpotency degree of finite groups, *Algebra Colloquium*, **12**:2 (2005),255-261.