Wide Diameter and Diameter of Networks

Bolian Liu*
Department of Mathematics
South China Normal University
Guangzhou,China

Xiankun Zhang
Department of Mathematics
West Virginia University
Morgantown WV,U.S.A.

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Abstract

A container $C(x, y)$ is a set of vertex–disjoint paths between vertices x and y in a graph G. The width $w(C(x, y))$ and length $L(C(x, y))$ are defined to be $|C(x, y)|$ and the length of the longest path in $C(x, y)$ respectively. The w–wide distance $d_w(x, y)$ between x and y is the minimum of $L(C(x, y))$ for all container $C(x, y)$ with width w. The w–wide diameter $d_w(G)$ of G is the maximum of $d_w(x, y)$ among all pairs of vertices x, y in $G, x \neq y$. In this paper, we investigate some problems on the relations between $d_w(G)$ and diameter $d(G)$ which raised by D.F.Hsu([1]). Some results about graph equation of $d_w(G)$ are proved.

Key words: network, diameter, wide diameter

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1 Introduction

Let G be a graph without loops or multiple edges. We use $k(G), d(G)$ and $V(G)$ to denote the connectivity, diameter and the set of vertices of G respectively. A container $C(x,y)$ is a set of vertex–disjoint paths between vertex $x, y \in V(G)$. The length of the $C(x,y)$, written as $L(C(x,y))$, is the length of the longest path in $C(x,y)$.

Definition 1.1: Given a graph G and vertices x and y, $x \neq y$, the w–wide distance(or simply w–distance) $d_w(x,y)$ is the minimum of $L(C(x,y))$ among all $C(x,y)$ with width w. The w–wide diameter(or simply w–diameter)$d_w(G)$ is the maximum of $d_w(x,y)$ among all pairs of vertices $x, y, x \neq y$.

The concepts of wide distance and wide diameter of a graph are natural generalizations of distance and diameter in a graph taking into consideration the connectivity of the graph. The notions will be explored due to their intrinsic importance in communication networks.

A graph G is said to be k–connected if $k(G) \geq k$. For a constant c and a k–connected graph G, we call G strongly c –resilient if $d_k(G) = d(G) + c$, and weakly c–resilient if $d_k(G) = cd(G)$. And we define $G^*(w,b)$ to be the class of graphs in which there exists a container of width w and length at most b between any pair of distinct vertices x and y.

Clearly, when $w = 1, d_w(G) = d(G)$. Hence we have $d_w(G) \geq d(G)$ and $d_w(x,y) \geq d(x,y)$ for any vertices $x, y (x \neq y)$. On the other hand we can assume $w \leq k(G)$. More often, we are interested in relation between $d_w(G)$ and $d(G)$, where G belong to an interconnection network for parallel and distributed systems. Some open problems were raised by D.F.Hsu in the survey [1]:

Problem 1: For $w = k(G)$ characterize strongly c–resilient graph in which $d_w(G) = d(G) + c$.(1.1)

Problem 2: For $w = k(G)$ characterize weakly c–resilient graph in which $d_w(G) = c \cdot d(G)$. (1.2)

In section 2, we characterize a specific class of graph G with $d_w(G) = d(G), w > 1$. In section 3, we prove that for any positive integer c, graph equations (1.1), (1.2) have solutions.

2 A characterization of graphs $G's$ with $d_w(G) = d(G) \quad (w > 1)$

As we know, when $w = 1$, $d_w(G) = d(G)$. The converse is not necessarily true. Clearly, if G is a complete graph on $k + 2$ vertices with one edge missing, then $d_k(G) = d(G) = 2$. We now characterize a specific class of G with $d_k(G) = d(G), \quad w > 1$.

We establish the following definitions.

Definition 2.1: Let $C(x, y) = \{p_1, p_2, \dots, p_w\}$ be a container with width w, where $p_1, p_2, \dots, p_w$ are vertex–disjoint paths between x and y, $x \neq y$. If $|p_1| = |p_2| = \cdots = |p_w| = p$, where $|p_i|$ denotes the number of edges of path $p_i, C(x, y)$ is called a p–uniform container. We call p the length of $C(x, y)$.

Definition 2.2: A pair of vertices $\{x, y\}$ in $V(G)$ is called paired w–poles of G if $d_w(x, y) = d_w(G)$. Namely, the distance of 1–poles of G is the diameter of G. Let $V^{(k)} = \{\{x, y\} \,|x, y \in V(G), d_k(x, y) = d_k(G)\}$.

The following is a characterization of graphs with $d_w(G) = d(G)$.

Theorem 2.1 *Let $d = d(G)$ and $w \leq k(G)$. Then the following three statements are equivalent.*

(1) $d_w(G) = d(G)$.

(2) $G \in G^*(w, d)$.

(3) There exists $(u, v) \in V^{(1)}(G) \cap V^{(w)}(G)$ such that there is a d–uniform container with width w between u and v.

Proof. (1) $\Rightarrow$ (2) If $d_w(G) = d(G) = d$, then for any pair of distinct vertices x and y in G, we have $d_w(x, y) \leq d$. Thus $G \in G^*(w, d)$.

(2) $\Rightarrow$ (3) Suppose that $G \in G^*(w,d)$. Let u, v be two vertices in G such that $d(u,v) = d$. Since $G \in G^*(w,d)$, we have $d_w(u,v) \le d$, $d_w(G) \le d$ and there exists a container of with w. Note that $d_w(u,v) \ge d$ and $d_w(G) \ge d$, thus $d_w(u,v) = d$ and $d_w(G) = d$. Hence there exists $(u,v) \in V^{(1)}(G) \cap V^{(w)}(G)$ such that there is a d–uniform container with width w between u and v.

(3) $\Rightarrow$ (1) Suppose that there exists $(u,v) \in V^{(1)}(G) \cap V^{(w)}(G)$ such that there is a d–uniform container with width w between u and v. We have $d(u,v) = d(G) = d$ and $d_w(u,v) = d_w(G)$. Thus $d_w(G) = d_w(u,v) \ge d(u,v) = d$. Since there exists a d–uniform container with width w between u and v, it follows that $d_w(u,v) \le d$. Thus $d_w(G) = d_w(u,v) \le d$, so $d_w(G) = d = d(G)$.

This completes the proof of Theorem 2.1. ∎

Corollary 2.1 If $uv \in E(G)$ for each $(u,v) \in V^{(w)}(G)$ $(w > 1)$, then $d_w(G) \ne d(G)$.

Proof. Suppose that $d_w(G) = d(G)$. By theorem 2.1, there exists $(u,v) \in V^{(1)}(G) \cap V^{(w)}(G)$. Since $uv \in E(G), d(u,v) = d(G) = 1$. Hence $G = K_n$ and $d_w(G) > d(G)$. This is a contradiction. ∎

For any positive integer d with $3 \le d \le n-1$, we can construct a graph with $n = k(d-1)+2$ vertices such that $d_k(G) = d(G) = d$.

Taking two vertices u and v, we construct k $(k \le d-1)$ vertex–disjoint paths with length d that compose a graph G as follows.

$P_i(u,v) : up_1^{(i)}p_2^{(i)} \dots p_{d-1}^{(i)}v, i = 1, 2, \dots, k$. Let

$V_1 = \{p_1^{(1)}, p_1^{(2)}, \cdots, p_1^{(k)}, \quad p_2^{(1)}, p_2^{(2)}, \cdots, p_2^{(k)}\}$,

$V_2 = \{p_2^{(1)}, p_2^{(2)}, \cdots, p_2^{(k)}, \quad p_3^{(1)}, p_3^{(2)}, \cdots, p_3^{(k)}\}$,

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$V_{d-2} = \{p_{d-2}^{(1)}, p_{d-2}^{(2)}, \cdots, p_{d-2}^{(k)}, \quad p_{d-1}^{(1)}, p_{d-1}^{(2)}, \cdots, p_{d-1}^{(k)}\}$.

Adding some edges to G such that each subgraph G_i with vertex set $V_i, i = 1, 2, ..., d-2$ is a complete graph, we obtain a graph G with

$V(G) = \{u, v\} \cup \{p_j^{(i)} : i = 1, 2, \dots, k \ \ and \ \ j = 1, 2, \dots, k\}$ and

$E(G) = \{up_1^{(i)} : i = 1, 2, \dots, k\} \cup \{p_{d-1}^{(i)}v : i = 1, 2, \dots, k\} \cup_{i=1}^{d-2} E(G_i)$, where $p_{j_1}^{(i_1)} \neq p_{j_2}^{(i_2)}$ if $j_1 \neq j_2$ or $i_1 \neq i_2$.

It is easy to verify that G satisfies $d_k(G) = d(G) = d$.

If $d = 2$, we take G to be a complete graph on $k+2$ vertices with one edge missing. Then $d_k(G) = d(G) = 2$.

3 The solutions of graph equations $d_w(G) = d(G) + c$ and $d_w(G) = c \cdot d(G)$

Certainly, for any a graph G, there exists a nonnegative integer c such that $d_w(G) = d(G) + c$, where $w \leq k(G)$. So we naturally raised the following converse question. Given positive integers c, d and w, can one always find a graph G that satisfies $d_w(G) = d(G) + c$ and $d(G) = d$?

We can give the following result:

Theorem 3.1 *Let c, d, k be any three positive integers. Then there exists a graph G with connectivity k, such that*

$$d_k(G) = d(G) + c \text{ and } d(G) = d.$$

Before constructing the graph G, we first give the following definition and lemma.

Definition 3.1: The k–th power G^k of a graph $G = (V, E)$ is the graph in which $V(G^k) = V(G)$ and for any $u, v \in V(G^k)$, $uv \in E(G^k)$ if $d(u, v) \leq k$ in G.

Proposition 3.1: Let G^k be the k–th power of a graph G. Then $E(G^(k-1)) \subseteq E(G^k)$.

Lemma 3.1 *Denote a path with n vertices by P_n and denote the k–th power of P_n by P_n^k. Then P_n^k is a k–connected graph, and*

$d_k(P_n^k) = \lceil \frac{n-2}{k} \rceil + 1, d(P_n^k) = \lceil \frac{n-1}{k} \rceil$, *namely*

$$d_k(P_n^k) = \begin{cases} d(P_n^k), & \text{if } k \mid (n-2) \\ d(P_n^k) + 1, & \text{otherwise.} \end{cases}$$

Proof. Let u, v be the two endpoints of P_n and $P_n = uu_1u_2\dots u_{n-2}v$. Then by definition 3.1, for the graph P_n^k we have $d(u,v) = d(P_n^k)$. Obviously $d(u,v) = \lceil \frac{n-1}{k} \rceil$ in P_n^k. Thus $d(P_n^k) = d(u,v) = \lceil \frac{n-1}{k} \rceil$.

Let $n-2 = kq+r$, where $q = \lfloor \frac{n-2}{k} \rfloor$ and $0 \le r \le k-1$. Then if $r=0$, the paths $W_i = uu_iu_{k+i}u_{2k+i}\dots u_{(q-1)k+i}v$, $i = 1,2,\dots,k$, and if $r=1$, the paths $W_i' = uu_iu_{k+i}u_{2k+i}\dots u_{(q-1)k+i}u_{kq+i}v$, $i = 1,2,\dots,r$, and $W_i' = uu_iu_{k+i}u_{2k+i}\dots u_{(q-1)k+i}v$, $i = r+1, r+2, \dots, k$, are just k vertex-disjoint paths between u and v. Hence $d_k(P_n^k) = d_k(u,v) = |P_k| = \lceil \frac{n-2}{k} \rceil + 1$. ∎

Now we construct a graph satisfying the condition of Theorem 3.1.

(The proof of Theorem 3.1)

Proof. If $c=0$, let $n = k(d-1)+2$. By Lemma 3.1, we have $d_k(P_n^k) = d(P_n^k) = d$.

If $c \ge 1, d' = d+c$, consider the $(k-1)$th power P_n^{k-1} of the path P_n with consecutive vertices $v_1, v_2, ..., v_n$, where $n = (k-1)(d+c-1)+2$. Then

$d_{k-1}(P_n^{k-1}) = \left\lceil \frac{n-2}{k-1} \right\rceil + 1 = d + c = d'$.

In addition, consider a path P_{d-1} with $d-1$ consecutive vertices $u_1, u_2, ..., u_{d-1}$ and define

$G = P_n^{k-1} \bigoplus P_{d-1}$,

where $V(G) = V(P_n^{k-1}) \cup V(P_{d-1})$ and

$E(G) = E(P_n^{k-1}) \cup E(P_{d-1}) \cup \{u_iv_j | v_j \in V(P_n^{k-1}), u_i \in V(P_{d-1})$. If $i = 1,2,...,d-2, (i-1)k-(i-2) \le j \le ik-(i-1)$; if $i = d-1, (d-2)k-(d-3) \le j \le n\}$.

We see that $d(v_j) \ge d(v_1) = d(v_n) = k$ for $1 \le j \le n$, $d(u_1) = k+1$, $d(u_i) = k+2$ for $2 \le i \le d-2$ and $d(u_{d-1}) = (k-1)c + k + 2$. Thus G is a graph with connectivity k.

By Lemma 3.1, there are exactly $k-1$ vertex-disjoint paths, say $W_1, W_2, \dots, W_{k-1}$, between vertices u and v. Let path $W_k = v_1u_1u_2\dots u_{d-1}v_n$. Then $W_1, W_2, \dots, W_{k-1}, W_k$ are k vertex-disjoint paths between vertices u and v. Thus,

$$d_k(G) = d_k(v_1, v_n) = d_{k-1}(P_n^{k-1}) = d + c = d', \text{ and}$$

$d(G) = d(v_1, v_n) = d.$

Hence $d_k(G) = d(G) + c$. The proof is complete. ∎

For problem 2, we can give a similar result.

Theorem 3.2 *Let c, d, k be any three positive integers and $d' = cd$. Then there exists a graph G with connectivity k, such that*

$$d_k(G) = cd(G) \quad \textit{and}$$
$$d_k(G) = d', d(G) = d.$$

Since $d' = cd = d + (c-1)d$, as the proof of Theorem 3.1, we construct a graph G with connectivity k as follows.

$G = P_n^{k-1} \bigoplus P_{d-1}$, where $n = (k-1)(cd-1)+2$, $c \geq 2$, $V(G) = V(P_n^{k-1}) \cup V(P_{d-1})$ and

$E(G) = E(P_n^{k-1}) \cup E(P_{d-1}) \cup \{u_i v_j | v_j \in V(P_n^{k-1}), u_i \in V(P_{d-1})$. If $i = 1, 2, ..., d-2, (i-1)k - (i-2) \leq j \leq ik - (i-1)$; if $i = d-1, (d-2)k - (d-3) \leq j \leq n\}$.

It follows from the proof of Theorem 3.1 that

$$d_k(G) = cd = d',$$
$$d(G) = d \qquad \text{and}$$
$$d_k(G) = cd(G).$$

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