A note on graphs with disjoint dominating and total dominating sets

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Abstract

A total dominating set of a graph is a set of vertices such that every vertex is adjacent to a vertex in the set. In this note, we show that the vertex set of every graph with minimum degree at least two and with no component that is a 5-cycle can be partitioned into a dominating set and a total dominating set.

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1 Introduction

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6]. A classical result in domination theory is that if S is a minimal dominating set of a graph $G = (V, E)$ without isolates, then $V \setminus S$ is also a dominating set of G. Thus, the vertex set of every graph without any isolates can be partitioned into two dominating sets.

However, it is not the case that the vertex set of every graph with at least four vertices can be partitioned into two total dominating sets, even if every vertex has degree at least 2. (Recall that a total dominating set is a set S such that every vertex in the graph is adjacent to some vertex of S; see [5].)

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A partition of the vertex set can also be thought of as a coloring. In particular, a partition into two total dominating sets is a 2-coloring of the graph such that no vertex has a monochromatic (open) neighborhood.

Zelinka [8, 9] showed that no minimum degree is sufficient to guarantee the existence of two total dominating sets. Consider the bipartite graph G_n^k formed by taking as one partite set a set A of n elements, and as the other partite set all the k-element subsets of A, and joining each element of A to those subsets it is a member of. Then G_n^k has minimum degree k. As observed in [8], if $n \geq 2k-1$ then in any 2-coloring of A at least k vertices must receive the same color, and these k are the neighborhood of some vertex.

In contrast, results of Calkin and Dankelmann [2] and Feige et al. [4] show that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree does suffice.

Heggernes and Telle [7] showed that the decision problem to decide for a given graph G if there is a partition of $V(G)$ into two total dominating sets is NP-complete, even for bipartite graphs.

Broere et al. [1] considered the question of how many edges must be added to G to ensure the partition of V into two total dominating sets in the resulting graph. They denote this minimum number by $td(G)$. It is clear that $td(G)$ can only exist for graphs with at least four vertices. In particular, it was shown that if T is a tree with ℓ leaves, then $\ell/2 \leq td(T) \leq \ell/2+1$.

Dorfling et al. [3] showed that given a graph of order $n \geq 4$ with minimum degree at least 2, one can add at most $(n-2\sqrt{n})/4+O(\log n)$ edges such that the resulting graph has two disjoint total dominating sets, and this bound is best possible.

In this note we consider the question of whether the vertex set of every graph with minimum degree at least two can be partitioned into a dominating set and a total dominating set.

2 Notation

We generally use the definitions and terminology of [5]. Let $G=(V,E)$ be a graph, and let $S \subseteq V$ be a set of vertices of G and $v \in V$ a vertex of G.

The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of S is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. Hence, S is a dominating set of G if $N[S]=V$, while S is a total dominating set of G if $N(S)=V$. A vertex $w \in V-S$ is an *S-external private neighbor of v* if $N(w) \cap S = \{v\}$; and the *S-external private neighbor set of v*, denoted $\text{epn}(v,S)$, is the set of all S-external private neighbors of v. The minimum degree among the vertices of G is denoted by $\delta(G)$.

We say that v is an ***S*-bad** vertex if $N[v] \subseteq S$. Further, we say that a vertex $u \in S$ is an ***S*-weak** vertex if u has degree 1 in $G[S]$ and its neighbor in S is an S-bad vertex.

3 Main Result

Clearly the vertex set of a 5-cycle C_5 cannot be partitioned into a dominating set and a total dominating set. We show that this is the only exception.

We shall prove:

Theorem 1 *If $G = (V, E)$ is a graph with $\delta(G) \geq 2$ that contains no C_5-component, then V can be partitioned into a dominating set and a total dominating set.*

Proof. Among all total dominating sets of G, let S be chosen so that

(1) the number of S-bad vertices is minimized, and
(2) subject to (1), the number of S-weak vertices is minimized.

Assume that there is at least one S-bad vertex. Let v be such a vertex. If v has no S-weak neighbor, then $S' = S \setminus \{v\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence we may assume that every S-bad vertex has at least one S-weak neighbor.

Let w be an S-weak vertex. Since $\delta(G) \geq 2$, w is adjacent to at least one vertex in $V \setminus S$. If $\text{epn}(w, S) = \emptyset$, then $S' = S \setminus \{w\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence, $|\text{epn}(w, S)| \geq 1$. For each S-weak vertex w, let $w' \in \text{epn}(w, S)$. Since $\delta(G) \geq 2$, w' is adjacent to at least one vertex in $V \setminus S$ and $N[w'] \setminus \{w\} \subseteq V \setminus S$.

We show next that every S-weak vertex has degree 2 in G. As defined earlier, let w be an S-weak vertex and suppose that $\deg w \geq 3$. Then, $S' = S \cup \{w'\}$ is a total dominating set of G that satisfies condition (1), but with fewer S'-weak vertices than S-weak vertices, contradicting our choice of S. Hence, every S-weak vertex has degree 2.

As defined earlier, let v be an S-bad vertex. Then, v has at least one S-weak neighbor. For $k \geq 1$, let $W = \{w_1, \ldots, w_k\}$ be the set of all S-weak neighbors of v. Then, $N(w_i) = \{v, w_i'\}$ for $i = 1, \ldots, k$. Let $W' = \{w_1', \ldots, w_k'\}$.

If every vertex in W' is adjacent to a vertex in $V \setminus (S \cup W')$, then $S' = (S \cup W') \setminus \{v\}$ is a total dominating set of G with fewer S'-bad vertices than S-

bad vertices, contradicting our choice of S. Hence, renaming vertices if necessary, we may assume that $N[w_1'] \subseteq W' \cup \{w_1\}$ and that $w_1'w_2'$ is an edge of G.

If $\deg v \geq 3$, then $S' = (S \cup \{w_1', w_2'\}) \setminus \{w_1, w_2\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. Hence each of v, w_1, w_1' and w_2 has degree 2 in G and $C\colon v, w_1, w_1', w_2', w_2, v$ is an induced 5-cycle in G.

Since G contains no C_5-component, the vertex w_2' is adjacent to some vertex not in the 5-cycle C. But then $S' = (S \cup \{w_1', w_2'\}) \setminus \{v, w_1\}$ is a total dominating set of G with fewer S'-bad vertices than S-bad vertices, contradicting our choice of S. We deduce, therefore, that the total dominating set S contains no S-bad vertices. Hence, $V \setminus S$ is a dominating set of G, and we are done. □

We close with the remark that the minimum degree condition of Theorem 1 cannot be relaxed to $\delta(G) \geq 1$. For example, the vertex set of a graph G obtained from any graph H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint (the graph G is called the 2-*corona* of H) cannot be partitioned into a dominating set and a total dominating set.

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