Disproof of a Conjecture on List Coloring

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Abstract

Let G be the graph obtained from $K_{3,3}$ by deleting an edge. We find a list assignment with $|L(v)| = 2$ for each vertex v of G, such that G is uniquely $L-$colorable, and show that for any list assignment L' of G, if $|L'(v)| \geq 2$ for all $v \in V(G)$ and there exists a vertex v_0 with $|L'(v_0)| > 2$, then G is not uniquely $L'-$colorable. However, G is not 2-choosable. This disproves a conjecture of Akbari, Mirrokni, and Sadjad (Problem 404 in Discrete Math. 266(2003) 441-451).

1 Introduction

Let $G = (V(G), E(G))$ be a graph and $\mathbb{N}$ be the set of natural numbers. A list assignment of G is a function L defined on $V(G)$ with $L(v) \subseteq \mathbb{N}$. For a list assignment L of G, an $L-$coloring of G is a proper coloring c, i.e., $c(u) \neq c(v)$ whenever $uv \in E(G)$, such that $c(w) \in L(w)$ for every $w \in V(G)$. A graph G admitting an $L-$coloring is said to be $L-$colorable. In particular, if there is a unique $L-$coloring, G is said to be uniquely

$L-$colorable; that G is not uniquely L-colorable means either G has no coloring with list L, or G has at least two distinct colorings using list L. The results concerning with uniquely list coloring of graphs can be found in [3,4,5,7].

Consider a graph $G = (V(G), E(G))$, and a function $f : V(G) \to \mathbb{N}$. G is $f-$choosable if it is $L-$colorable for every list assignment L of G with $|L(v)| \geq f(v)$. In particular, G is $k-$choosable if it is $f-$choosable for the constant function $f(v) = k$. G is said to be maximal uniquely $f-$colorable if

(1) there is a list assignment L of G such that $|L(v)| = f(v)$, and G is uniquely L-colorable, and

(2) for any list assignment L' of G, if $|L'(v)| \geq f(v)$ for all $v \in V(G)$ and there exists a vertex v_0 such that $|L'(v_0)| > f(v_0)$, then G is not uniquely $L'-$colorable.

In [1], Akbari et al. made two conjectures on list coloring, one of which is the following.

Conjecture (Akbari, Mirrokni, and Sadjad [1]) Suppose G is a graph and $f : V(G) \to \mathbb{N}$ is a function, where $\mathbb{N}$ is the set of natural numbers. If G is maximal uniquely $f-$colorable, then G is $f-$choosable.

In this note, we disprove this conjecture.

2 Counterexample

At first, we need two useful results.

Lemma 1. (Mahdian and Mahmoodian [7]) A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.

Lemma 2. (Gravier and Maffray [6]) Let G be the graph $K_{3,3}$ and L a list assignment to its vertices such that $|L(v)| = 2$ for each $v \in V(G)$ except for one vertex z which has $|L(z)| = 3$. Then G is $L-$colorable.

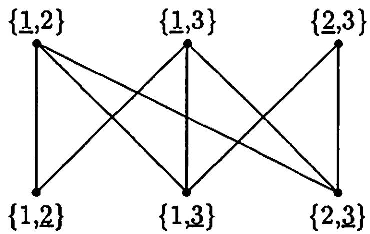
Now we are ready for giving our counterexample. It is exactly the graph $K_{3,3} - e$ obtained from $K_{3,3}$ by deleting an edge. For more accuracy, we have

Theorem 3. Let G be the graph $K_{3,3} - e$. Then the following hold:

(1) G is uniquely $L-$colorable for the list assignment L shown below, where all vertices of G have the list of size 2.

(2) For any list assignment L' of G, if $|L'(v)| \geq 2$ for all $v \in V(G)$ and there exists a vertex v_0 such that $|L'(v_0)| > 2$, then G has at least two distinct colorings using list L'.

(3) G is not 2-choosable.



Proof. For (1), one easily verifies that for the list L given, G is uniquely $L-$colorable.

To prove (2), let L' be a list assignment of G satisfying the conditions in (2). We can see that L' is also a list assignment of $K_{3,3}$. So, by Lemma 2, $K_{3,3}$ is $L'-$colorable, and thus by Theorem 1, $K_{3,3}$ has at least two $L'-$colorings, which are $L'-$colorings of G, too.

From the well-known characterization of 2-choosable graphs in [2], we have G is not 2-choosable. □

We do not know if there is other counterexample to the conjecture.

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References

[1] P. J. Cameron, Reasearch problems from the 18th British Combinatorial Conference, Discrete Math. 266(2003) 441-451.

[2] P. Erdős, A. Rubin and H. Taylor, Choosability in graphs, Congr. Numer. 26(1979) 125-157.

[3] Ch. Eslahchi, M. Ghebleh, and H. Hajiabolhassan, Some concepts in list coloring, J. Combin. Math. Combin. Comput. 41(2002) 151-160.

[4] Y. G. Ganjali, M. Ghebleh, H. Hajiabolhassan, M. Mirzazadeh, B. S. Sadjad, Discrete Appl. Math. 119(2002) 217-225.

[5] M. Ghebleh and E.S. Mahmoodian, On uniquely list colorable graphs, Ars Combin. 59(2001) 307-318.

[6] S. Gravier, F. Maffray, Graphs whose choice number is equal to their chromatic number, J. Graph Theory 27(1998) 87-97.

[7] M. Mahdian and E.S. Mahmoodian, A characerization of uniquely 2-list colorable graphs, Ars Combin. 51(1999) 295-305.