# New results on the eccentric digraphs of the digraphs

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**Abstract** Let $G$ be a digraph. For two vertices $u$ and $v$ in $G$, the distance $d(u, v)$ from $u$ to $v$ in $G$ is the length of the shortest directed path from $u$ to $v$. The *eccentricity* $e(v)$ of $v$ is the maximum distance of $v$ to any other vertex of $G$. A vertex $u$ is an *eccentric vertex* of $v$ if the distance from $v$ to $u$ is equal to the eccentricity of $v$. The *eccentric digraph* $ED(G)$ of $G$ is the digraph that has the same vertex set as $G$ and the arc set defined by: there is an arc from $u$ to $v$ if and only if $v$ is an eccentric vertex of $u$. In this paper, we determine the eccentric digraphs of digraphs for various families of digraphs and we get some new results on the eccentric digraphs of the digraphs.

**Keywords** Eccentricity; Eccentric vertex; Distance; Directed graph

## 1. Introduction

Let $G$ be a digraph with vertex set $V(G)$ and arc set $A(G)$. For two vertices $u$ and $v$ in $G$, if there is a directed path from $u$ to $v$, then we say that $v$ is reachable from $u$ and the distance $d(u, v)$ from $u$ to $v$ is the length of the shortest directed path from $u$ to $v$. If there is no directed path from $u$ to $v$ in $G$, then we define $d(u, v) = \infty$. The *eccentricity* $e(v)$ of $v$ in $G$, is the distance from $v$ to a vertex farthest from $v$. A vertex $u$ in $G$ is an *eccentric vertex* of vertex $v$ if the distance from $v$ to $u$ is equal to $e(v)$. The *eccentric digraph* of $G$, denoted $ED(G)$, is the digraph on the same vertex set as $G$, in which there is an arc from $v$ to $u$ if and only if $u$ is an eccentric vertex of $v$.

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Given a positive integer $k \geq 1$, the $k$th iterated eccentric digraph of $G$ is defined as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^1(G) = ED(G)$ and $ED^0(G) = G$. Since the number of the digraphs on $n$ vertices is finite, there is a positive integer $p$ and a non-negative integer $k$ such that $ED^t(G) = ED^{p+t}(G)$. The smallest $p$ and $t$, which make the equality hold, are called the *period* and the *tail* of $G$ respectively. The period and tail of $G$ are denoted by $p(G)$ and $t(G)$ respectively. We say that a graph is *periodic* if $t(G) = 0$.

Besides, we define the following digraphs in this paper.

The directed path $P_n = v_1 v_2 ... v_n$ is a directed graph with vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and arc set $A(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$.

The directed cycle $C_n = v_1 v_2 ... v_n v_1$ is a directed graph with vertex set $V(C_n) = \{v_1, v_2, ..., v_n\}$ and arc set $A(C_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$.

The in-directed fan $F_n^i$ is the digraph with vertex set $V(F_n^i) = \{c, v_1, v_2, ..., v_n\}$ and arc set $A(F_n^i) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{v_1c, ..., v_nc\}$.

The out-directed fan $F_n^o$ is the digraph with vertex set $V(F_n^o) = \{c, v_1, v_2, ..., v_n\}$ and arc set $A(F_n^o) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{cv_1, ..., cv_n\}$.

Let $F_n^\bullet$ be the digraph with vertex set $V(F_n^\bullet) = \{c, v_1, v_2, ..., v_n\}$ and arc set $A(F_n^\bullet)=\{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup\{cv_1, ..., cv_n\} \cup \{v_nc\}$.

The out-directed wheel $W_n^o$ is the digraph with vertex set $V(W_n^o)=\{c, v_1, v_2, ..., v_n\}$ and arc set $A(W_n^o)=\{v_1v_2, v_2v_3 \ ..., v_{n-1}v_n, v_nv_1\} \cup\{cv_1, ..., cv_n\}$.

The in-directed wheel $W_n^i$ is the digraph with vertex set $V(W_n^i)=\{c, v_1, v_2, ..., v_n\}$ and arc set $A(W_n^i)=\{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\} \cup\{v_1c, ..., v_nc\}$.

For a graph $G$, $G^*$ is the digraph obtained from $G$ by replacing each edge of $G$ by a symmetric pair of arcs.

For two vertex disjoint digraphs $G_1$ and $G_2$, $G_1 \oplus G_2$ is the digraph obtained by joining each vertex of $G_1$ to each vertex of $G_2$.

The complement of a digraph $G$ with $n$ vertices is the digraph $(K_n)^* - A(G)$, denoted $\overline{G}$.

In [1], Bolland and Miller introduced the concept of the eccentric digraph of a digraph and obtained some useful results as follow.

**Proposition 1.1** For the complete digraph $(K_n)^*$, $ED((K_n)^*) = (K_n)^*$.
**Proposition 1.2** For the complete multipartite digraph $G$, $ED^2(G) = G$.
**Proposition 1.3** For a directed cycle $C_n$, $ED(C_n) = C_n$.

Note that the direction of any arc in $ED(C_n)$ is opposite to that in $C_n$.

**Proposition 1.4** A non-trivial eccentric digraph has no vertex of out-degree zero.

**Proposition 1.5**

1. $p = 1, t = 0$ if and only if $G = K_n$.
2. $p = 1, t = 1$ if and only if $G = \overline{K_n}$.
3. $p = 2, t = 0$ when $G = K_{n_1,n_2,...,n_k}$ or $G = K_{n_1} \cup K_{n_2} \cup ... \cup K_{n_k}$.
4. $p = 2, t = 1$ when $G = H_{n_1,n_2,...,n_k}$ or $G = H_{n_1} \cup H_{n_2} \cup ... \cup H_{n_k}$ where $H_{n_1,n_2,...,n_k}$ is a strongly connected subdigraph of $K_{n_1,n_2,...,n_k}$ of order $n_1 + n_2 + ... n_k$, $H_{n_1} \cup H_{n_2} \cup ... \cup H_{n_k}$ is a strongly connected subdigraph of

$K_{n_1} \cup K_{n_2} \cup ... \cup K_{n_k}$ of order $n_1 + n_2 + ...n_k$.

**Proposition 1.6** Let $G$ be a digraph with $|V(G)| = n$ and no vertex of out-degree 0. Then $G$ has a vertex of out-degree $n-1$ if and only if $ED(G)$ has a vertex of out-degree $n-1$.

In this paper, we have obtained some results on the eccentric digraphs of the digraphs.

## 2. New results

**Lemma 2.1** The eccentric digraph of a directed path $P_n$ is a directed graph $G$, where $V(G) = V(P_n)$ and $A(G) = \{v_i v_j : i > j,\ i, j = 1, 2, ..., n\}$.

**Lemma 2.2** $ED((K_m \cup K_n)^*) = (K_{m,n})^*$ and $ED((K_{m,n})^*) = (K_m \cup K_n)^*$. So $t((K_m \cup K_n)^*)$=$p((K_m \cup K_n)^*) = 1$.

**Lemma 2.3** The digraph $G_1$ in Figure 1 satisfies that $ED(G_1) = K_1 \oplus (K_n)^*$ and $ED^2(G_1) = G_1$. So $t(G_1) = 0$, $p(G_1) = 2$, i.e. $G_1$ is periodic.

Proof: Suppose $V(G_1) = \{v_1, v_2, ..., v_n, c\}$ and $c$ is the vertex of in-degree $n$ and out-degree $n$. Since $e(c) = 1$ then the other vertex $v_i$ is the eccentric vertex of $c$ for any $i = 1, 2, ..., n$. Since $indegree(c) = outdegree(c) = n$ then $e(v_i) = 2$ for any $i = 1, 2, ..., n$. Thus, $v_i$ is the eccentric vertex of $v_j$ if $i \neq j$ and $i, j = 1, 2, ..., n-1$. So $ED(G_1) = K_1 \oplus (K_n)^*$. Furthermore, since $ED(K_1 \oplus (K_n)^*) = G_1$ then $ED^2(G_1) = G_1$. □

**Lemma 2.4** Let $G = K_1 \oplus (K_n)^*$, then $ED(G) = G_1$ and $ED^2(G) = G$. So $t(G) = 0$, $p(G) = 2$, i.e. $G$ is periodic, where $G_1$ is the digraph in the Figure 1.

**Lemma 2.5** Let $G = rK_1 \oplus (K_n)^*$, then $ED^2(G) = G$. So $t(G) = 0$, $p(G) = 2$, i.e. $G$ is periodic, where $r$ is a positive integer.

**Lemma 2.6** The eccentric digraph of $F_n^{\mathrm{i}}$ is the digraph in Figure 2. Furthermore, $ED^2(F_n^{\mathrm{i}}) = G_1$ and $ED^2(F_n^{\mathrm{i}}) = ED^4(F_n^{\mathrm{i}})$. So $t(F_n^{\mathrm{i}}) = p(F_n^{\mathrm{i}}) = 2$.

**Lemma 2.7** The eccentric digraph of $F_n^{\circ}$ is the digraph in Figure 3. Furthermore, $ED(F_n^{\circ}) = ED^3(F_n^{\circ})$. So $t(F_n^{\circ}) = 1$, $p(F_n^{\circ}) = 2$.

**Lemma 2.8** The eccentric digraph of $F_n^{\bullet}$ is the digraph in Figure 4. Furthermore, $ED(F_n^{\bullet}) = ED^3(F_n^{\bullet})$. So $t(F_n^{\bullet}) = 1$, $p(F_n^{\bullet}) = 2$.

**Lemma 2.9** The eccentric digraph of $W_n^o$ is also the digraph $G_1$ in Figure 1. Furthermore, $ED(W_n^o) = ED^3(W_n^o)$. So $t(W_n^o) = 1$, $p(W_n^o) = 2$.

**Lemma 2.10** The eccentric digraph of $W_n^i$ is the out-directed wheel $W_n^o$, while the direction of the rim of $W_n^o = ED(W_n^i)$ is opposite to that in $W_n^i$ and it satisfies that $ED^1(W_n^i) = W_n^o$ and $ED^2(W_n^i) = ED^4(W_n^i)$. So $t(W_n^i) = p(W_n^i) = 2$.

**Lemma 2.11** $ED(\overline{C_n}) = C_n$.

Note that the direction of any arc in $ED(\overline{C_n})$ is the same to that in the given cycle $C_n$. By proposition 1.3, $ED^2(\overline{C_n}) = C_n^*$, where the direction of any arc of $C_n^*$ is opposite to that in the directed cycle $C_n$.

**Lemma 2.12** The eccentric digraph of the complement of $P_n$ satisfies that $ED(\overline{P_n})=F_{n-1}^\bullet$ and $ED^2(\overline{P_n})=ED^4(\overline{P_n})=ED(F_{n-1}^\bullet)$, where $v_n$ is the center of $F_{n-1}^\bullet$. So $t(\overline{P_n}) = p(\overline{P_n}) = 2$.

**Lemma 2.13** Let $rP_2$ be a digraph in the following Figure 5, then $ED(rP_2) = (K_{2r})^* - E(rP_2)$.

**Lemma 2.14** Let the digraph $K_{m,n} - E(rP_2) = (mK_1 \oplus nK_1) - E(rP_2)$, then $ED(K_{m,n} - E(rP_2)) = ED^3(K_{m,n} - E(rP_2))$. So $t(K_{m,n} - E(rP_2)) = 1$, $p(K_{m,n} - E(rP_2)) = 2$, where $1 \leq r \leq \min\{m, n\}$.

**Lemma 2.15** Let $S_{m,n}^i$ $(i = 1, 2)$ be a directed double-star in the Figure 6 and Figure 7, then

(1) $ED^k(S_{m,n}^1) = \begin{cases} G_{1.1}, & \text{if k odd,} \\ G_{1.2}, & \text{if k even.} \end{cases}$

Note that $G_{1.1}$ is isomorphic to $G_{1.2}$ (See Figure 7 and Figure 8).

(2) $ED^3(S_{m,n}^2) = K_1 \oplus (K_{m+n+1})^*$ and $ED^2(S_{m,n}^2) = ED^4(S_{m,n}^2)$.

**Theorem 2.1** Let $G$ be a digraph with $|V(G)| = n$. If there is one vertex of in-degree $n-1$ and out-degree 0, then the vertex has out-degree $n-1$ in the eccentric digraph $ED(G)$.

**Proof:** Let $V(G) = \{v_1, v_2, ..., v_{n-1}, c\}$ and $c$ be the vertex of in-degree $n-1$ and out-degree 0 in $G$. Then $e(c) = \infty$. Hence, every other vertex $v_i$ $(i = 1, 2, ..., n-1)$ is the eccentric vertex of $c$. Thus, $c$ is a vertex of out-degree $n-1$ in $ED(G)$. □

**Theorem 2.2** Let $G$ be a digraph with $|V(G)| = n+1$. If there is one vertex of out-degree $n$ and in-degree 0 and others are reachable each other, then $ED(G) = G_1$ and $ED(G) = ED^3(G)$.

**Proof:** Suppose that $V(G) = \{v_1, v_2, ..., v_n, c\}$ and $c$ is the vertex of out-degree

$n$ and in-degree 0. Since $e(c) = 1$ then the other vertex $v_i$ is the eccentric vertex of $c$ for any $i = 1, 2, ..., n$. Since $indegree(c) = 0$, then $e(v_i) = \infty$ for $i = 1, 2, ..., n$. Since $v_i$ and $v_j$ are reachable for $i \neq j$, then $c$ is the only eccentric vertex of $v_i$ for any $i = 1, 2, ..., n$. From the above, we get that $ED(G) = G_1$. Furthermore, by lemma 2.3 we know that $ED^3(G) = ED(G)$. □

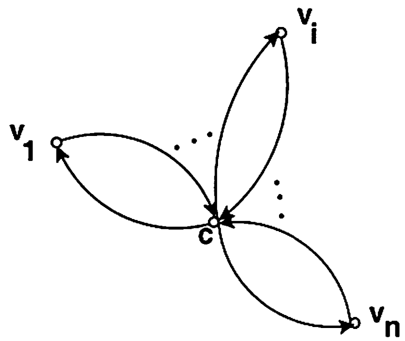


Figure 1 : $G_1$

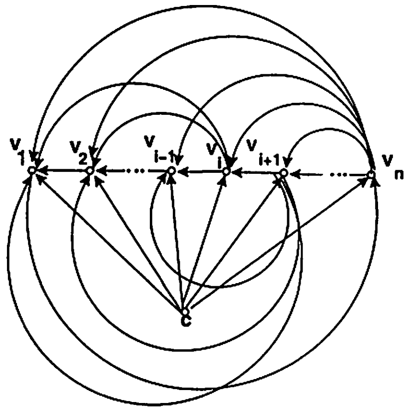


Figure 2 : $ED(F_n^i)$

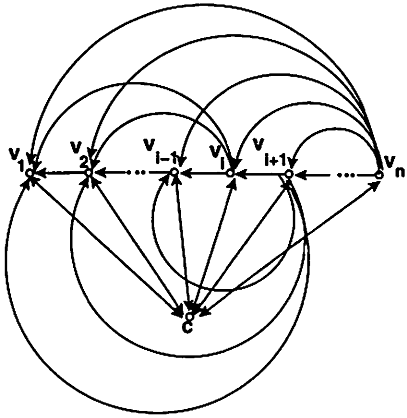


Figure 3 : $ED(F_n^{\circ})$

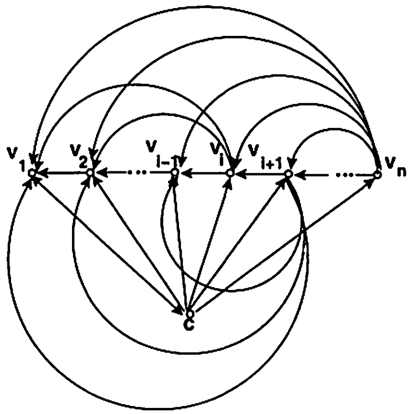


Figure 4 : $ED(F_n^{\bullet})$

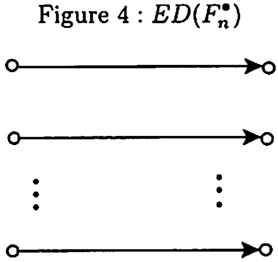

Figure 5 : $rP_2$

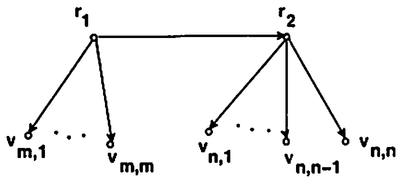


Figure 6 : $S^1_{m,n}$

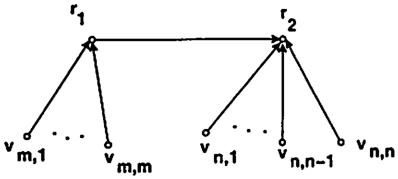


Figure 7 : $S^2_{m,n}$

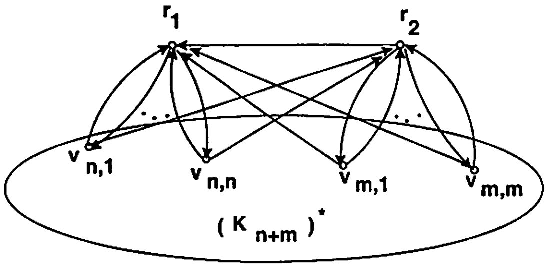


Figure 8 : $G_{1,1}$

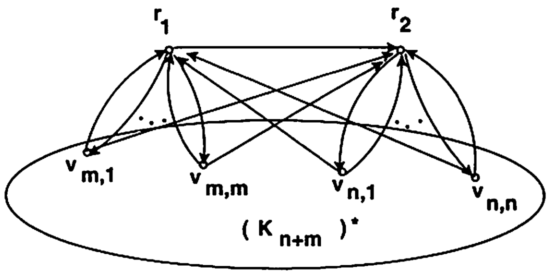


Figure 9 : $G_{1,2}$

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