$(\in, \in \vee q)-$ fuzzy ideals of K-algebras

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Abstract

Our work in this paper is concerned with a new kind of fuzzy ideal of a K-algebra called an $(\in, \in \vee q)$- fuzzy ideal. We investigate some interesting properties of $(\in, \in \vee q)$-fuzzy ideal of K-algebras. We study fuzzy ideals with thresholds which is a generalization of both fuzzy ideals and $(\in, \in \vee q)$-fuzzy ideals. We also present characterization theorems of implication-based fuzzy ideals.

Keywords: K-algebras; Fuzzy point; Level set; $(\in, \in \vee q)$- fuzzy ideals; Fuzzy ideals with thresholds; Implication-based fuzzy ideals.
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1 Introduction

The notion of a K-algebra $(G, \cdot, \odot, e)$ was introduced by Dar and Akram in [9]. A K-algebra was built on a group $(G, \cdot, e)$ with identity element e, by adjoining the induced binary operation $\odot$ on $(G, \cdot, e)$. It is attached to an abstract K-algebra $(G, \cdot, \odot, e)$, which is non-commutative and non-associative with right identity element e. It is proved in [3, 9] that a K-algebra on an abelian group is equivalent to a p-semisimple BCI-algebra. For the convenience of study, authors renamed *a K-algebra built on a group G* as a $K(G)$-algebra [10]. The $K(G)$-algebra has been characterized by using its left and right mappings in [10]. Dar and Akram [11] have further proved that the class of $K(G)$-algebras is a generalized class of B-algebras [18] when $(G, \cdot, e)$ is a non-abelian group, and they also proved that

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the $K(G)$-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [14, 15, 16] when $(G, \cdot, e)$ is an abelian group.
The concept of a fuzzy set was introduced by Zadeh [24], and it is now a rigorous area of research with manifold applications ranging from engineering and computer science to medical diagnosis and social behavior studies. Rosenfeld [20] introduced notion of fuzzy subgroup of a group in 1971. Since then, many scholars have studied the theories of fuzzy subgroups of a group. On the other hand, the concept of quasi-coincidence of a fuzzy point in a fuzzy subset was introduced by Pu and Liu [19]. Bhakat and Das [6] introduced the concept of $(\in, \in \vee q)$-fuzzy subgroups by using the *belongs to* relation $(\in)$ and *quasi-coincident* with relation (q) between a fuzzy point and a fuzzy subgroup. Akram *et al.* [1] introduced fuzzy structures of K-algebras. Since then, the concepts and results of K-algebras have been broadened to the fuzzy setting frames (see, [3,4, 8, 17]). In this paper we introduce a new kind of fuzzy ideal of a K-algebra called $(\in, \in \vee q)$- fuzzy ideal which is a useful generalization of Akram *et al.'s* fuzzy ideal. We also give the definition of an implication-based fuzzy ideal and discuss relations between two fuzzy ideals.

2 Preliminaries

In this section we review some elementary aspects that are necessary for this paper.
A K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ is an algebra of type (2, 2, 0) defined on a group $(G, \cdot, e)$ in which each non-identity element is not of order 2 and observes the following $\odot$-axioms:

(K1) $(x \odot y) \odot (x \odot z)=(x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(K3) $x \odot x = e$,

(K4) $x \odot e = x$,

(K5) $e \odot x = x^{-1}$

for all x, y, $z \in G$.
If the group $(G, \cdot, e)$ is abelian , then the above axioms $(K1)$ and $(K2)$ can be replaced by:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* [9] of the K-algebra $\mathcal{K}$ if $a \odot b \in H$ for all a, $b \in H$. Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot, e)$. A mapping $f : \mathcal{K}_1 = (G_1, \cdot, \odot, e_1) \rightarrow \mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ of K-algebras is called a *homomorphism* [11] if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. We note that if f is a homomorphism, then $f(e) = e$. A nonempty subset I of a K-algebra $\mathcal{K}$ is called an *ideal* [1] of $\mathcal{K}$ if it satisfies:

(i) $e \in I$,

(ii) $x \odot y \in I$, $y \odot (y \odot x) \in I \Rightarrow x \in I$ for all x, $y \in G$.

Let μ be a *fuzzy set* on G, i.e., a map $\mu : G \rightarrow [0, 1]$.

Definition 2.1. [1] A fuzzy ideal of a K-algebra $\mathcal{K}$ is a mapping $\mu : G \rightarrow [0, 1]$ such that

(i) $(\forall x \in G)\ (\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y \in G)\ (\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Definition 2.2. [19] A fuzzy set μ in a set G of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if y=x ,} \\ 0, & \text{y} \neq \text{x} \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $F(x; t)$.

Definition 2.3. A fuzzy point $F(x; t)$ is said to "*belonging to*" a fuzzy set μ, written as $F(x; t) \in \mu$ if $\mu(x) \geq t$. A fuzzy point $F(x; t)$ is said to be "*quasicoincident with*" a fuzzy set μ, denoted by $F(x; t)q\mu$ if $\mu(x) + t > 1$.

Throughout this paper, the following notations will be used.

(i) "$F(x; t) \in \mu$" or "$F(x; t) \in qF(x; t)$" will be denoted by $F(x; t) \in \vee q\mu$.

(ii) "$F(x; t)\overline{\in}\mu$ and $F(x; t)\overline{\in \vee q}\mu$" mean that "$F(x; t) \in \mu$ and $F(x; t) \in \vee q\mu$" do not hold, respectively.

3 $(\in, \in \vee q)$- fuzzy ideals

Definition 3.1. A fuzzy set μ in G is called an $(\in, \in \vee q)$-*fuzzy ideal* of $\mathcal{K}$, if it satisfies the following conditions:

(i) $F(x; s) \in \mu \Longrightarrow F(e; s) \in \vee q\mu$,

(ii) $F(x \odot y; s) \in \mu$ and $F(y \odot (y \odot x); t) \in \mu \Longrightarrow F(x; \min(s,t)) \in \vee q\mu$

for all x, $y \in G$, s, $t \in (0,1]$.

Noted that an $(\in, \in)$-fuzzy ideal is in fact, a fuzzy ideal.

Example 3.2. Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on the cyclic group $G = \{e, a, b, c, d, f\}$ of order 6, where $b = a^2$, $c = a^3$, $d = a^4$, $f = a^5$ and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	b	c	d	f
e	e	f	d	c	b	a
a	a	e	f	d	c	b
b	b	a	e	f	d	c
c	c	b	a	e	f	d
d	d	c	b	a	e	f
f	f	d	c	b	a	e

Define a fuzzy set μ in $\mathcal{K}$ by

$$\mu(x) = \begin{cases} 0.77, & \text{if x =e,} \\ 0.85, & \text{if x =a, b,} \\ 1.0, & \text{if x =c, d, f.} \end{cases}$$

It is easy to see that μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$.

Proposition 3.3. *Every $(\in, \in)$-fuzzy ideal is an $(\in, \in \vee q)$-fuzzy ideal.*

Proof. Let μ be an $(\in, \in)$- fuzzy ideal of $\mathcal{K}$. Condition (i) of Definition 3.1 is trivial. Let x, $y \in G$ and t_1, $t_2 \in (0,1]$ be such that $F(x \odot y; t_1) \in \mu$ and $F(y \odot (y \odot x); t_2) \in \mu$. Then $F(x \odot y; t_1) \in \mu$ and $F(y \odot (y \odot x); t_2) \in \mu$ $\Longrightarrow F(x; \min(t_1, t_2)) \in \vee q\mu$. Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$. □

Converse of Proposition 3.3 may not be true as seen in the following example.

Example 3.4. Consider the K-algebra $\mathcal{K} = (S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Define a fuzzy set μ in $\mathcal{K}$ by

$$\mu(t) = \begin{cases} 0.6, & \text{if t =e,} \\ 0.7, & \text{if t =a,} \\ 0.8, & \text{if t =b, x} \\ 0.4, & \text{if t =y, z.} \end{cases}$$

By routine computations, it is easy to see that μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$. But, μ is not an $(\in, \in)$- fuzzy ideal of $\mathcal{K}$ since $F(x \odot y; 0.65) \in \mu$ and $F(y \odot (y \odot x); 0.37) \in \mu$, but $F(x; \min(0.65, 0.37)) = F(x; 0.37)\overline{\in}\mu$.

Remark. If μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$, then $0.5 \leq \mu(e) \leq 1$.

The following propositions are obvious.

Proposition 3.5. *Let μ be an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$.*

(i) *If there exists $x \in G$ such that $0.5 \leq \mu(x) \leq 1$, then $0.5 \leq \mu(e) \leq 1$.*

(ii) *If $0 \leq \mu(e) < 0.5$, then μ is an $(\in, \in)$-fuzzy ideal of $\mathcal{K}$.*

Proposition 3.6. *Let $\{\mu_i : i \in I\}$ be any family of $(\in, \in \vee q)$-fuzzy ideals of $\mathcal{K}$. Then $\mu = \cap_i \mu_i$ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$.*

Theorem 3.7. *Let $G_0 \subset G_1 \subset \cdots \subset G_n = G$ be a strictly increasing chain of an $(\in, \in)$-fuzzy ideals of a K-algebras $\mathcal{K}$, then there exists $(\in, \in)$-fuzzy ideal μ of $\mathcal{K}$ whose level ideals are precisely the members of the chain with $U(\mu; 0.5) = G_0$.*

Proof. Let $\{t_i : t_i \in (0, 0.5], i = 1, 2, \cdots, n\}$ be such that $t_1 > t_2 > t_3 > \cdots > t_n$. Let $\mu : G \longrightarrow [0, 1]$ defined by

$$\mu(x) = \begin{cases} t, & \text{if x =0,} \\ n, & \text{if x =0, } x \in G_0 \\ t_1, & \text{if x} \in G_1 \setminus G_0, \\ t_2, & \text{if x} \in G_2 \setminus G_1, \\ \vdots & \\ t_n, & \text{if x} \in G_n \setminus G_{n-1}. \end{cases}$$

Condition (i) of Definition 3.1 is trivial. Let $x, y \in G$. If $x \in G_0$, then

$$\mu(x) \geq 0.5 \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5).$$

On the other hand, If $x \notin G_0$, then there exists i, $1 \leq i \leq n$ such that $x \in G_i \setminus G_{i-1}$ so that $\mu(x) = t_i$. Now there exists $j(\geq i)$ such that $x \odot y \in G_j$

or $y \odot (y \odot x) \in G_j$. If $x \odot y$, $y \odot (y \odot x) \in G_k (k < i)$, then G_k is an ideal of $\mathcal{K}$, $x \in G_k$ which contradicts $x \notin G_{i-1}$. Thus

$$\mu(x) \geq t_i \geq t_j \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5).$$

Hence μ is an $(\in, \in)$-fuzzy ideal of $\mathcal{K}$. It follows from the contradiction of μ that $U(\mu; 0.5) = G_0$, $\mu_{t_i} = G_i$ for $i = 1, 2, \cdots, n$. The proof is completed. □

Definition 3.8. An $(\in, \in)$-fuzzy ideal of a K-algebra $\mathcal{K}$ is said to be *proper* if Imμ has at least two elements. Two $(\in, \in)$- fuzzy ideals μ and λ are said to be *equivalent* if they have same family of level ideals. Otherwise, they are said to be non- equivalent.

Theorem 3.9. *A proper $(\in, \in)$-fuzzy ideal of $\mathcal{K}$ such that cardinality of Im $\mu \geq 3$, can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy ideals of $\mathcal{K}$.*

Proof. Let μ be a proper $(\in, \in)$-fuzzy ideal of $\mathcal{K}$ with $\{\mu(x) : \mu(x) < 0.5\} = \{t_1, t_2, \cdots, t_n\}$ where $t_1 > t_2 > \cdots > t_n$ and $n \geq 2$. Then

$$U(\mu; 0.5) \subseteq U(\mu; t_1) \subseteq \cdots \subseteq U(\mu; t_n) = G$$

is the chain of $(\in, \in)$-ideals of μ. Define two fuzzy sets λ_1, $\lambda_2 \leq \mu$ defined by

$$\lambda_1(x) = \begin{cases} t_1, & \text{if x} \in U(\mu; t_1), \\ t_2, & \text{if x} \in U(\mu; t_2) \setminus U(\mu; t_1), \\ \vdots & \\ t_n, & \text{if x} \in U(\mu; t_n) \setminus U(\mu; t_{n-1}), \end{cases}$$

$$\lambda_2(x) = \begin{cases} \mu(x), & \text{if x} \in U(\mu; 0.5), \\ n, & \text{if x} \in U(\mu; t_2) \setminus U(\mu; 0.5), \\ t_3, & \text{if x} \in U(\mu; t_3) \setminus U(\mu; t_2), \\ \vdots & \\ t_n, & \text{if x} \in U(\mu; t_n) \setminus U(\mu; t_{n-1}), \end{cases}$$

respectively, where $t_3 < n < t_2$. Then λ_1 and λ_2 are $(\in, \in)$-fuzzy ideals of $\mathcal{K}$ with

$$U(\mu; t_1) \subseteq U(\mu; t_2) \subseteq \cdots \subseteq U(\mu; t_n)$$

and

$$U(\mu; t_{0.5}) \subseteq U(\mu; t_2) \subseteq \cdots \subseteq U(\mu; t_n)$$

being respectively chains of $(\in, \in)$-fuzzy ideals. Hence λ_1 and λ_2 are non-equivalent and $\mu = \lambda_1 \cup \lambda_2$. □

Theorem 3.10. *A fuzzy set μ in $\mathcal{K}$ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$ if and only if it satisfies*

$$\mu(x) \quad \geq \quad \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5) \tag{1}$$

for all x, $y \in G$.

Proof. Suppose that μ is an $(\in, \in \vee q)$-fuzzy ideal of K-algebra $\mathcal{K}$ and let x, $y \in G$. If $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) < 0.5$, then $\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$. Assume that $\mu(x) < \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$ and choose t such that $\mu(x) < t < \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$. Then $F(x \odot y; t) \in \mu$ and $F(y \odot (y \odot x); t) \in \mu$ but $F(x; \min(t, t)) = F(x; t) \overline{\in \vee q}\, \mu$, a contradiction. Thus $\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$ whenever $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) < 0.5$. Now suppose that $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) \geq 0.5$. Then $F(x \odot y; 0.5) \in \mu$ and $F(y \odot (y \odot x); 0.5) \in \mu$ which imply that

$$F(x; \min(0.5, 0.5)) = F(x; 0.5) \in \vee q\mu.$$

Hence $\mu(x) \geq 0.5$. Otherwise, $\mu(x) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5)$ for all x, $y \in G$.
Conversely, assume that (1) is valid. Let x, $y \in G$ and t_1, $t_2 \in (0, 1]$ be such that $F(x \odot y; t_1) \in \mu$ and $F(y \odot (y \odot x); t_2) \in \mu$. Then $\mu(x \odot y) \geq t_1$ and $\mu(y \odot (y \odot x)) \geq t_2$. If $\mu(x) < \min(t_1, t_2)$, then $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) \geq 0.5$. Otherwise, we have

$$\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)) \geq \min(t_1, t_2)),$$

a contradiction. If follows that

$$\mu(x) + \min(t_1, t_2) > 2\mu(x) \geq 2 \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5) = 1$$

so that $F(x; \min(t_1, t_2)) q\mu$. Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$.

□

Theorem 3.11. *Let μ be a fuzzy set of fuzzy ideal of $\mathcal{K}$. Then μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$ if and only if $U(\mu; t)$, $t \in (0, 0.5]$, is an ideal of $\mathcal{K}$ when it is nonempty.*

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$ and let $t \in (0, 0.5]$ be such that $U(\mu; t) \neq \emptyset$. Let $x \odot y$, $y \odot (y \odot x) \in U(\mu; t)$. Then $\mu(x \odot y) \geq t$ and $\mu(y \odot (y \odot x)) \geq t$. It follows that

$$\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5) \geq \min(t, 0.5) = t$$

so that $x \in U(\mu; t)$. Hence $U(\mu; t)$ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$.
Conversely, suppose that the nonempty set $U(\mu; t)$ is an ideal of $\mathcal{K}$ for all

$t \in (0, 0.5]$. If the condition (1) is not true, then there exists a, $b \in G$ such that $\mu(a) < \min(\mu(a \odot b), \mu(b \odot (b \odot a)), 0.5)$. Choose

$$t_1 := \tfrac{1}{2}(\mu(a) + \min(\mu(a \odot b), \mu(b \odot (b \odot a)), 0.5))$$

such that $\mu(a) < t_1 < \min(\mu(a \odot b), \mu(b \odot (b \odot a)), 0.5)$. It follows that $a \odot b$, $b \odot (b \odot a) \in U(\mu; t_1)$ and $a \notin U(\mu; t_1)$. This is a contradiction. Therefore the condition (1) is valid, and so μ is an $(\in, \in \vee q)$-fuzzy ideal of $\mathcal{K}$. □

Theorem 3.12. *Let μ be a fuzzy set of a K-algebra $\mathcal{K}$. Then $U(\mu; t)(\neq \emptyset)$ is an ideal of $\mathcal{K}$ for all $t \in (0, 0.5]$ if and only if μ satisfies the following assertion:*

$$\max(\mu(x), 0.5) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$$

for all x, $y \in G$.

Proof. If there exist x, $y \in G$ such that

$$\max(\mu(x \odot y), 0.5) < t = \min(\mu(x \odot y), \mu(y \odot (y \odot x))),$$

then $t \in (0.5, 1]$, $\mu(x) < t$ and $x \odot y$, $y \odot (y \odot x) \in U(\mu; t)$. Since $U(\mu; t)$ is an ideal of $\mathcal{K}$ for all $t \in (0.5, 1]$, it follows that $x \in U(\mu; t)$ so that $\mu(x) \geq t$, a contradiction. Hence assertion is valid.
Conversely, let $t \in (0.5, 1]$ and $x \odot y$, $y \odot (y \odot x) \in U(\mu; t)$. Then

$$\max(\mu(x), 0.5) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x))) \geq t > 0.5$$

and so $\mu(x) \geq t$. Hence $x \in U(\mu; t)$ and $U(\mu; t)$ is an ideal of $\mathcal{K}$. This ends the proof. □

Remark. From Theorem 3.12, we know that a fuzzy set μ in $\mathcal{K}$ may satisfy the condition that $U(\mu; t)$ is an ideal of $\mathcal{K}$ for some $t \in (0, 1]$; but $U(\mu; t)$ is not an ideal of $\mathcal{K}$ for some $t \in (0, 1]$. Let

$$G^* := \{t \in (0, 1] \mid U(\mu, t) \textit{ is an ideal of } \mathcal{K}\}.$$

(i) If $G^* = (0, 1]$; then μ is a fuzzy ideal of $\mathcal{K}$.

(ii) If $G^* = (0.5, 1]$; then μ is an $(\in, \in \vee q)-$ fuzzy ideal of $\mathcal{K}$.

(iii) If $G^* = (\rho_1, \rho_2]$, $\rho_1 < \rho_2$, then μ is fuzzy ideal with thresholds.

Yuan *et al.*[23] gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld[20], and Bhakat-Dass fuzzy subgroup [5]. Based on Yuan *et al.*, we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy K-algebras with thresholds in the following way:

Definition 3.13. Let ρ_1,$\rho_2 \in [0,1]$ and $\rho_1 < \rho_2$. Let μ be a fuzzy set of a K-algebra $\mathcal{K}$. Then μ is called *fuzzy subalgebra with thresholds* (ρ_1, ρ_2) of $\mathcal{K}$, if

$$\max(\mu(x \odot y), \rho_1) \geq \min(\mu(x), \mu(y), \rho_2)$$

for all x, $y \in G$.

Definition 3.14. Let ρ_1,$\rho_2 \in [0,1]$ and $\rho_1 < \rho_2$. Let μ be a fuzzy set of a K-algebra $\mathcal{K}$. Then μ is called *fuzzy ideal with thresholds* (ρ_1, ρ_2) of $\mathcal{K}$, if

$$\max(\mu(x), \rho_1) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), \rho_2)$$

for all x, $y \in G$.

Example 3.15. Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3	a^4
e	e	a^4	a^3	a^2	a
a	a	e	a^4	a^3	a^2
a^2	a^2	a	e	a^4	e^3
a^3	a^3	a^2	a	e	e^4
a^4	a^4	a^3	a^2	a	e

If we define a fuzzy set μ in $\mathcal{K}$ by

$$\mu(x) = \begin{cases} 0.5, & \text{if x =e}, \\ 0.7, & \text{if x =a}, \\ 0.4, & \text{if x =}a^2, \\ 0.3, & \text{if x =}a^3,\ a^4. \end{cases}$$

Then it is easy to see that:

(1) μ is fuzzy subalgebra and ideal with thresholds $(\rho_1 = 0.3\ ,\ \rho_2 = 0.56)$ of $\mathcal{K}$.

(2) μ is not fuzzy subalgebra with thresholds $(\rho_1 = 0.55,\ \rho_2 = 0.78)$ of $\mathcal{K}$ since

$$\max(\mu(a \odot a), 0.55) = 0.55 < 0.7 = \min(\mu(a), \mu(a), 0.78).$$

(3) μ is not an $(\in, \in)$-fuzzy subalgebra of $\mathcal{K}$ since

$$\mu(a \odot a^2) = \mu(a^4) = 0.3 < 0.4 = \min(\mu(a), \mu(a^2)).$$

(4) μ is fuzzy subalgebra with thresholds ($\rho_1 = 0.75$, $\rho_2 = 0.85$) of $\mathcal{K}$.

Remark. In what follows let ρ_1, $\rho_2 \in [0,1]$ and $\rho_1 < \rho_2$ unless otherwise specified.
(1) when $\rho_1 = 0$ and $\rho_2 = 1$ in fuzzy ideal with thresholds (ρ_1, ρ_2), μ is an ordinary fuzzy ideal.
(2) when $\rho_1 = 0$ and $\rho_2 = 0.5$ in fuzzy ideal with thresholds (ρ_1, ρ_2), μ is an $(\in, \in \vee q)$-fuzzy ideal.
(3) A fuzzy ideal is a fuzzy ideal with some thresholds (ρ_1, ρ_2).
(4) An $(\in, \in \vee q)$-fuzzy ideal is a fuzzy ideal with some thresholds.

Theorem 3.16. *A fuzzy set μ of K-algebra $\mathcal{K}$ is a fuzzy ideal with thresholds (ρ_1, ρ_2) of $\mathcal{K}$ if and only if $U(\mu;t)(\neq \emptyset)$, $t \in (\rho_1, \rho_2]$, is a ideal of $\mathcal{K}$.*

Proof. Assume that μ is a fuzzy ideal with thresholds of $\mathcal{K}$. Let x, $y \in U(\mu;t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, $t \in (\rho_1, \rho_2]$. It follows that

$$\max(\mu(x), \rho_1) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), \rho_2) = t \Longrightarrow \mu(x) \geq t$$

so that $x \in U(\mu;t)$. Hence $U(\mu;t)$ is an ideal of $\mathcal{K}$.
Conversely, Assume that fuzzy set μ such that $U(\mu;t) \neq \emptyset$ is an ideal of $\mathcal{K}$ for ρ_1, $\rho_2 \in [0,1]$ and $\rho_1 < \rho_2$. Suppose that $\max(\mu(x), \rho_1) < \min(\mu(x \odot y), \mu(y \odot (y \odot x)), \rho_2) = t$, then $\mu(x) < t$, $x \odot y \in U(\mu;t)$, $y \odot (y \odot x) \in U(\mu;t)$, $t \in (\rho_1, \rho_2]$. Since $x \odot y$, $y \odot (y \odot x) \in U(\mu;t)$ and $U(\mu;t)$ is an ideal, $x \in U(\mu;t)$, i.e., $\mu(x) \geq t$, a contradiction. This ends the proof. □

4 Implication-based fuzzy ideals

Fuzzy logic is an extension of set theoretic multi-valued logic in which the truth values are linguistic variables (or terms of the linguistic variable truth). Some operators, like $\wedge$, $\vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators.
In fuzzy logic, the truth value of fuzzy proposition p is denoted by $[p]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this paper.

1. $[x \in p] = p(x)$,
2. $[p \wedge q] = \min([p], [q])$,
3. $[p \rightarrow q] = \min(1, 1 - [p] + [q])$,
4. $[\forall x p(x)] = \inf_{x \in U}[p(x)]$,

5. $\models p$ if and only if $[p] = 1$ for all valuations.

The truth valuation rules given in (4) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following:

A. Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(x, y) := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise} . \end{cases}$$

B. Gödel implication operator (I_G):

$$I_G(x, y) := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise} . \end{cases}$$

C. The contraposition of Gödel implication operator ($\overline{I}_G$):

$$\overline{I}_G(x, y) := \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x & \text{otherwise} . \end{cases}$$

Ying [21] introduced the concept of fuzzifying topology. We can extend this concept to K-algebras, and we define a fuzzifying ideal as follows:

Definition 4.1. A fuzzy set μ in G is called a *fuzzfying ideal* of $\mathcal{K}$ if for any x, $y \in G$:

a. $\models [x \in \mu] \rightarrow [e \in u]$,

b. $\models \min([x \odot y \in \mu], [y \odot (y \odot x) \in \mu]) \rightarrow [x \in u]$.

Obviously, Definition 4.1 is equivalent to the Definition 2.1. Hence a fuzzifying ideal is an ordinary fuzzy ideal. Ying [22] introduced the concept of t-topology, i.e., $\models_t p$ if and only if $[p] \geq t$ for all valuations.

Definition 4.2. Let μ be a fuzzy set of G and $t \in (0, 1]$. Then μ is called a *t-implication-based ideal* of $\mathcal{K}$ if for any x, $y \in G$:

c. $\models_t [x \in \mu] \rightarrow [e \in u]$,

d. $\models_t \min([x \odot y \in \mu], [y \odot (y \odot x) \in \mu]) \rightarrow [x \in u]$.

Proposition 4.3. *Let I be an implication operator. A fuzzy set μ of $\mathcal{K}$ is a t-implication based fuzzy ideal of $\mathcal{K}$ if and only if $I(\min(\mu(x \odot y), \mu(y \odot (y \odot x))), \mu(x)) \geq t$ for all x, $y \in G$.*

Proof. Straightforward. □

Theorem 4.4. *Let μ be a fuzzy set in G. If $I = I_G$, then μ is a 0.5-implication- based fuzzy ideal of $\mathcal{K}$ if and only if μ is a fuzzy ideal with thresholds ($\rho_1 = 0$, $\rho_2 = 0.5$) of $\mathcal{K}$.*

Proof. Suppose that μ is a 0.5-implication based ideal of $\mathcal{K}$. Then $I_G(\min(\mu(x \odot y), \mu(y \odot (y \odot x))), \mu(x)) \geq 0.5$, and hence $\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$ or $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) \geq \mu(x) \geq 0.5$. It follows that

$$\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5).$$

so that μ is a fuzzy ideal with thresholds ($\rho_1 = 0$, $\rho_2 = 0.5$) of $\mathcal{K}$.
Conversely, if μ is a fuzzy ideal with thresholds ($\rho_1 = 0$, $\rho_2 = 0.5$) of $\mathcal{K}$. Then
$\mu(x) = \max(\mu(x), 0) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5)$. If $\min(\mu(x \odot y), \mu(y \odot (y \odot x)), 0.5) = \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$, then

$$I_G(\min(\mu(x \odot y), \mu(y \odot (y \odot x)), \mu(x)) = 1 \geq 0.5.$$

Otherwise, $I_G(\min(\mu(x \odot y), \mu(y \odot (y \odot x)), \mu(x)) \geq 0.5$.
Hence μ is a 0.5-implication based ideal of $\mathcal{K}$. □

Theorem 4.5. *Let μ be a fuzzy set in G. If $I = \overline{I}_G$, then μ is a 0.5-implication- based fuzzy ideal of $\mathcal{K}$ if and only if μ is a fuzzy ideal with thresholds ($\rho_1 = 0.5$, $\rho_2 = 1$) of $\mathcal{K}$.*

Proof. Suppose that μ is a 0.5-implication based ideal of $\mathcal{K}$. Then $\overline{I}_G(\min(\mu(x \odot y), \mu(y \odot (y \odot x)), \mu(x)) \geq 0.5$ which implies that $\mu(x) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)))$ or $1 - \min(\mu(x \odot y), \mu(y \odot (y \odot x))) \geq 0.5$, i.e, $\min(\mu(x \odot y), \mu(y \odot (y \odot x))) \leq 0.5$.
Thus

$$\max(\mu(x), 0.5) \geq \min(\mu(x \odot y), \mu(y \odot (y \odot x)), 1).$$

Hence μ is a fuzzy ideal with thresholds ($\rho_1 = 0.5$, $\rho_2 = 1$) of $\mathcal{K}$
The proof of converse part is obvious. □

Theorem 4.6. *Let μ be a fuzzy set in G. If $I = I_{GR}$, then μ is a 0.5-implication- based fuzzy ideal of $\mathcal{K}$ if and only if μ is a fuzzy ideal with thresholds ($\rho_1 = 0$, $\rho_2 = 1$) of $\mathcal{K}$.*

Proof. Obvious. □

As a consequence of the above Theorems we obtain the following corollary.

Corollary 4.7. (1) *Let $\mathcal{K} = \mathcal{K}_{GR}$. Then μ is an implication-based fuzzy ideal of $\mathcal{K}$ if and only if μ is a Akram et al.'s fuzzy ideal of $\mathcal{K}$.*

(2) *Let $\mathcal{K} = \mathcal{K}_G$. Then μ is an implication-based fuzzy ideal of $\mathcal{K}$ if and only if μ is an $(\in, \in \vee q)$- fuzzy ideal of $\mathcal{K}$.*

5 Conclusions

In the present paper, we have introduced a new kind of fuzzy ideal of a K-algebra, namely, an $(\in, \in \vee q)$- fuzzy ideal and have explored some of its properties. It is known that logic is an essential tool for giving applications in mathematics and computer science and is also a technique for laying foundation. Non-classical logic has become a formal tool for computer science and computational intelligence so that we can use this tool to deal with fuzzy information and uncertain information. Thus our obtained results can be applied in: (1) engineering; (2) computer science: artificial intelligence, signal processing, genetic algorithm, neural networks, expert systems (3) medical diagnosis.

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