# Greedy Defining Sets in Latin Squares

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**Abstract**

Greedy defining sets have been studied first time by the author for graphs. In this paper we consider greedy defining sets for Latin squares and study the structure of these sets in Latin squares. We give a general bound for greedy defining numbers and linear bounds for greedy defining numbers of some infinite families of Latin squares. Greedy defining sets of circulant Latin squares are also discussed in the paper.

**AMS Classification:** 05B15, 05C15.

**Keywords:** Latin square, critical set, greedy coloring, greedy defining set.

## 1 Introduction

Let $K_n$ be the complete graph on $n$ vertices with vertex set $\{1, 2, \ldots, n\}$. By $K_n \Box K_n$ we mean the graph with vertex set $\{(i, j) : i, j = 1, 2, \ldots, n\}$, where two distinct vertices $(i_1, j_1)$ and $(i_2, j_2)$ are adjacent if and only if either $i_1 = i_2$ or $j_1 = j_2$. A Latin square is an $n \times n$ array from the numbers $1, 2, \ldots, n$ such that each of these numbers occurs in each row and column exactly once. We note that any $n \times n$ Latin square is equivalent to a proper vertex $n$-coloring of the graph $K_n \Box K_n$.

In a graph $G$, a set of vertices $S$ with an assignment of colors is said to be a *defining set*, if there exists a unique extension of the colors of $S$ to a $\chi(G)$-coloring of the vertices of $G$. Since any $n \times n$ Latin square is equivalent to an $n$-coloring of the graph $K_n \square K_n$, then defining sets can also be defined for Latin squares. In this case a defining set for a Latin square which does not contain properly another defining set is known as a *critical set*. Defining sets of graphs are widely studied in the literature, see [6, 7, 8]. Also critical sets of Latin squares and their various aspects have been much studied, see for example [1, 2, 5, 8, 9]. For a recent survey on defining sets and critical sets see [4]. The concept of greedy defining set in graph colorings was studied for the first time in [10]. This new concept arises when we consider the greedy coloring of graphs. Given a graph $G$ and an order $\sigma$ on the vertex set of $G$. Let $\sigma$ order the vertices of $G$ as $v_1 \leq v_2 \leq \ldots \leq v_n$. The greedy coloring of $G$ corresponding to $\sigma$ starts with the first vertex $v_1$ and colors it by 1, and then scans the other vertices in turn according to the order $\sigma$ and at each time gives the minimum available color to the vertex to be colored. Greedy defining sets can be considered as a variation of ordinary defining sets when the vertices of graph are to be colored by a greedy coloring. We know that a greedy coloring of a graph $G$ does not always color it with minimum number of colors and sometimes it uses many more colors than $\chi(G)$. For example the greedy coloring of the tree (with given order) in Figure 1 uses four colors and there is a greedy coloring of the bipartite graph $K_{n,n} \setminus nK_2$ (complete bipartite graph minus a perfect matching) which uses exactly $n$ colors. Greedy defining sets can be used to reduce these extra number of colors in order for graph $G$, be colored greedily with $\chi(G)$ colors. We first begin with the definition of greedy defining sets for graphs.

**Definition 1.** *For a graph $G$ and an order $\sigma$ on $V(G)$, a greedy defining set is a subset $S$ of $V(G)$ with an assignment of colors to vertices in $S$, such that the pre-coloring can be extended to a $\chi(G)$- coloring of $G$ by the greedy coloring of $(G, \sigma)$ and fixing the colors of $S$. The greedy defining number of $G$ is the size of a greedy defining set which has minimum cardinality, and is denoted by* $\mathrm{GDN}(G, \sigma)$. *A greedy defining set for a $\chi(G)$-coloring $C$ of $G$ is a greedy defining set of $G$ which results in $C$. The size of a greedy defining set of $C$ with the smallest cardinality is denoted by* $\mathrm{GDN}(G, \sigma, C)$.

An example of a greedy defining set of size 2 for the graph $G$ is given in Figure 1, while the greedy coloring of $G$ uses four colors.

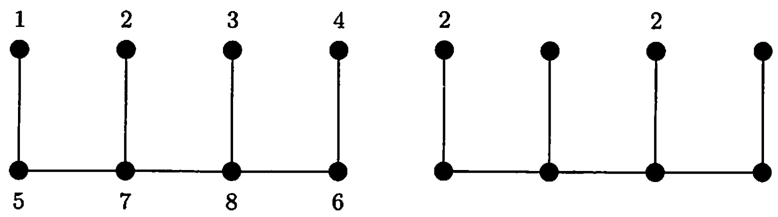



Figure 1: An ordered graph $G$ with GDN(G)=2

In [10], the computational complexity of determining the greedy defining number of a coloring of an ordered graph has been studied.

**Theorem 1.**([10]) *The following problem is $\mathcal{NP}$-complete:*

**Instance:** *An ordered graph $(G, \sigma)$, a $\chi(G)$-coloring $C$ and integer $k$.*

**Question:** $\mathrm{GDN}(G, \sigma, C) \leq k$?

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 3 | 1 | 2 | 5 | 4 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | **3** | 4 | 1 | 2 |
| **2** | 4 | 5 | 3 | 1 |

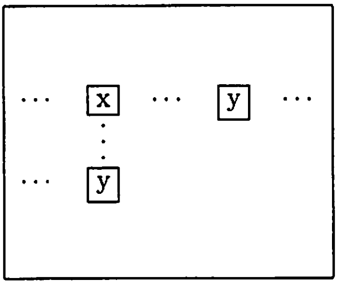
| | | | | | |
|---|---|---|---|---|---|
| **6** | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 1 | 5 | 6 | 4 |
| 3 | 4 | 5 | **6** | 1 | 2 |
| 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 6 | 4 | 2 | 3 | 1 |

Figure 2: Greedy defining sets for $5 \times 5$ and $6 \times 6$ Latin squares

# 2 Latin Squares

In this paper we study greedy defining sets and numbers for Latin squares. Throughout the paper the cells of Latin square are ordered lexicographically. It means that we scan cells from top row to below and in each row from left to right. Let $L$ be a Latin square. Then $L$ can be considered as a vertex coloring of $K_n \square K_n$ for some $n$. By a greedy defining set (or simply GDS) for $L$ we mean any GDS for $K_n \square K_n$ which results in the square $L$, where the order on $V(K_n \square K_n)$ is lexicographic order. Greedy defining sets of size 2 for $5 \times 5$ and $6 \times 6$ Latin squares are given in Figure 2.

**Definition 2.** *Let $L$ be any Latin square whose cells is ordered lexicographically. We define a descent to be three cells in the Latin square $L$ such as the following:*




*where $x$ and $y$ are any entries with $x > y$.*

An example of a descent is displayed in Figure 3, where the entries of the descent is specified by boxes.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 3 | 1 | 2 | 5 | 4 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | 3 | 4 | 1 | 2 |
| 2 | 4 | 5 | 3 | 1 |

Figure 3: An example of descent in a $5 \times 5$ Latin square

In the following we explain the structure of greedy defining sets in Latin squares in terms of hypergraph theory using the concept of descents. By a hypergraph $\mathcal{H} = (V, E)$, we mean any nonempty set $V$ (as the vertex set of $\mathcal{H}$) and a collection $E$ consisting of some subsets of $V$. We call every member of $E$ a hyperedge of $\mathcal{H}$. In any hypergraph $\mathcal{H}$, a blocking set is any subset of vertices $B$ such that $B$ intersects every hyperedge of $\mathcal{H}$.

**Definition 3.** *Let $L$ be a Latin square of order $n$. The hypergraph on $n^2$ entries of $L$ and consisting of all descents in $L$ is denoted by $\mathcal{H}_L$.*

**Theorem 2.** *Suppose a Latin square $L$ is given. A partial Latin square $D$ of $L$, is a greedy defining set (or GDS) of $L$ if and only if $D$ is a blocking set for $\mathcal{H}_L$.*

**Proof.** Suppose $D$ is a greedy defining set for $L$. Then obviously $D$ is a blocking set for $\mathcal{H}_L$, because $D$ should intersect any descent of $L$. Otherwise if $D$ does not intersect a descent with entries $x$, $y$ and $y$, where $x > y$, then when we color greedily the cells of $L$ and when we arrive at the position of $x$ in $D$, this position can not be colored by $x$ because $y < x$ and no neighbor of $x$ is previously colored by $y$ (the only entries of $y$ in the same row and column of $x$ appear after $x$). Therefore the resulting coloring is not the same as $L$, and hence $D$ can not be a greedy defining set.

Conversely, while coloring an arbitrary position $(i, j)$ with entry $x$ in $L$, we observe that any value $y < x$ either appears in the same row or column of $x$ and before $x$, or otherwise -forming a descent in $L$ which is intersected by $D$- is given in the defining set itself. Therefore the only admissible value to color the position $(i, j)$ is $x$ itself. □

Let $g_n$ denote the smallest size of a greedy defining set in an $n \times n$ Latin square. The following proposition shows when $g_n$ is zero.

**Proposition 1.** *$g_n = 0$ if and only if $n$ is a power of 2.*

**Proof.** $g_n = 0$ if and only if the greedy coloring of $K_n \square K_n$ with respect to lexicographic order results in a minimum coloring with $n$ colors. If $n = 2^m$, for some integer $m$, then the Latin square obtained by the greedy coloring of $K_n \square K_n$, is isomorphic to the Cayley table of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, which is the direct sum of $m$ copies of the additive cyclic group $\mathbb{Z}_2$ of order 2. For $n = 2^3$ this is displayed in Figure 4. Another way of seeing this is that in that Cayley table there exists no descent (because of the structure of the group), therefore no need to have an entry in a minimum greedy defining set, namely $g_n = 0$ in this case.

Now let $2^{k-1} < n < 2^k$, for some $k$. Let $A$ be the array obtained by the first $n$ rows and columns of the Cayley table of $k$ copies of $\mathbb{Z}_2$. Because the order in greedy coloring is the lexicographic order, we note that $A$ is in fact equivalent to the greedy coloring of $K_n \square K_n$. But we note that the number of colors used in $A$ is $2^k$. Since $2^k$ appears in the $(2^{k-1}+1)$-th row of that Cayley table and $2^{k-1} < n$ , hence $A$ contains this entry $2^k$. Therefore unless $n = 2^k$, we will not have $g_n = 0$. □

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

Figure 4: Cayley table of size 8

Besides the powers of 2, the only known exact values for $g_n$ are $g_3 = 1$, $g_5 = g_6 = 2$. It can be easily seen that $g_3 = 1$ and for $n = 5, 6$ we have checked by hand that one can not block all of the descents in an $n \times n$ Latin square by only one entry.

In the proof of the following theorem we use a construction based on tensor product of Latin squares. In order to define this concept, let $L$ and $K$ be two Latin squares with entries $\{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, m\}$ and sizes $n$ and $m$, respectively. By $L^r$, for each $r \in \{0, 1, \cdots, m-1\}$, we mean the $n \times n$ array obtained by adding $rn$ to any entry of $L$ (in other words $L^r = rnJ + L$, where $rnJ$ is an $n \times n$ array in which each entry is $rn$, and the operation + performs just like the sum of matrices). The tensor product of $L$ and $K$, denoted by $L \otimes K$, is the $nm \times nm$ Latin square obtained by replacing any entry in $K$ which is $r+1$ by $L^r$. There is an example in Figure 5.

Defining sets of cartesian product of graphs and direct product of Latin squares have been studied in [8] and [3, 5], respectively.

$$\begin{array}{|cc|}\hline 1 & 2 \\ 2 & 1 \\ \hline\end{array} \otimes \begin{array}{|ccc|}\hline 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ \hline\end{array} = \begin{array}{|cccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \\ \hline\end{array}$$

Figure 5: Tensor product of two Latin squares

**Theorem 3.** *Suppose $n = rs$, then*

$$g_n \leq r^2 g_s + (s^2 - g_s) g_r.$$

**Proof.** Suppose $L$ and $K$ are two Latin squares which have greedy defining sets of size $g_r$ and $g_s$, respectively. We prove the theorem by showing that $L \otimes K$ has a greedy defining set of size $s^2 g_r + (r^2 - g_r) g_s$. The Latin square $L \otimes K$ contains $s^2$ subsquares of the form $L^t$ in such a way that the subsquare in position $(i, j)$ is $L^t$ if the $(i, j)$-th entry in $K$ is $t + 1$.

Let $R$ and $S$ be GDS's for $L$ and $K$ with $g_r$ and $g_s$ entries respectively. Associated with any entry $i + 1$ in $S$ we have the subsquare $L^i$ in $L \otimes K$. We choose all entries of these subsquares, which are $|S||L|$ entries, i.e. $r^2 g_s$ entries. Now there remain $s^2 - g_s$ subsquares in $L \otimes K$. From each one we choose those entries whose positions correspond to the position of entries of the GDS $R$ in $L$. This provides $(s^2 - g_s) g_r$ new entries for our GDS for $L \otimes K$. Let us denote the resulting partial Latin square of $L \otimes K$ which consists of these $r^2 g_s + (s^2 - g_s) g_r$ entries by $F$. For example in Figure 6 our construction for $r = 2$ and $s = 5$ is displayed, where for $r = 2$ we have $g_2 = 0$ and for $s = 5$ we use the greedy defining set introduced in Figure 2.

We now prove that $F$ consisting of $r^2 g_s + (s^2 - g_s) g_r$ entries is in fact a GDS for $L \otimes K$. Using Theorem 2 it is enough to show that $F$ blocks any descent $D$ in $L \otimes K$. Noting that $L \otimes K$ consists of $s^2$ subsquares of the form $L^i$, there are two possibilities for $D$:

**1.** Three entries of $D$ lie together in a subsquare having type $L^i$, say. It can be easily seen that $\mathcal{H}_L$ is isomorphic to $\mathcal{H}_{L^i}$ since $L^r = rnJ + L$. Therefore in this case some entry of $D$ will also be in $F$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | 10 | 9 |
| 5 | 6 | 1 | 2 | 3 | 4 | 9 | 10 | 7 | 8 |
| 6 | 5 | 2 | 1 | 4 | 3 | 10 | 9 | 8 | 7 |
| 7 | 8 | 9 | 10 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 7 | 10 | 9 | 2 | 1 | 4 | 3 | 6 | 5 |
| 9 | 10 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 10 | 9 | 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 3 | 4 | 7 | 8 | 9 | 10 | 5 | 6 | 1 | 2 |
| 4 | 3 | 8 | 7 | 10 | 9 | 6 | 5 | 2 | 1 |

Figure 6: A GDS for $10 \times 10$ Latin square using the tensor product of Latin square of order 2 in Figure 5 and the Latin Square of order 5 in Figure 2

**2.** Three entries of $D$ lie in three different subsquares of $L \otimes K$. Suppose the largest entry $x$ of $D$ lies in a subsquare having the form $L^t$. Because other two entries of $D$ have the same value, they should be in different subsquares which have an identical form $L^p$, say. Considering $L \otimes K$ as $s^2$ subsquares, assume that the position of $L^t$ as a subsquare in $L \otimes K$ is $(i, j)$. Similarly, let the positions of the subsquares associated with two other entries of $D$ be $(i_1, j)$ and $(i, j_1)$ where $i > i_1$ and $j > j_1$. We have $t > p$ since $D$ is a descent. It turns out that the three entries of $K$ located at the positions $(i, j), (i_1, j)$ and $(i, j_1)$ form a descent in $K$. Consequently $F$ intersects $D$.

The case that two entries of $D$ place in a subsquare and another entry in a different subsquare is impossible, because let two entries $x$ and $y$ be in a subsquare $L^t$. This means that $x = tr + i_1$ and $y = tr + i_2$ for some $i_1 > i_2$. Let the position of $L^t$ in $L \otimes K$ be $(i, j)$. Hence the entry of $K$ in position $(i, j)$ is $t + 1$. Now the third entry of $D$ with value $y = tr + i_2$ lies in a different subsquare of $L \otimes K$ but in the same column of $x = tr + i_1$. It turns out that there is another entry $t + 1$ in the column $j$ of $K$, a contradiction. □

Determining $g_n$ for a given $n$ is not an easy task and therefore one important goal in studying $g_n$ is to find good upper bounds (in particular linear bounds) for $g_n$ even when $n$ belongs to some infinite subsets of integers. The following three theorems are obtained for this purpose.

**Theorem 4.** *Suppose* $n = 2^k - 1$ *for some integer* $k > 1$, *then*

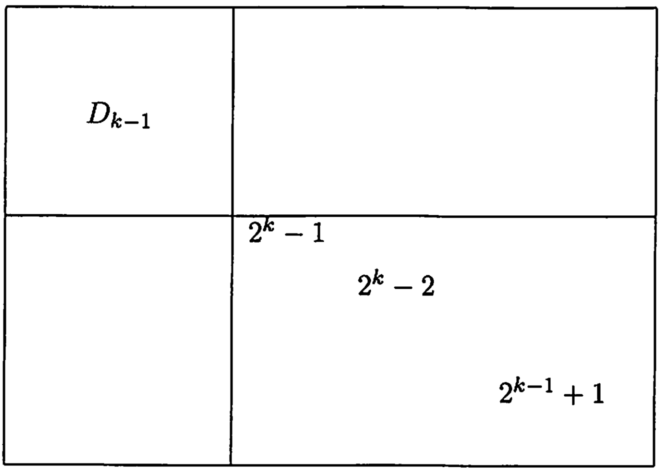
$$g_n \leq n - \log(n+1).$$

**Proof.** To prove the theorem it suffices to obtain a GDS say, $D_k$ of size $2^k-k-1$ for a Latin square $L_k$ of order $2^k-1$, since $2^k-k-1 = n-\log(n+1)$ when $n = 2^k - 1$. For this purpose we will construct our greedy defining sets recursively, i.e. $D_k$ will be obtained by $D_{k-1}$.

For $k = 2$ the Latin square $L_2$ and its greedy defining set $D_2$ is

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | [3] | 1 |
| 3 | 1 | 2 |

Let $D_{k-1}$ be a GDS of size $2^{k-1} - (k-1) - 1$ for a Latin square $L_{k-1}$ of order $2^{k-1} - 1$. We consider a partial Latin square $F$ of size $2^k - 1 \times 2^k - 1$ consisting of the entries of $D_{k-1}$ and extra entries in the principal diagonal, which are $2^k - 1, 2^k - 2, \ldots, 2^{k-1} + 1$ (from up to down) as follows:



We prove that $F$ is a GDS for a Latin square of order $2^k - 1$. In fact the resulting Latin square denoted by $L_k$ will consist of the entries illustrated in Figure 7. We consider the greedy coloring of the array $F$ where the order on cells is lexicographic. Let the resulting array be $H$. As the order on cells is lexicographic and $D_{k-1}$ is a GDS for $L_{k-1}$, it is obvious that the subarray generated by the first $2^{k-1} - 1$ rows and columns in $H$ will be the same as $L_{k-1}$. Now let $R_1$ be the subarray generated by the rows $1, 2, \ldots, 2^{k-1} - 1$ and columns $2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 1$ in $H$. As the Latin square $L_{k-1}$ appears before $R_1$, therefore any entry in $R_1$ is larger that $2^{k-1} - 1$, which is the order of $L_{k-1}$. On the other hand as the length of $R_1$ is $2^{k-1}$ (i.e. a power of two), using the proof of Proposition 1, we conclude that any entry in $R_1$ belongs to $\{2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 1\}$. Of course we note that the entries of the lower right subarray of $F$ don't effect the greedy coloring of $R_1$. Using a similar argument we obtain that the lower left subarray $R_2$ of $F$ will be the transpose of $R_1$. Therefore in two subarrays $R_1$ and $R_2$, there will be no entry larger than $2^k - 1$.

In the rest of proof we consider the lower right subarray $R_3$ of $H$ which is a square of order $2^{k-1}$. The entries in the principal diagonal of $R_3$ (except the last one) are given in the GDS and these entries are all larger than $2^{k-1}$ which is the order of $R_3$. On the other hand all the entries in $R_1$ and $R_2$ are larger than $2^{k-1} - 1$. Let $\tilde{R}_3$ be obtained by $R_3$ by replacing any entry in the diagonal of $R_3$ (except the last one) by zero. We note that $\tilde{R}_3$ is equivalent to the greedy coloring of $K_n \times K_n$, $n = 2^{k-1}$, where colors begin from 0 instead of 1. Hence in $\tilde{R}_3$ the entries are belong to $\{0, 1, \ldots, 2^{k-1} - 1\}$, where the diagonal entries are all zero. On the other hand the entries in the diagonal of $R_3$ (except the last one) are all larger than $2^{k-1}$. It turns out that in $R_3$ except its entries in the diagonal, all other entries belong to $\{1, 2, \ldots, 2^{k-1} - 1\}$ and the last entry in the diagonal is $2^{k-1}$. This shows that $H$ is in fact a Latin square of order $2^k - 1$ and $F$ is a GDS for it. Now we take $D_k = F$.

By our construction $|D_k| = |D_{k-1}| + 2^{k-1} - 1$. Now suppose $\tilde{g}_k$ stands for $g_n$ when $n = 2^k - 1$, then using an induction we have the following:

$$\tilde{g}_k \leq \tilde{g}_{k-1} + 2^{k-1} - 1 \leq 2^{k-1} - (k-1) - 1 + 2^{k-1} - 1 = 2^k - k - 1.$$

□

The array in Figure 8 illustrates the construction of above theorem for a $15 \times 15$ Latin square, which also includes a GDS of size 4 for a $7 \times 7$ Latin square.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | $2^{k-1}$ | $2^{k-1}+1$ | $\cdots$ | $\cdots$ | $2^k-1$ |
| | $L_{k-1}$ | | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| | | | $2^k-2$ | $2^k-1$ | $\cdots$ | $\cdots$ | $2^{k-1}+1$ |
| $2^{k-1}$ | $\cdots$ | $2^k-2$ | $\boxed{2^k-1}$ | $1$ | $2$ | $\cdots$ | $2^{k-1}-1$ |
| | | | $1$ | $\boxed{2^k-2}$ | $3$ | $\cdots$ | $2^{k-1}-2$ |
| | | | | | $\cdots$ | | |
| | | | | | | $\boxed{2^{k-1}+1}$ | |
| $2^k-1$ | $\cdots$ | $2^{k-1}+1$ | $2^{k-1}+1$ | $2^{k-1}-2$ | $\cdots$ | $1$ | $2^{k-1}$ |

Figure 7: The Latin square $L_k$

Using Theorem 3 and 4 we obtain the following result.

**Theorem 5.** *Suppose $n = 2^k - 2$ for some integer $k$, then*

$$g_n \leq 2n - 4(\log(n+2) - 1).$$

**Proof.** We have $n = 2(2^{k-1} - 1)$. By taking $r = 2$ and $s = 2^{k-1} - 1$, we apply Theorem 3, where we use the GDS obtained in Theorem 4 for the size $2^{k-1} - 1$. The result is a GDS for an $n \times n$ Latin square with the required size, i.e.,

$$g_n \leq 4(2^{k-1} - 1 + 1 - k) = 2(2^k - 2 + 2 - 2k) = 2n - 4(\log(n+2) - 1).$$

□

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 2 | **3** | 1 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 | 15 | 14 |
| 3 | 1 | 2 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 | 14 | 15 | 12 | 13 |
| 4 | 5 | 6 | **7** | 1 | 2 | 3 | 11 | 10 | 9 | 8 | 15 | 14 | 13 | 12 |
| 5 | 4 | 7 | 1 | **6** | 3 | 2 | 12 | 13 | 14 | 15 | 8 | 9 | 10 | 11 |
| 6 | 7 | 4 | 2 | 3 | **5** | 1 | 13 | 12 | 15 | 14 | 9 | 8 | 11 | 10 |
| 7 | 6 | 5 | 3 | 2 | 1 | 4 | 14 | 15 | 12 | 13 | 10 | 11 | 8 | 9 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | **15** | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 8 | 11 | 10 | 13 | 12 | 15 | 1 | **14** | 3 | 2 | 5 | 4 | 7 | 6 |
| 10 | 11 | 8 | 9 | 14 | 15 | 12 | 2 | 3 | **13** | 1 | 6 | 7 | 4 | 5 |
| 11 | 10 | 9 | 8 | 15 | 14 | 13 | 3 | 2 | 1 | **12** | 7 | 6 | 5 | 4 |
| 12 | 13 | 14 | 15 | 8 | 9 | 10 | 4 | 5 | 6 | 7 | **11** | 1 | 2 | 3 |
| 13 | 12 | 15 | 14 | 9 | 8 | 11 | 5 | 4 | 7 | 6 | 1 | **10** | 3 | 2 |
| 14 | 15 | 12 | 13 | 10 | 11 | 8 | 6 | 7 | 4 | 5 | 2 | 3 | **9** | 1 |
| 15 | 14 | 13 | 12 | 11 | 10 | 9 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 8 |

Figure 8: A greedy defining set constructed by the method of Theorem 4

There is another construction for a GDS of size $2n - 4(\log(n+2) - 1)$ in an $n \times n$ Latin square where $n = 2^k - 2$, for some integer $k$. This construction can be explained recursively. The method to extend one GDS to another one is similar to the construction in Theorem 4.

For $k = 3$ the GDS is

$$\begin{array}{|cccccc|} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & \boxed{5} & \boxed{6} & 1 & 2 \\ 4 & 3 & \boxed{6} & \boxed{5} & 2 & 1 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{array}$$

Suppose we have a GDS $D_{k-1}$ of size $2^{k+1} - 4(k-1) - 4$ for a Latin square $L_{k-1}$ of order $2^{k-1} - 2$.

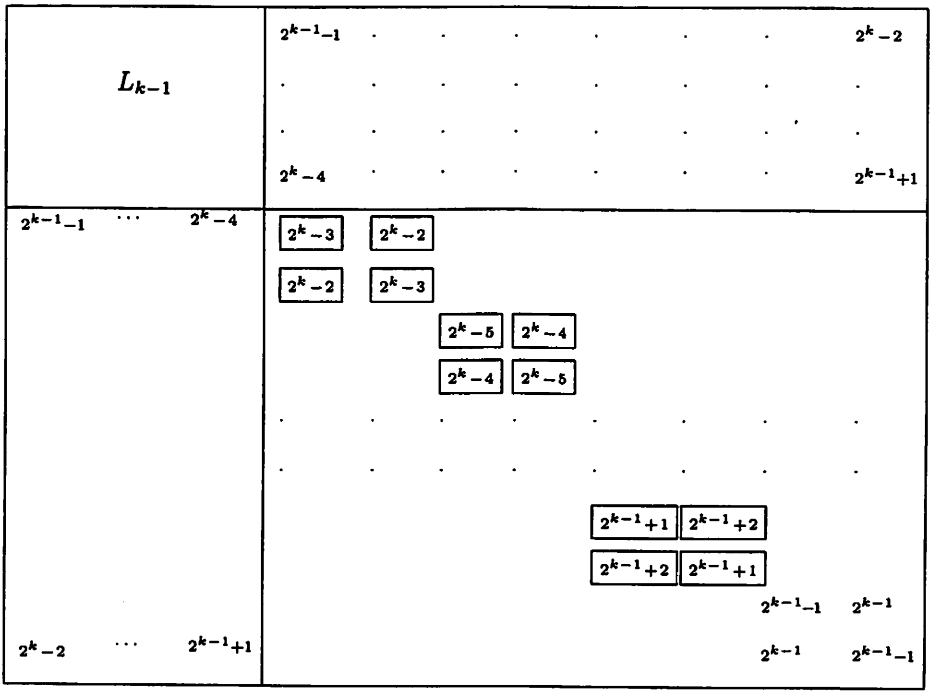



Figure 9: The GDS for a Latin square of order $2^k - 2$ constructed by the GDS of $L_{k-1}$ and $2 \times 2$ blocks specified in the array.

A GDS $D_k$ for a Latin square $L_k$ of order $2^k - 2$ can be obtained by considering the entries of $D_{k-1}$ and extra entries which are shown in the array of Figure 9. These new entries are entries of diagonal $2 \times 2$ subsquares in the opposite subrectangle of $L_{k-1}$ in that arrary. These subsquares are as follows:

$$\begin{array}{|c|c|}\hline 2^k-3 & 2^k-2 \\ \hline 2^k-2 & 2^k-3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2^k-5 & 2^k-4 \\ \hline 2^k-4 & 2^k-5 \\ \hline \end{array}, \quad \dots, \quad \begin{array}{|c|c|}\hline 2^{k-1}+1 & 2^{k-1}+2 \\ \hline 2^{k-1}+2 & 2^{k-1}+1 \\ \hline \end{array}.$$

The result is our GDS $D_k$, as indicated in Figure 9.

The square in Figure 10 illustrates the above construction for a $14 \times 14$ Latin square.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | 10 | 9 | 12 | 11 | 14 | 13 |
| 3 | 4 | $\boxed{5}$ | $\boxed{6}$ | 1 | 2 | 9 | 10 | 7 | 8 | 13 | 14 | 11 | 12 |
| 4 | 3 | $\boxed{6}$ | $\boxed{5}$ | 2 | 1 | 10 | 9 | 8 | 7 | 14 | 13 | 12 | 11 |
| 5 | 6 | 1 | 2 | 3 | 4 | 11 | 12 | 13 | 14 | 7 | 8 | 9 | 10 |
| 6 | 5 | 2 | 1 | 4 | 3 | 12 | 11 | 14 | 13 | 8 | 7 | 10 | 9 |
| 7 | 8 | 9 | 10 | 11 | 12 | $\boxed{13}$ | $\boxed{14}$ | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 7 | 10 | 9 | 12 | 11 | $\boxed{14}$ | $\boxed{13}$ | 2 | 1 | 4 | 3 | 6 | 5 |
| 9 | 10 | 7 | 8 | 13 | 14 | 1 | 2 | $\boxed{11}$ | $\boxed{12}$ | 5 | 6 | 3 | 4 |
| 10 | 9 | 8 | 7 | 14 | 13 | 2 | 1 | $\boxed{12}$ | $\boxed{11}$ | 6 | 5 | 4 | 3 |
| 11 | 12 | 13 | 14 | 7 | 8 | 3 | 4 | 5 | 6 | $\boxed{9}$ | $\boxed{10}$ | 1 | 2 |
| 12 | 11 | 14 | 13 | 8 | 7 | 4 | 3 | 6 | 5 | $\boxed{10}$ | $\boxed{9}$ | 2 | 1 |
| 13 | 14 | 11 | 12 | 9 | 10 | 5 | 6 | 3 | 4 | 1 | 2 | 7 | 8 |
| 14 | 13 | 12 | 11 | 10 | 9 | 6 | 5 | 4 | 3 | 2 | 1 | 8 | 7 |

Figure 10: A GDS for a $14 \times 14$ Latin square

It is clear that we can apply Theorem 3 and 4 for all sizes of the form $n = 2^k - 2^l$, where $k$ varies in positive integers and $l$ is arbitrary fixed positive integer and $0 \leq l \leq k$. What is important for us, this time, is not just to obtain a GDS, but to see how fast the function $g_n$ grows when $n$ is restricted to those infinite subfamilies of $\mathbb{N}$. The result is that $g_n$ grows at most linearly for these infinite families.

**Theorem 6.** *Suppose $n = 2^k - 2^t$, where $k$ is an arbitrary positive integer and $t$ an arbitrary fixed integer with $0 \leq t \leq k$. Setting $\lambda = 2^t$, we have*

$$g_n \leq \lambda n - \lambda^2(k-t).$$

**Proof.** We have $n = 2^t(2^{k-t} - 1)$. Therefore it turns out by Theorem 3 and 4 that

$$g_n \leq 2^{2t}(2^{k-t} - 1 - k + t) = 2^t(2^k - 2^t) - k2^{2t} + t2^{2t} = \lambda n - \lambda^2(k-t).$$

□

# 3 Circulant Squares

Circulant Latin squares form a well-known family of Latin squares which are commonly studied in theories of Latin squares. For critical sets of circulant squares see [8, 9]. In this paper by the $n \times n$ circulant Latin square we mean the square in Figure 11.

**Theorem 7.** *There exists a greedy defining set of size $\lfloor \frac{(n-1)^2}{4} \rfloor$ for the $n \times n$ circulant square.*

**Proof.** A GDS of size $\lfloor \frac{(n-1)^2}{4} \rfloor$ is specified by the following positions in the square:

$$\{(i,j) : j = 1, 2, \ldots, \lfloor \frac{(n-1)}{2} \rfloor \ \text{ and } \ j+1 \leq i \leq n-j\}.$$

The above-specified entries in the $n \times n$ circulant square intersect any descent in the square. Therefore they form a GDS with $\lfloor \frac{(n-1)^2}{4} \rfloor$ entries. □

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | ⋯ | ⋯ | n |
| n | 1 | 2 | ⋯ | ⋯ | n-1 |
| n-1 | n | 1 | ⋯ | ⋯ | n-2 |
| . | . | . | | | . |
| . | . | . | | | . |
| . | . | . | | | . |
| 2 | 3 | 4 | ⋯ | ⋯ | 1 |

Figure 11: The $n \times n$ circulant Latin square

In Figure 12, a greedy defining set with 12 entries is displayed. We note that the bound $\lfloor \frac{(n-1)^2}{4} \rfloor$ for the greedy defining number of the $n \times n$ circulant Latin square is quite close to $\lfloor \frac{n^2}{4} \rfloor$ which is the upper bound for the smallest size of a critical set in the same Latin square.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\boxed{8}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $\boxed{7}$ | $\boxed{8}$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $\boxed{6}$ | $\boxed{7}$ | $\boxed{8}$ | 1 | 2 | 3 | 4 | 5 |
| $\boxed{5}$ | $\boxed{6}$ | $\boxed{7}$ | 8 | 1 | 2 | 3 | 4 |
| $\boxed{4}$ | $\boxed{5}$ | 6 | 7 | 8 | 1 | 2 | 3 |
| $\boxed{3}$ | 4 | 5 | 6 | 7 | 8 | 1 | 2 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 |

Figure 12: A GDS for $8 \times 8$ circulant square

# 4 Concluding remarks

We end the paper with introducing two conjectures and two open problems. First, in this paper we provided linear bounds for $g_n$ for some infinite families of positive integers in Theorems 4, 5 and 6. We believe that this is true for the whole $\mathbb{N}$. In other words we have the following conjecture.

**Conjecture 1.** $g_n = O(n)$.

Our second problem concerns the complexity of determining the greedy defining number of Latin squares. We know from [10] that determining the minimum greedy number of a coloring of a given graph is $\mathcal{NP}$-complete problem.

**Problem 1.** *Is it true that determining the greedy defining number of a Latin square is an $\mathcal{NP}$-complete problem?*

Another problem concerns the smallest size of a GDS in the $n \times n$ circulant Latin square. In Theorem 7 it is shown that the $n \times n$ circulant square contains a GDS of size $\lfloor \frac{(n-1)^2}{4} \rfloor$. It can be checked by hand that for $n = 3, 4, 5$ the minimum size of a GDS for the $n \times n$ circulant square is the same as $\lfloor \frac{(n-1)^2}{4} \rfloor$. But no proof has been obtained so far for general cases. We make the following conjecture.

**Conjecture 2.** *The minimum size of a* GDS *in the $n \times n$ circulant Latin square is $\lfloor \frac{(n-1)^2}{4} \rfloor$.*

Our last problem concerns the descents in Latin squares. Proposition 1 shows that a Latin square $L$ is descent-free if and only if

$$L = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline\end{array} \otimes \cdots \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline\end{array}.$$

An interesting topic is to determine the minimum or maximum number of descents in an $n \times n$ Latin square.

**Problem 2.** *What is the minimum or maximum number of descents in an $n \times n$ Latin square?*

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