Stability Number and f-factor in $K_{1,n}$-free Graphs *

Jiansheng Cai[a] Guizhen Liu[b] †
[a] School of Mathematics and Information Sciences
Weifang University, Weifang 261061, P.R.China.
[b] School of Mathematics and System Sciences
Shandong University, Jinan 250100, P. R. China.

Abstract

Let G be a graph with vertex set V(G) and let f be a nonnegative integer-valued function defined on V(G). A spanning subgraph F of G is called an f-factor if $d_F(x) = f(x)$ for every $x \in V(F)$. In this paper we present some sufficient conditions for the existence of f-factors and connected $(f-2, f)$-factors in $K_{1,n}$-free graphs. The conditions involve the minimum degree, the stability number and the connectivity of graph G.

Key words: f-factor; connected $(f-2, f)$-factor; $K_{1,n}$-free graph
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1 Introduction

The graphs considered in this paper will be simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(x)$ the degree of a vertex x in G. Let g and f be two integer-valued functions defined on V(G) such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then a (g,f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = f(x)$ for all $x \in V(G)$, then a (g,f)-factor is called an f-factor. Let a and b be two integers such that $0 \leq a \leq b$. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g,f)-factor is called an [a,b]-factor.

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†The corresponding author: Guizhen Liu, mailaddress: School of Mathematics and System Sciences, Shandong University, Jinan 250100, P.R.China. E-mail: gzliu@sdu.edu.cn

If $a = b = k$, then an [a,b]-factor is called a k-factor. Denote by $\alpha(G)$ the stability number of a graph G, by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of a vertex in G, respectively. For $A \subseteq V(G)$, denote by $N_G(A)$ the set of neighbors in G of vertices in A and denote by $\kappa(G)$ the connectivity of graph G. If A and B are disjoint subsets of V(G), then $e_G(A, B)$ denotes the number of edges that join a vertex in A and a vertex in B. If $A = \{x\}$, then $e_G(x, B)$ denotes the number of edges that join x and a vertex in B. The number of connected components of G is denoted by $\omega(G)$. Let $S, T \subseteq V(G)$ and $S \bigcap T = \emptyset$. If C is a component of $G - (S \bigcup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(C, T) \equiv 1 \pmod 2$, then we say that C is an odd component of $G - (S \bigcup T)$ and we denote by $h(S, T)$ the number of odd components of $G - (S \bigcup T)$. For a subset S of V(G), we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with edges incident with vertices in S. In the following we write $f(W) = \sum_{x \in W} f(x)$ and $f(\emptyset) = 0$ for any $W \subseteq V(G)$. In particular, we set $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ for $S, T \subseteq V(G)$ and $S \bigcap T = \emptyset$. We also set $N_G[A] = N_G(A) \bigcup A$. Notations and definitions not given here can be found in [1].

Many authors have investigated (g, f)-*factors* and f-*factors* [4,6]. There is a well-known necessary and sufficient condition for a graph G to have an f-*factor* which was given by Tutte.

Theorem A. [6] *(1) A graph G has an f-factor if and only if*

$$\delta(S, T) = f(S) + d_{G-S}(T) - f(T) - h(S, T) \geq 0$$

for any disjoint subsets S and T of V(G), where $h(S, T)$ denotes the number of odd components C of $G - (S \bigcup T)$.

(2) $\delta(S, T) \equiv f(V(G)) \pmod 2$.

Mekkia Kouider and Zbigniew Lonc investigated the relationship between stability number and the existence of an [a,b]-factor. In the following we always assume that a and b are integers and $a \leq b$.

Theorem B. [3] *Let $b \geq a + 1$ and let G be a graph with the minimum degree δ, if*

$$\alpha(G) \leq \begin{cases} \frac{4b(\delta - a + 1)}{(a+1)^2} & \text{for a odd,} \\ \frac{4b(\delta - a + 1)}{a(a+2)} & \text{for a even,} \end{cases}$$

Then G has an [a,b]-factor.

Studying connected factors was initiated by *M.Kano* [3]. This topic is related to the Hamiltonian cycle problem because a connected 2-factor is obviously a Hamiltonian cycle. The following result is also essential to the proof of our main theorem.

Theorem C. [2] *Let G be a simple graph with* $|V(G)| \geq 3$, *if* $\kappa(G) \geq \alpha(G)$, *then G has a Hamiltonian cycle. In particular, G has a 2-factor.*

In this paper, we will give a sufficient condition for the existence of an f-*factor* in a $K_{1,n}$-free graph in terms of its stability number and minimum degree δ, where $a \leq f(x) \leq b$ for every vertex $x \in V(G)$. The following theorems are our main results.

Theorem 1. *Let G be a connected* $K_{1,n}$*-free graph and let f be a nonnegative integer- valued function defined on V(G) such that* $1 \leq n-1 \leq a \leq f(x) \leq b$ *for every* $x \in V(G)$. *If* $f(V(G))$ *is even,* $\delta(G) \geq b+n-1$, *and* $\alpha(G) \leq \frac{4a(\delta-b-n+1)}{(b+1)^2(n-1)}$, *then G has an f-factor.*

This Theorem can be regarded as a partial generalization of Theorem 1 of [5] in $K_{1,n}$-free graphs.

Theorem 2. *Let G be a connected* $K_{1,n}$*-free graph with* $|V(G)| \geq 3$ *and let f be a nonnegative integer-valued function defined on V(G) such that* $1 \leq n-1 \leq a \leq f(x)-2 \leq b$ *for every* $x \in V(G)$. *If* $f(V(G))$ *is even,* $\delta(G) \geq b+n-1$, *and* $\alpha(G) \leq min\{\kappa(G), \frac{4a(\delta-b-n+1)}{(b+1)^2(n-1)}\}$, *then G has a connected* $(f-2, f)$*-factor.*

2 Proof of Theorem 1

Let G be a graph satisfying the hypothesis of Theorem 1, we prove the theorem by contradiction. Suppose that G has no f-factors. Then $\delta(S,T) < 0$ for some disjoint subsets S and T of V(G) by Theorem A. We take S and T such that $\delta(S,T)$ is minimum and $|S \bigcup T|$ is as large as possible. Then we know that $\delta(S,T) < 0$. At first, we prove the following lemma.

Lemma 1. *Choose S and T as the above. Then the order of each component C of* $G-(S \bigcup T)$ *is greater than 1.*

Proof. We suppose that there exists a component C satisfying $|C| = 1$. Let $V(C) = \{v\}$. If $e_G(v,T) > f(v)$, then set $S' = S \bigcup \{v\}$. We obtain

$$\begin{aligned}\delta(S',T) &= f(S') + d_{G-S'}(T) - f(T) - h(S',T)\\ &\leq f(S) + f(v) + d_{G-S}(T) - e_G(v,T) - f(T) - (h(S,T)-1)\\ &= \delta(S,T) + f(v) + 1 - e_G(v,T)\\ &\leq \delta(S,T).\end{aligned}$$

This is a contradiction to the maximality of $|S \bigcup T|$. If $e_G(v,T) \leq f(v)$, then set $T' = T \bigcup \{v\}$. We get

$$\delta(S,T') = f(S) + d_{G-S}(T') - f(T') - h(S,T')$$

$$\begin{aligned} &\le f(S) + d_{G-S}(T) + e_G(v,T) - f(T) - f(v) - (h(S,T) - 1) \\ &= \delta(S,T) + e_G(v,T) - f(v) + 1 \\ &\le \delta(S,T) + 1. \end{aligned}$$

Since $\delta(S,T') \equiv \delta(S,T) \pmod 2$, we have $\delta(S,T') \le \delta(S,T)$. Again this is a contradiction to the maximality of $| S \bigcup T |$. Therefore, the order of each component C of $G - (S \bigcup T)$ is greater than 1. ∎

For convenience, we set $U = G - (S \bigcup T)$ and denote by $\omega(U)$ the number of components of $G - (S \bigcup T)$

Since $\delta(S,T) = f(S) + d_{G-S}(T) - f(T) - h(S,T) \ge a \mid S \mid - b \mid T \mid + d_{G-S}(T) - h(S,T)$, to prove Theorem 1, we need only to prove that $a \mid S \mid - b \mid T \mid + d_{G-S}(T) - h(S,T) \ge 0$. Then we get a contradiction.

For $A \subseteq T$, let $\gamma(A)$ denote the number of components C of $G - (S \bigcup T)$ satisfying $e_G(C, A) > 0$. If $A = \{x\}$, denote $\gamma(A)$ by $\gamma(x)$.

First we claim that $T \ne \emptyset$. Otherwise $T = \emptyset$. If $T = S = \emptyset$, since G is connected and $\delta(S,T) < 0$, we know that $h(S,T) = 1$. Therefore $f(V(G))$ is odd according to Theorem A. This contradicts the assumption that $f(V(G))$ is even. If $S \ne \emptyset$, since G is a connected $K_{1,n}$-free graph, $h(S,T) \le (n-1) \mid S \mid$. Then $\delta(S,T) \ge a \mid S \mid - h(S,T) \ge 0$, which is a contradiction to our assumption that G has no f-factor. Therefore $T \ne \emptyset$. We take $x_1 \in T$ such that $x_1$is the vertex with the least degree in T. Let $N_1 = N_G[x_1] \bigcap T$ and $T_1 = T$. For $i \ge 2$, if $T - \bigcup_{j<i} N_j \ne \emptyset$, let $T_i = T - \bigcup_{j<i} N_j$. Then take $x_i \in T_i$ such that x_i is the vertex with the least degree in T_i , and set $N_i = N_G[x_i] \bigcap T_i$. We continue this procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = k+1$. Then from the above definition we know that $x_1, x_2, \cdots, x_k$ is an independent set of G. Since $T \ne \emptyset$, we have $k \ge 1$.

At first we prove that the following two claims hold.

Claim 1. $d_{G-S}(x) \le b + n - 1$ for any $x \in T$.

For any $x_0 \in T$, let $T' = T - \{x_0\}$. From the minimum of $\delta(S,T)$ we can see that $\delta(S,T') \ge \delta(S,T)$. Since G is $K_{1,n}$-free graph, x_0 connects at most $n-1$ components of $G - (S \bigcup T)$. Then $h(S,T') \ge h(S,T) - (n-1)$. Therefore

$$\begin{aligned} \delta(S,T) &\le \delta(S,T') = f(S) - f(T') + d_{G-S}(T') - h(S,T') \\ &\le f(S) - f(T) + d_{G-S}(T) - d_{G-S}(x_0) + f(x_0) - h(S,T) + n - 1 \\ &= \delta(S,T) - d_{G-S}(x_0) + f(x_0) + n - 1. \end{aligned}$$

Thus

$$d_{G-S}(x_0) \le b + n - 1.$$

Claim 2. $|S| \geq \frac{1}{n-1} \sum_{i=1}^{k} e(x_i, S) + \frac{1}{a}(\omega(U) - \gamma(T))$.

Let $l = \omega(U) - \gamma(T)$, then there exist l components $C_1, C_2, \cdots, C_l$ of $G - (S \bigcup T)$ such that $e_G(C_j, T) = 0 \ \ (1 \leq j \leq l)$.

If $S = \emptyset$, since G is connected, $l = 0$. Then the claim clearly holds. If $S \neq \emptyset$, since G is connected, $e_G(C_j, S) > 0, (1 \leq j \leq l)$. Therefore for any j, we can choose a vertex $z_j \in C_j$ such that $e_G(z_j, S) > 0$. Let $Z = \{x_1, x_2 \cdots, x_k, z_1, z_2, \cdots, z_l\}$. Since Z is an independent set of G and G is a $K_{1,n}$-free graph, For any vertex $v \in S$ there exist at most $(n-1)$ vertices in Z adjacent to v. Thus,

$$(n-1) \mid S \mid \geq e(Z, S) = \sum_{i=1}^{k} e(x_i, S) + \sum_{j=1}^{l} e(z_j, S) \geq \sum_{i=1}^{k} e(x_i, S) + l,$$

and $a \geq n-1$, we have

$$|S| \geq \frac{1}{n-1} \sum_{i=1}^{k} e(x_i, S) + \frac{1}{a}(\omega(U) - \gamma(T)).$$

Let $|N_i| = n_i$. From the definition of N_i, we can get the following properties.

$$\alpha(G[T]) \geq k,$$

$$|T| = \sum_{1 \leq i \leq k} n_i, \tag{1}$$

$$\sum_{1 \leq i \leq k} \Big(\sum_{x \in N_i} d_{T_i}(x)\Big) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i). \tag{2}$$

It is easy to see that

$$d_{G-S}(T) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i) + \sum_{1 \leq i < j \leq k} e_G(N_i, N_j) + e_G(T, U). \tag{3}$$

Let $\kappa(G - S) = t$, we have

$$e_G(N_i, \bigcup_{j \neq i} N_j) + e_G(N_i, U) \geq t$$

for each $N_i (1 \leq i \leq k)$. Summing up these inequalities for all $i (1 \leq i \leq k)$, we have

$$\sum_{1 \leq i \leq k} \Big(e_G(N_i, \bigcup_{j \neq i} N_j) + e_G(N_i, U)\Big) = 2 \sum_{1 \leq i < j \leq k} e_G(N_i, N_j) + e_G(T, U) \geq kt. \tag{4}$$

By the results of (3) and (4), the following inequality can be obtained.

$$d_{G-S}(T) \geq \sum_{1 \leq i \leq k} (n_i(n_i - 1)) + (kt + e_G(T, U))/2. \tag{5}$$

From (1), (5) and obvious inequalities $n_i(n_i - b - 1) \geq -(b+1)^2/4$ and $h(S,T) \leq \omega(U)$, we get

$$\begin{aligned} 0 > \delta(S,T) &\geq a \mid S \mid + d_{G-S}(T) - b \mid T \mid - h(S,T) \\ &\geq a \mid S \mid + \sum_{1 \leq i \leq k} (n_i(n_i - b - 1)) + (kt + e_G(T,U))/2 - \omega(U) \\ &\geq a \mid S \mid - (b+1)^2 k/4 + (kt + e_G(T,U))/2 - \omega(U). \end{aligned}$$

We now estimate $e_G(T,U)$. First note that $T \neq \emptyset$. This implies that $e_G(C_i,T) \geq t$ for all components C_i of U. Moreover, by Lemma 1, for C_i that satisfy the condition $C_i \subseteq N_G(\{x_1, x_2, \cdots, x_k\})$, we have $e_G(C_i,T) \geq 2$. Let us denote by c the number of C_i that satisfy the condition $C_i \subseteq N_G(\{x_1, x_2, \cdots, x_k\})$ and by d the number of C_j that satisfy the condition $C_j \not\subseteq N_G(\{x_1, x_2, \cdots, x_k\})$, then we have

$$e_G(T,U) \geq 2c + td.$$

So

$$\omega(U) = c + d \geq \gamma(T), \tag{6}$$

$$\alpha(G) \geq k + d. \tag{7}$$

Since $e(x_i,S) = d_G(x_i) - d_{G-S}(x_i)$, thus according to Claim 1,

$$e(x_i,S) \geq \delta(G) - b - n + 1. \tag{8}$$

From (6), (7), (8) and Claim 2, we get

$$\begin{aligned} 0 > \delta(S,T) &\geq a \mid S \mid - (b+1)^2 k/4 + (kt + e_G(T,U))/2 - \omega(U) \\ &\geq \frac{a}{n-1}\sum_{i=1}^{k} e(x_i,S) - (b+1)^2 k/4 - \gamma(T) + (k+d)t/2 + c \\ &\geq \frac{a}{n-1}\sum_{i=1}^{k} e(x_i,S) - (b+1)^2 k/4 - d + (k+d)t/2 \\ &\geq \frac{a}{n-1}\sum_{i=1}^{k} e(x_i,S) - (b+1)^2 (k+d)/4 + (k+d)t/2. \end{aligned}$$

Since $\delta(S,T) < 0$, $(k+d)t/2 - (b+1)^2(k+d)/4$ is negative. We replace $(k+d)$ by $\frac{4a(\delta-b-n+1)}{(b+1)^2(n-1)} (\geq \alpha(G))$, to get

$$0 > \delta(S,T) \geq \frac{ak(\delta - b - n + 1)}{n-1} - \frac{a(\delta - b - n + 1)}{n-1} + \frac{2a(\delta - b - n + 1)}{(b+1)^2(n-1)}t \geq 0.$$

Thus we get a contradiction and complete the proof. ∎

In Theorem 1 if $a = b$, then we obtain the following result.

Corollary 1. *Let G be a connected $K_{1,n}$-free graph and let a be a nonnegative integer such that $1 \leq n - 1 \leq a$. If $a \mid V(G) \mid$ is even, $\delta(G) \geq a + n - 1$ and $\alpha(G) \leq \frac{4a(\delta - a - n + 1)}{(a+1)^2(n-1)}$, then G has an a-factor.*

3 Proof of Theorem 2

By our assumptions $\alpha(G) \leq min\{\kappa(G), \frac{4a(\delta - b - n + 1)}{(b+1)^2(n-1)}\}$, then according to Theorem 1 and Theorem C, we know that graph G has an $(f - 2)$-factor F_1 as well as a connected 2-factor F_2. Clearly, the union of F_1 and F_2 is a connected $(f - 2, f)$-*factor*. So the proof of Theorem 2 is completed. ∎

Remark. The condition that $f(V(G))$ is even in Theorem 1 is necessary. If $f(V(G))$ is odd, then when $S = \emptyset$ and $T = \emptyset$, we have $\delta(S, T) = -h(\emptyset, \emptyset) = -1$. Thus graph G has no f-factors. Similarly, the condition that $f(V(G))$ is even is also necessary in Theorem 2. But we do not know whether the condition $\alpha(G) \leq \frac{4a(\delta - b - n + 1)}{(b+1)^2(n-1)}$ can be improved.

We conjecture that for general graph the result similar to Theorem 1 also holds, so we made the following conjecture.

Conjecture *Let G be a connected graph and let f be a nonnegative integer-valued function defined on V(G) such that $a \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq b$, f(V(G)) is even and $\alpha(G) \leq \frac{4a(\delta - b)}{(b+1)^2}$, then G has an f-factor.*

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