On Ramsey Numbers of Short Paths versus Large Wheels*

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Abstract: For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let P_n denote a path of order n and W_m a wheel of order $m+1$. Chen et al. determined all values of $R(P_n, W_m)$ for $n \geq m-1$. In this paper, we establish the best possible upper bound and determine some exact values for $R(P_n, W_m)$ with $n \leq m-2$.

Key words: Ramsey number, Path, Wheel

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree*, *components number* and *connectivity* of G are denoted by $\delta(G)$, $\omega(G)$ and $\kappa(G)$, respectively. C_n and P_n denote a cycle and a path of

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order n, respectively. A path or cycle of G is *hamiltonian* if it includes all vertices of G. A *complete graph* of order n is denoted by K_n. A *complete bipartite graph* of order $m+n$ is denoted by $K_{m,n}$. A *Wheel* $W_n = \{x\}+C_n$ is a graph of $n+1$ vertices, namely, a vertex x, called the *hub* of the wheel, adjacent to all vertices of C_n. mK_n denotes the union of m vertex-disjoint copies of K_n. The orders of the longest cycle and path of G are denoted by $c(G)$ and $p(G)$, respectively. We use I_k to denote an independent set of order k. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$.

In [4], Faudree et al. considered the Ramsey numbers for all path-cycle pairs and obtained the following.

Theorem 1 (Faudree et al. [4]). $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$ for even m and $2 \leq n \leq m$. $R(P_n, C_m) = max\{m + \lfloor n/2 \rfloor - 1, 2n - 1\}$ for odd m and $2 \leq n \leq m$.

In [5], Surahmat et al. obtained the Ramsey numbers of a path versus W_4 or W_5.

Theorem 2 (Surahmat et al. [5]). $R(P_n, W_5) = 3n - 2$ for $n \geq 4$ and $R(P_n, W_4) = 2n - 1$ for $n \geq 3$.

In [2], Chen et al. obtained the Ramsey numbers $R(P_n, W_m)$ for $n \geq m - 1$.

Theorem 3 (Chen et al. [2]). $R(P_n, W_m) = 3n - 2$ for odd m and $n \geq m - 1 \geq 2$ and $R(P_n, W_m) = 2n - 1$ for even m and $n \geq m - 1 \geq 3$.

In this paper, we consider the Ramsey numbers $R(P_n, W_m)$ for $4 \leq n \leq m - 2$. In the following, we always let $m = k(n - 1) + r$, where $k \geq 1$ and $0 \leq r \leq n - 2$ are integers.

The main results of this paper are the following.

Theorem 4. If m is odd and $n+2 \leq m \leq 2n-1$, then $R(P_n, W_m) = 3n-2$.

Theorem 5. Let $n \geq 4$. If m is even and $n \leq m - 2$ or m is odd and $n \leq (m - 1)/2$, then $R(P_n, W_m) \leq n + m - \mu$, where $\mu = 1$ if $r = 1$ and $\mu = 2$ otherwise.

The proofs of Theorems 4 and 5 will be given in Section 3. We now use Theorem 5 to determine some values of the Ramsey numbers $R(P_n, W_m)$

for $4 \leq n \leq m-2$.

If m is even and $m/2 \leq n \leq m-2$, then $n+2 \leq m \leq 2n$ which implies $r \neq 1$. Thus by Theorem 5 we have $R(P_n, W_m) \leq n+m-2$. On the other hand, the graph $G = K_{n-1} \cup 2K_{m/2-1}$ shows $R(P_n, W_m) \geq n+m-2$. Thus we have the following.

Corollary 1. If m is even and $m/2 \leq n \leq m-2$, then $R(P_n, W_m) = n+m-2$.

By Theorems 3 and 4, and Corollary 1, the Ramsey numbers $R(P_n, W_m)$ for $n \geq \lceil m/2 \rceil$ are determined.

If $n \leq \lceil m/2 \rceil - 1$ and $r = 0$, then $k \geq 2$. Noting that the graph $G = (k-1)K_{n-1} \cup 2K_{n-2}$ shows $R(P_n, W_m) \geq n+m-2$, by Theorem 5 we have the following.

Corollary 2. If $n \leq \lceil m/2 \rceil - 1$ and $r = 0$, then $R(P_n, W_m) = n+m-2$.

Let $G = (k+1)K_{n-1}$. If $r = 1$ or 2, then it is easy to see that $|G| = n+m-r-1$ and neither G contains a P_n nor $\overline{G}$ contains a W_m. This shows that $R(P_n, W_m) \geq n+m-r$. Thus by Theorem 5 we obtain the following.

Corollary 3. If $n \leq \lceil m/2 \rceil - 1$ and $r = 1, 2$, then $R(P_n, W_m) = n+m-r$.

If $r \geq 3$ and $k+r \geq n-1$, then since $n+m-3 = (k+1)(n-1)+(r-2)$, we have $(k+1)+(r-2) \geq n-2$ which implies there are two non-negative integers k_1 and k_2 such that $k_1+k_2 = k+2$ and $n+m-3 = k_1(n-1)+k_2(n-2)$. Thus, the graph $G = k_1K_{n-1} \cup k_2K_{n-2}$ shows $R(P_n, W_m) \geq n+m-2$. Thus by Theorem 5 we get the following.

Corollary 4. If $n \leq \lceil m/2 \rceil - 1$, $r \geq 3$ and $k+r \geq n-1$, then $R(P_n, W_m) = n+m-2$.

Combining Theorems 3, 4 and 5, and Corollary 3, we have the following.

Theorem 6. Let $m \geq 3$ be odd. If $n \geq (m+1)/2$, then $R(P_n, W_m) = 3n-2$ and if $n \leq (m-1)/2$, then $R(P_n, W_m) \leq n+m-1$ and $R(P_n, W_m) = n+m-1$ if and only if $r = 1$.

Combining Theorems 3 and 5, and Corollaries 1 and 3, we have the following.

Theorem 7. Let $m \geq 4$ be even. If $n \geq m-1$, then $R(P_n, W_m) = 2n-1$, if $m/2 \leq n \leq m-2$, then $R(P_n, W_m) = n+m-2$ and if $n \leq m/2-1$, then $R(P_n, W_m) \leq n+m-1$ and $R(P_n, W_m) = n+m-1$ if and only if $r = 1$.

2. Lemmas

Lemma 1 (Dirac [3]). Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.

Lemma 2 (Dirac [3]). Let G be a connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $p(G) \geq \min\{2\delta + 1, n\}$.

Lemma 3 (Bondy [1]). Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.

Lemma 4 (Chen et al. [2]). Let G be a graph with $|G| \geq R(P_n, C_m) + 1$. If there is a vertex $v \in V(G)$ such that $|N[v]| \leq |G| - R(P_n, C_m)$ and G contains no P_n, then $\overline{G}$ contains a W_m.

Lemma 5. Let $n \geq 4$ be even and G a connected graph of order $n^* \geq n$. If $p(G) \leq n-1$ and $\delta(G) \geq n/2-1$, then $G = K_1 + (n^*-1)/(n/2-1)K_{n/2-1}$ or $G = G_0 + I_{n^*-n/2+1}$, where G_0 is a graph of order $n/2-1$.

Proof. If G is 2-connected, then by Lemma 1, $c(G) \geq n-2$. Since $p(G) \leq n-1$, we have $c(G) = n-2$. And if $C = C_{n-2}$, then $G - V(C)$ contains no edges. Thus since $\delta(G) \geq n/2-1$, we have $G = G_0 + I_{n^*-n/2+1}$, where G_0 is a graph of order $n/2-1$. If $\kappa(G) = 1$, we let v_0 be a cut-vertex of G and H any component of $G - v_0$. Since $\delta(G) \geq n/2-1$, we have $\delta(H) \geq n/2-2$ and then $|H| \geq n/2-1$. Let $P = v_0v_1\cdots v_l$ be a longest path such that $P - \{v_0\} \subseteq H$. Since $\delta(G) \geq n/2-1$ and $p(G) \leq n-1$, we have $|P| = n/2$. Thus we have $N[v_l] = P$, which implies $N[v_i] = P$ for all $1 \leq i \leq l$ and hence $H = K_{n/2-1}$. Therefore, $G = K_1 + (n^*-1)/(n/2-1)K_{n/2-1}$. ∎

Lemma 6. Let n_i $(1 \leq i \leq k)$ be positive integers, $A = \{n_i \mid 1 \leq i \leq k\}$ and (A_1, A_2) be a partition of A such that $|a_1 - a_2|$ is as small as possible, where $a_l = \sum_{n_i \in A_l} n_i$ and $l = 1, 2$. If $n_i \leq m$, then $|a_1 - a_2| \leq m$ and the equality holds if and only if $n_1 = \cdots = n_k = m$ and k is odd.

Proof. If $k = 1$, then the conclusion holds trivially. If $k \geq 2$, we assume $n_k \leq n_i$ for all $1 \leq i \leq k-1$. If $n_k = m$, then $|a_1 - a_2| = 0$ if k is even and $|a_1 - a_2| = m$ if k is odd, and hence the conclusion holds. Assume now

$n_k < m$. Let $B = A - \{n_k\}$ and (B_1, B_2) be a partition of B such that $|b_1 - b_2|$ is as small as possible, where $b_l = \sum_{n_i \in B_l} n_i$ and $l = 1, 2$. Assume $b_1 \geq b_2$. By induction hypothesis, we have $b_1 - b_2 \leq m$. Now, set $C_1 = B_1$, $C_2 = B_2 \cup \{n_k\}$ and $c_l = \sum_{n_i \in C_l} n_i$, $l = 1, 2$. Since $1 \leq n_k < m$, it is easy to see that $|c_1 - c_2| = |(b_1 - b_2) - n_k| \leq |m - n_k| < m$. Thus, by the choice of (A_1, A_2), we can see the conclusion holds. ∎

3. Proofs of Theorems

Proof of Theorem 4. Let G be a graph of order $3n - 2$. Suppose to the contrary neither G contains a P_n nor $\overline{G}$ contains a W_m. If $m \leq \lceil 3n/2 \rceil$, then by Theorem 1, $R(P_n, C_m) = 2n-1$. By Lemma 4 we have $\delta(G) \geq n-1$. By Lemma 2, we have $p(G) \geq n$, a contradiction. If $\lceil 3n/2 \rceil + 1 \leq m \leq 2n - 1$, then by Theorem 1, $R(P_n, C_m) = m + \lfloor n/2 \rfloor - 1$. Since $3n - 2 \geq m + n - 1$, by Lemma 4 we have $\delta(G) \geq n/2$. Let $H_1, \ldots, H_t$ be the components of G with $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$. If $|H_1| \geq n$, then by Lemma 2 we have $p(H_1) \geq n$, a contradiction. Thus we have $|H_1| \leq n-1$. Since $|G| = 3n-2$, we have $t \geq 4$ and $|H_t| \leq n - 2$. Let $G_0 = G - H_t$. It is easy to see that $|G_0| = 3n - 2 - |H_t|$ and $\delta(\overline{G_0}) \geq |G_0| - (n - 1) = 2n - 1 - |H_t|$. Since $|H_t| \leq n-2$, we have $\delta(\overline{G_0}) > |G_0|/2$. By Lemma 3, $\overline{G_0}$ is pancyclic. Since $|G_0| \geq m + 1$, $\overline{G_0}$ contains a C_m and hence $\overline{G}$ contains a W_m with a hub $x \in V(H_t)$, also a contradiction. Thus we have $R(P_n, W_m) \leq 3n - 2$. On the other hand, the graph $3K_{n-1}$ shows $R(P_n, W_m) \geq 3n - 2$ and hence $R(P_n, W_m) = 3n - 2$. ∎

Proof of Theorem 5. Let G be a graph of order $n+m-\mu$, where $\mu = 1$ if $r = 1$ and $\mu = 2$ otherwise. Suppose $H_1, H_2, \ldots, H_t$ are the components of G with $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$. Obviously, $|G| \geq R(P_n, C_m)+1$. Suppose to the contrary neither G contains a P_n nor $\overline{G}$ contains a W_m. By Lemma 4, we have $|N[v]| \geq |G| - R(P_n, C_m) + 1$ for any vertex $v \in V(G)$, which implies $\delta(G) \geq |G| - R(P_n, C_m)$. By Theorem 1, $\delta(G) \geq \lceil n/2 \rceil + 1 - \mu$.

Claim 1. $|H_1| \geq n$.

Proof. If $|H_1| \leq n - 1$, then since $m \geq n + 2$, we have $t \geq 3$. If $r = 1$, then since $|G| = n + m - 1 = (k + 1)(n - 1) + 1$, we have $|H_t| \leq n - 2$. If $r \neq 1$, then since $|G| = n+m-2 = (k+1)(n-1)+(r-1)$, we have $|H_t| \leq n-2$. Set $G' = G - H_t$. Obviously, $|G'| \geq m$ and $\delta(\overline{G'}) \geq |G'| - |H_1|$. If m is odd and $n \leq (m - 1)/2$, then $\delta(\overline{G'}) > |G'|/2$. By Lemma 3, $\overline{G'}$ is pancyclic. Since

$|G'| \geq m$, $\overline{G'}$ contains a C_m and hence $\overline{G}$ contains a W_m, a contradiction. Now, assume m be even and $n \leq m-2$. Note that $|H_t| \leq n-2$, $|G'| = m+n-\mu-|H_t|$ and $\delta(\overline{G'}) \geq |G'|-|H_1| = m+n-\mu-|H_t|-|H_1|$. If $k \geq 2$ or ($k=1$ and $|H_1| \leq m/2-1$), then we have $\delta(\overline{G'}) \geq |G'|/2$, which implies $\overline{G'}$ contains a C_m by Lemma 3 and hence $\overline{G}$ contains a W_m, a contradiction. Hence we may assume $k=1$ and $|H_1| \geq m/2$. If $t=3$, then since $|G| \geq n+m-2$ and $|H_1| \leq n-1$, we have $|H_2| \geq m/2$. Thus, $\overline{G}[V(H_1) \cup V(H_2)]$ contains a complete bipartite graph between $V(H_1)$ and $V(H_2)$ and therefore contains a C_m. So $\overline{G}$ contains a W_m with the hub $x \in V(H_t)$, a contradiction. If $t \geq 4$, then $|H_t| \leq (n+m-\mu)/4$. Let $\cup_{i=2}^{t-1} V(H_i) = U$. Then we have $|U| \geq (n+m-\mu)-[(n+m-\mu)/4+(n-1)] = (3m-n+4-3\mu)/4 \geq (3m-n-2)/4$. Since $m \geq n+2$, we have $|U| \geq m/2$. Thus, $\overline{G}-V(H_t)$ contains a complete bipartite graph between $V(H_1)$ and U, which implies $\overline{G}-V(H_t)$ contains a C_m. So $\overline{G}$ contains a W_m with the hub $x \in V(H_t)$, also a contradiction. ∎

If $\mu = 1$ or ($\mu = 2$ and n is odd), then $\delta(H_1) \geq \delta(G) \geq (n-1)/2$. Thus by Lemma 2 and Claim 1, we have $p(H_1) \geq n$, a contradiction. Hence we may assume n is even and $\mu = 2$. In this case, $\delta(G) \geq n/2-1$. Let $|H_i| = n_i$ for $1 \leq i \leq t$. Define

$$A = \{H_i \mid n_i \geq n \text{ and } H_i = K_1 + (n_i-1)/(n/2-1)K_{n/2-1}\} \text{ and}$$
$$B = \{H_i \mid n_i \geq n \geq 6 \text{ and } H_i = G_i + I_{n_i-n/2+1}, \text{ where } |G_i| = n/2-1\}.$$

Since G contains no P_n, by Lemma 5 and Claim 1 we have $A \cup B \neq \emptyset$ and if $n_i \geq n$, then $H_i \in A$ or $H_i \in B$. If $H_i \in A$, we let $H_i = \{v_i\} + (n_i-1)/(n/2-1)K_{n/2-1}$ and H_{ij} the components of $H_i - \{v_i\}$, where $1 \leq j \leq (n_i-1)/(n/2-1)$. If $H_i \in B$, we let $H_i = G_i + I(i)$, where $I(i) = I_{n_i-n/2+1}$.

Now, let $H_i \in A \cup B$ be a given component of G, $u \in H_{i1}$ if $H_i \in A$ and $u \in I(i)$ if $H_i \in B$. Set $G_0 = \cup_{n_j \leq n-1} H_j$, $G_S = H_i \cup G_0 - N[u]$ and $G_L = \cup_{n_j \geq n \text{ and } j \neq i} H_j$.

Claim 2. If $H_i \in A$, then $p(\overline{G_S}) = |G_S|$. Furthermore, if $|A \cup B| = 1$, then $\overline{G_S}$ contains a C_m.

Proof. Let $G_M = G_0 \cup (\cup_{j \geq 4} H_{ij})$. If $G_M = \emptyset$, then obviously $p(\overline{G_S}) = |G_S|$. If $|A \cup B| = 1$, then $|G| = 3n/2-2$ which implies $m = n/2$, a contradiction. Hence we may assume $G_M \neq \emptyset$. By Lemma 6, there are

G'_M, G''_M such that $G_M = G'_M \cup G''_M$ and $0 \leq |G'_M| - |G''_M| \leq n-1$. Choose $V_M \subseteq V(G'_M)$ such that $|V_M| = |G'_M| - |G''_M|$ and $|E(\overline{G}[V_M])|$ is as large as possible. Set $F = \overline{G}[V(H_{i2}) \cup V(H_{i3}) \cup V_M]$. Let $v \in H_{i2}$ and $w \in H_{i3}$. If $V_M \neq \emptyset$ and $|V_M| \leq n-3$, then since $|H_{i2}| = |H_{i3}| = n/2 - 1$, and $V(H_{i2}), V(H_{i3})$ and V_M induce a complete 3-partite graph in $\overline{G}$, we can see

$$F \text{ contains a } (v,w)\text{-path } P_l \text{ for } 2 \leq l \leq |F|. \tag{1}$$

If $|V_M| = n-2$, we let $x \in V_M$. If $|V_M| = n-1$ and $|E(\overline{G}[V_M])| \geq 1$, then since $n \geq 4$, there is some vertex $x \in V_M$ such that $|E(\overline{G}[V_M - \{x\}])| \geq 1$. Thus, since $V_M - \{x\} \neq \emptyset$, we can see

$$F - \{x\} \text{ contains a } (v,w)\text{-path } P_l \text{ for } 2 \leq l \leq |F| - 1. \tag{2}$$

We now show that $p(\overline{G_S}) = |G_S|$. If $V_M = \emptyset$, then it is easy to see that both $\overline{G_M}$ and F are hamiltonian, which implies $p(\overline{G_S}) = |G_S|$. If $V_M \neq \emptyset$ and $|V_M| \leq n-3$, then since $\overline{G_M} - V_M = \emptyset$ or $\overline{G_M} - V_M$ is hamiltonian, by (1) we have $p(\overline{G_S}) = |G_S|$. If $|V_M| = n-2$ or $|V_M| = n-1$ and $|E(\overline{G}[V_M])| \geq 1$, then since $\overline{G_M} - \{V_M - \{x\}\}$ is hamiltonian, by (2) we have $p(\overline{G_S}) = |G_S|$. If $|V_M| = n-1$ and $|E(\overline{G}[V_M])| = 0$, then we have $G_M = G'_M = K_{n-1}$ and $G_S = 2K_{n/2-1} \cup K_{n-1}$. Obviously, $p(\overline{G_S}) = |G_S|$.

Let $|A \cup B| = 1$. We shall show that $\overline{G_S}$ contains a C_m. If m is odd, then since $|G| = m+n-2$ and $n \leq (m-1)/2$, we have $|G_M| \geq 3n/2+1$ and hence

$$|G_M - V_M| \geq n/2 + 2. \tag{3}$$

Since $n \geq 4$, we have $|G_S| = |G| - |N[u]| = n+m-2-n/2 = m+n/2-2 \geq m$. If $V_M = \emptyset$, then since F has a hamiltonian (v,w)-path and $\overline{G_M}$ has a hamiltonian path, we can choose a (v,w)-path P_1 in F and a path P_2 in $\overline{G_M}$ such that $|P_1| + |P_2| = m$. Obviously, P_1 and P_2 form a C_m in $\overline{G_S}$. If $V_M \neq \emptyset$ and $|V_M| \leq n-3$, then by (1), a (v,w)-path of order m and the edge vw gives a C_m in $\overline{G_S}$ if $\overline{G_M} - V_M = \emptyset$, and a (v,w)-path of order $|F| - (n/2-2)$ together with a hamiltonian path in $\overline{G_M} - V_M$ form a C_m in $\overline{G_S}$ if $\overline{G_M} - V_M \neq \emptyset$. If $|V_M| = n-2$ or $|V_M| = n-1$ and $|E(\overline{G}[V_M])| \geq 1$, then since $\overline{G_M} - \{V_M - \{x\}\}$ is hamiltonian, analogously, we can obtain a C_m in $\overline{G_S}$ by (2). If $|V_M| = n-1$ and $|E(\overline{G}[V_M])| = 0$, then $G_S = 2K_{n/2-1} \cup K_{n-1}$. By (3), m is even. Since $|A \cup B| = 1$, we have $n \geq 6$ for otherwise $n \geq m-1$, which contradicts $n \leq m-2$. In this case, it is easy to see that $\overline{G_S}$ contains a C_m. ▮

Claim 3. If $H_i \in B$, then $p(\overline{G_S}) \geq |G_S| - (n/2 - 2)$. Furthermore, if $|A \cup B| = 1$, then $\overline{G_S}$ contains a C_m.

Proof. Let $I = I(i) - \{u\}$. If $G_0 = \emptyset$, then the conclusion holds. Hence we may assume $G_0 \neq \emptyset$. Choose G_0', G_0'' such that $G_0 = G_0' \cup G_0''$ and $||G_0'| - |G_0''||$ is as small as possible. By Lemma 6, $||G_0'| - |G_0''|| \leq n - 1$. Noting that I is an independent set of at least $n/2$ vertices in G_S, there are G_S', G_S'' such that $G_S = G_S' \cup G_S''$ and $||G_S'| - |G_S''|| \leq max\{1, ||G_0'| - |G_0''|| - |I|\} \leq n/2 - 1$, which implies $p(\overline{G_S}) \geq |G_S| - (n/2 - 1) + 1$, that is, $p(\overline{G_S}) \geq |G_S| - (n/2 - 2)$.

Let $|A \cup B| = 1$. We now show that $\overline{G_S}$ contains a C_m. Obviously, $|G_S| = |G| - |N[u]| = n + m - 2 - n/2 = m + n/2 - 2$. Since I is an independent set of order at least $n/2$, $\overline{G}[I]$ is a complete graph. If $||G_0'| - |G_0''|| \leq 1$, then $\overline{G_0}$ has a hamiltonian path because $\overline{G_0}$ contains a complete bipartite graph between $V(G_0')$ and $V(G_0'')$. If $||G_0'| - |G_0''|| = n - 1$, then by Lemma 6, $|G_0| = \omega(G_0)(n - 1)$ and $\omega(G_0)$ is odd. If $\omega(G_0) \geq 3$, then it is easy to see that $\overline{G_0}$ is hamiltonian. Thus, a path of order $|I| - (n/2 - 2)$ in $\overline{G}[I]$ and a hamiltonian path of $\overline{G_0}$ give a C_m in $\overline{G_S}$. If $\omega(G_0) = 1$, then since $m \geq min\{n + 2, 2n + 1\} = n + 2$, we have $|I| = |G_S| - |G_0| = (m + n/2 - 2) - (n - 1) = m - n/2 - 1 \geq n/2 + 1$. Let $Y \subseteq V(G_0)$ and $|Y| = n/2 + 1$. Since I is an independent set of order at least $n/2 + 1$ in G_S, we can see that $\overline{G_S}[I \cup Y]$ contains a hamiltonian cycle, which implies $\overline{G_S}$ contains a C_m since $|I| + |Y| = m$. Now we may assume $2 \leq ||G_0'| - |G_0''|| \leq n - 2$. Without loss of generality, we assume $|G_0'| - |G_0''| \geq 2$. Choose $V_0 \subseteq V(G_0')$ such that $|V_0| = |G_0'| - |G_0''| - 1$. Obviously, $1 \leq |V_0| \leq n - 3$ and $\overline{G_0} - V_0$ has hamiltonian path. Let $v, w \in I$. If $|V_0| \leq n/2 - 1$, then since $|I| \geq n/2$, $\overline{G_S}[I \cup V_0]$ contains a (v, w)-path P_l for $2 \leq l \leq |I| + |V_0|$ and if $|V_0| \geq n/2$, then $\overline{G_S}[I \cup V_0]$ contains a (v, w)-path P_l for $2 \leq l \leq |I| + n/2 - 1$. Thus, $\overline{G_S}[I \cup V_0]$ contains a (v, w)-path of order $|I \cup V_0| - (n/2 - 2)$, which together with a hamiltonian path of $\overline{G_0} - V_0$ give a cycle of length $|G_S| - (n/2 - 2) = m$, that is, $\overline{G_S}$ contains a C_m. ▮

If $A \neq \emptyset$, we let $H_i \in A$. If $|A \cup B| = 1$, $\overline{G_S}$ contains a C_m by Claim 2 and hence $\overline{G}$ contains a W_m with the hub u. So we may assume $|A \cup B| \geq 2$. If $|A \cup B| = 2$, we let $H_j \in A \cup B$ with $j \neq i$. In this case,

$G_L = H_j$. Let $x = v_j$ if $H_j \in A$ and $x \in G_j$ if $H_j \in B$. Then by Claim 2, $\overline{G_S \cup \{x\}}$ contains a hamiltonian path P with its end vertices in G_S. If $H_j \in A$, then $G_L - x = mK_{n/2-1}$ and it's clear that $\overline{G_L - x}$ is hamiltonian. So $p(\overline{G_L}) = |G_L - x| = |G_L| - 1$ and $|P| + |p(\overline{G_L})| = |G_S| + |G_L| = n + m - 2 - n/2 = m + n/2 - 2 \geq m$. Then the path P and a path in $\overline{G_L}$ with appropriate length give a C_m, which implies $\overline{G}$ contains a W_m with hub u. If $H_j \in B$, then $|P| + |I(j)| = n + m - 2 - n/2 - (n/2 - 2) = m$ and so this P together with $\overline{I(j)}$ and u form a W_m with hub u in $\overline{G}$, a contradiction. If $|A \cup B| \geq 3$, then it is easy to check that $p(\overline{G_L}) = |G_L|$. Thus by Claim 2 we can see that a hamiltonian path in $\overline{G_S}$ and a path in $\overline{G_L}$ with appropriate length form a C_m, and then $\overline{G}$ contains a W_m with hub u, also a contradiction.

If $A = \emptyset$, we let $H_i \in B$ and P_S a longest path in $\overline{G_S}$. By Claim 3, $|P_S| \geq |G_S| - (n/2 - 2)$. Set $I = I(i) - \{u\}$. By Claim 3 we may assume $|B| \geq 2$. If $|B| = 2$, we let $H_j \in B$ with $j \neq i$. Because $\overline{G_S \cup G_j}$ contains a complete bipartite graph between $V(P_S)$ and $V(G_j)$, and by Claim 3 we can see that $|P_S| + |G_j| \geq |G_S| - (n/2 - 2) + (n/2 - 1) \geq |G_S| + 1$, so $\overline{G_S \cup G_j}$ contains a path P of order $|G_S| + 1$ such that the end vertices of P are in G_S. Noting that $I(j)$ is an independent set of order at least $n/2 + 1$ in G and $|P| + |I(j)| = |G_S| + 1 + |I(j)| = |G| - (|G_i| + 1) - |G_j| + 1 = n + m - 2 - n/2 - (n/2 - 1) + 1 = m$, so this P and a hamiltonian path in $\overline{I(j)}$ give a C_m, which implies that $\overline{G}$ contains a W_m with hub u, a contradiction. If $|B| \geq 3$, then $p(G_L) = |G_L|$. Thus, since $|P_S| \geq |G_S| - (n/2 - 2)$ and $|P_S| + |P(G_L)| \geq |G_S| - (n/2 - 2) + |G_L| = n + m - 2 - n/2 - (n/2 - 2) = m$, the path P_S and a path in $\overline{G_L}$ with appropriate length give a C_m in $\overline{G}$, which implies that $\overline{G}$ contains a W_m with hub u, also a contradiction. ∎

4. Problem

By Corollaries 2 and 3, the Ramsey numbers $R(P_n, W_m)$ are determined for $n \leq \lceil m/2 \rceil - 1$ and $0 \leq r \leq 2$. By Corollary 4, the Ramsey numbers $R(P_n, W_m)$ are determined for $n \leq \lceil m/2 \rceil - 1$, $r \geq 3$ and $k + r \geq n - 1$. Noting that $k \geq 2$ if $n \leq \lceil m/2 \rceil - 1$, we can see that the Ramsey numbers $R(P_n, W_m)$ are still unknown for $n \leq \lceil m/2 \rceil - 1$, $r \geq 3$ and $5 \leq k + r \leq n - 2$. Motivated by the results of this paper, we have the following:

Conjecture. If $4 \leq n \leq \lceil m/2 \rceil - 1$, $r \geq 3$ and $5 \leq k + r \leq n - 2$, then $R(P_n, W_m) \leq m + n - 3$.

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