ON THE COCIRCUITS OF A SPLITTING MATROID

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ABSTRACT. The cocircuits of a splitting matroid $M_{i,j}$ are described in terms of the cocircuits of the original matroid M.

1. INTRODUCTION

The matroid notation and terminology used here will follow Oxley [2]. The ground set of a matroid M will be denoted by $E(M)$ while the collections of circuits and cocircuits of M will be denoted by $\mathcal{C}(M)$ and $\mathcal{C}^*(M)$, respectively.

Fleischner [1] introduced the idea of splitting a vertex of degree at least three in a connected graph and used the operation to characterize Eulerian graphs. Figure 1 shows the graph $G_{6,7}$ that is obtained from G by splitting away the edges 6 and 7 from the vertex v. Raghunathan, Shikare, and Waphare [3] extended the splitting operation from graphs to binary matroids using the following definition.

Definition 1.1. Let $M = M[A]$ be a binary matroid with ground set $\{1, 2, \ldots, n\}$ and suppose $i, j \in E(M)$. Let $A_{i,j}$ be the matrix obtained from A by adjoining the row $\delta_{i,j}$ that is zero everywhere except for entries of 1 in the columns labeled by i and j. The splitting matroid $M_{i,j}$ is defined to be the vector matroid of the matrix $A_{i,j}$.

Using Definition 1.1, Raghunathan, Shikare, and Waphare [3] showed that $(M(G))_{i,j} = M(G_{i,j})$ for each graphic matroid $M(G)$. These authors also characterized the circuits of a splitting matroid $M_{i,j}$ as shown in the next result.

Lemma 1.2. *Let M be a binary matroid and suppose $i, j \in E(M) = \{1, 2, \ldots, n\}$. Then $\mathcal{C}(M_{i,j}) = \mathcal{C}_0 \cup \mathcal{C}_1$ where*
$\mathcal{C}_0 = \{C \in \mathcal{C}(M) \mid i, j \in C \text{ or } i, j \notin C\}$; *and*
$\mathcal{C}_1 = \{C_1 \cup C_2 \mid C_1, C_2 \in \mathcal{C}(M), C_1 \cap C_2 = \emptyset, i \in C_1, j \in C_2$; *and there is no $C \in \mathcal{C}_0$ such that $C \subseteq C_1 \cup C_2\}$.*

Shikare and Azadi [4] characterized the bases of a splitting matroid. This paper describes the cocircuits of a splitting matroid $M_{i,j}$ in terms of the cocircuits of the original matroid M.

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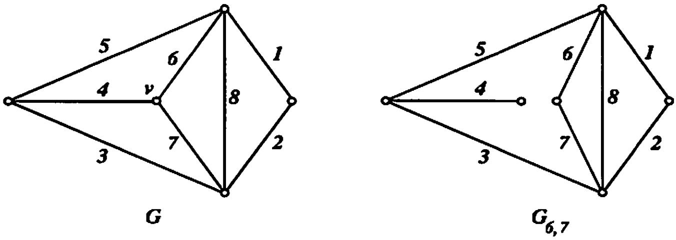

FIGURE 1. $G_{6,7}$ is obtained by splitting edges 6 and 7 from vertex v of G.

2. COCIRCUITS OF A SPLITTING MATROID

For a field F and a natural number n, the n-dimensional vector space over F is denoted by $V(n,F)$. The *support* of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is $\{i \,|\, \mathbf{v}_i \neq 0\}$ and is denoted by *supp*($\mathbf{v}$). Recall that the row space of an $m \times n$ matrix A over a field F, denoted $\mathcal{R}(A)$, is the subspace of $V(n,F)$ that is spanned by the rows of A. The next lemma is a result of Tutte [5] (or see [2, Proposition 9.2.4]) that relates the cocircuits of a vector matroid $M[A]$ to the minimal supports of vectors in $\mathcal{R}(A)$.

Lemma 2.1. *Let A be an $m \times n$ matrix over a field F and $M = M[A]$. Then the set of cocircuits of M coincides with the set of minimal non-empty supports of vectors from the row space of A.*

The next lemma is a basic linear algebra result and its straightforward proof is omitted.

Lemma 2.2. *Let A' be the matrix obtained from A by adjoining the row vector $\mathbf{z}$. Then $\mathcal{R}(A') = \mathcal{R}(A) \cup \{\mathbf{y} + \mathbf{z} \,|\, \mathbf{y} \in \mathcal{R}(A)\}$.*

The next lemma establishes that each cocircuit of M is either a cocircuit of the splitting matroid $M_{i,j}$ or the disjoint union of 2 cocircuits of $M_{i,j}$.

Lemma 2.3. *Suppose $M = M[A]$ is a binary matroid with ground set $E(M) = \{1, 2, \ldots, n\}$ and $i, j \in E(M)$. Let $M_{i,j} = M[A_{i,j}]$ be a splitting matroid of M. Then for each cocircuit C^* of M, the set C^* is either a cocircuit of $M_{i,j}$ or a disjoint union of two cocircuits of $M_{i,j}$.*

Proof. Suppose C^* is a cocircuit of M. It follows from Lemma 2.1 that C^* is a minimal non-empty support of a vector $\mathbf{v}$ in $\mathcal{R}(A)$. If *supp*($\mathbf{v}$) is also minimal in the collection of non-empty supports of vectors in $\mathcal{R}(A_{i,j})$, then C^* is a cocircuit of both M and $M_{i,j}$.

Now assume that *supp*($\mathbf{v}$) is not minimal in the collection of non-empty supports of vectors in $\mathcal{R}(A_{i,j})$. Then there is a vector $\mathbf{u}$ in $\mathcal{R}(A_{i,j})$ so that *supp*(**u**) $\subset$ *supp*(**v**) and *supp*(**u**) is minimal in the collection of non-empty supports of vectors in $\mathcal{R}(A_{i,j})$. Let C_1^* be the cocircuit of $M_{i,j}$ corresponding to *supp*(**u**). Evidently *supp*(**u** + **v**) $\subset$ *supp*(**v**). If *supp*(**u** + **v**) is minimal in the collection of non-empty supports of vectors in $\mathcal{R}(A_{i,j})$, then $M_{i,j}$ has a cocircuit $C_2^* =$ *supp*(**u** + **v**) $= C^* - C_1^*$. Thus C^* is the disjoint union of C_1^* and C_2^*. If *supp*(**u** + **v**) is not minimal, then there is a vector **w** in $\mathcal{R}(A_{i,j})$ having minimal support and *supp*(**w**) $\subset$ *supp*(**u** + **v**). Since **u** and **w** are elements of $\mathcal{R}(A_{i,j})$, Lemma 2.2 implies that $\mathbf{u} = \mathbf{x} + \delta_{i,j}$ and $\mathbf{w} = \mathbf{y} + \delta_{i,j}$ where $\mathbf{x}, \mathbf{y} \in \mathcal{R}(A)$ and $\delta_{i,j}$ is the vector whose only non-zero entries are ones appearing in coordinates i and j. Thus $\mathbf{u} + \mathbf{w} = (\mathbf{x} + \delta_{i,j}) + (\mathbf{y} + \delta_{i,j}) = \mathbf{x} + \mathbf{y}$. Therefore $\mathbf{u} + \mathbf{w} \in \mathcal{R}(A)$. However *supp*(**u** + **w**) $\subset$ *supp*(**v**) contradicting the minimality of *supp*(**v**) in the collection of non-empty supports of vectors in $\mathcal{R}(A)$. As a result of this contradiction we conclude that *supp*(**u** + **v**) is minimal in the collection of non-empty supports of vectors in $\mathcal{R}(A_{i,j})$ and the lemma holds. □

It follows from Lemma 1.2 that if each circuit of M contains both i and j, or neither, then $\mathcal{C}(M_{i,j}) = \mathcal{C}(M)$ and $M_{i,j} = M$. The fact that $M_{i,j} \neq M$ only if there is a circuit of M containing exactly one of i and j is the basis of the next two results.

Proposition 2.4. *If* $\{i,j\}$ *is a cocircuit of* M *or if* $\{i\}$ *and* $\{j\}$ *are cocircuits of* M*, then* $M = M_{i,j}$.

Proof. In both cases there is no circuit of M containing exactly one of i and j. Hence $M = M_{i,j}$. □

Proposition 2.5. *If exactly one of the sets* $\{i\}$ *and* $\{j\}$ *is a cocircuit of* M*, then both* $\{i\}$ *and* $\{j\}$ *are cocircuits of* $M_{i,j}$.

Proof. Suppose $\{i\}$ is a cocircuit of M and $\{j\}$ is not. Then j is in a circuit of M that does not contain i. Since i is in no circuits of M, it follows from Lemma 1.2 that i is in no circuits of $M_{i,j}$. Thus j is in no circuits of $M_{i,j}$ and we conclude that $\{j\}$ is a cocircuit of $M_{i,j}$. □

The next result characterizes the cocircuits of $M_{i,j}$ if exactly one of the sets $\{i\}$ and $\{j\}$ is a cocircuit of M.

Theorem 2.6. *Suppose* $M = M[A]$ *is a binary matroid with ground set* $E(M) = \{1, 2, \ldots, n\}$ *and* $i, j \in E(M)$. *Let* $M_{i,j} = M[A_{i,j}]$ *be a splitting matroid of* M. *If* $\{i\}$ *is a cocircuit of* M *and* $\{j\}$ *is not, then* $\mathcal{C}^*(M_{i,j})$ *consists of* $\{i\}$, $\{j\}$, *and the sets in the following collections of sets:*

a) $\{C^* - \{j\} \mid C^*$ *is a cocircuit of* M *containing* $\{j\}$ *as a proper subset*$\}$,

b) $\{C^* \in \mathcal{C}^*(M) \mid j \notin C^*$ *and* C^* *is the only cocircuit of* M *contained in* $C^* \cup \{j\}\}$.

Proof. We first show that each of the sets described in the result is a cocircuit of $M_{i,j}$. It follows from Proposition 2.5 that $\{i\}$ and $\{j\}$ are cocircuits of $M_{i,j}$. Since $\{j\}$ is a cocircuit of $M_{i,j}$, Lemma 2.3 establishes that for each cocircuit of M containing $\{j\}$ as a proper subset, the set $C^* - \{j\}$ is a cocircuit of $M_{i,j}$. Now suppose C^* is a cocircuit of M that does not contain j and C^* is the only cocircuit of M contained in $C^* \cup \{j\}$. If C^* is not a cocircuit of $M_{i,j}$, then Lemma 2.3 implies that $C^* = C_1^* \cup C_2^*$ where C_1^* and C_2^* are cocircuits of $M_{i,j}$. Suppose $\mathbf{w_1} \in \mathcal{R}(A_{i,j})$ and $supp(\mathbf{w_1}) = C_1^*$. Since $\mathbf{w_1} \notin \mathcal{R}(A)$, Lemma 2.2 implies that $\mathbf{w_1} = \mathbf{u_1} + \delta_{i,j}$ for some vector $\mathbf{u_1}$ in $\mathcal{R}(A)$. Now $j \in supp(\mathbf{u_1})$, for otherwise $j \in C^*$; a contradiction. Lemma 2.1 implies that either $supp(\mathbf{u_1})$ is a cocircuit of M or a proper subset of $supp(\mathbf{u_1})$ is a cocircuit of M. In either case it follows that $C^* \cup \{j\}$ contains a cocircuit of M other than C^*; a contradiction.

We now show that each cocircuit of $M_{i,j}$ other than $\{i\}$ and $\{j\}$ belongs to one of the sets listed in the result. Assume C^* is a cocircuit of $M_{i,j}$ other than $\{i\}$ or $\{j\}$ and $C^* = supp(\mathbf{v})$ for some vector $\mathbf{v}$ in $\mathcal{R}(A_{i,j})$. If $\mathbf{v} \in \mathcal{R}(A)$, then C^* is a cocircuit of M not containing j. Suppose X is a proper subset of C^* and $X \cup \{j\}$ is a cocircuit of M. Since $\{j\}$ is a cocircuit of $M_{i,j}$, it follows from Lemma 2.3 that X is a cocircuit of $M_{i,j}$. However X is a proper subset of the cocircuit C^* of $M_{i,j}$; a contradiction. Therefore C^* is a cocircuit of M not containing j and is the only cocircuit of M contained in $C^* \cup \{j\}$.

We now assume that $\mathbf{v} \notin \mathcal{R}(A)$. Then Lemma 2.2 implies that $\mathbf{v} = \mathbf{u} + \delta_{i,j}$ for some vector $\mathbf{u}$ in $\mathcal{R}(A)$. If $i \notin supp(\mathbf{u})$, then $i \in supp(\mathbf{v})$. However $\{i\}$ is a cocircuit of $M_{i,j}$ and $supp(\mathbf{v})$ is a cocircuit of $M_{i,j}$ other than $\{i\}$ or $\{j\}$. Thus $i \notin supp(\mathbf{v})$; a contradiction. We conclude that $i \in supp(\mathbf{u})$. Similarly, if $j \notin supp(\mathbf{u})$, then $j \in supp(\mathbf{v})$ which leads to a contradiction. Hence i and j are elements of $supp(\mathbf{u})$. Since $\{i\}$ is a cocircuit of M, there exists a vector $\mathbf{z}$ in $\mathcal{R}(A)$ such that $supp(\mathbf{z}) = \{i\}$. Hence $\mathbf{u} + \mathbf{z} \in \mathcal{R}(A)$. Note that $supp(\mathbf{u} + \mathbf{z})$ is a proper subset of $supp(\mathbf{u})$. Now $supp(\mathbf{u} + \mathbf{z})$ or a proper subset of it is a cocircuit of M. Hence there exists a vector $\mathbf{w}$ in $\mathcal{R}(A)$ so that $supp(\mathbf{w})$ is a cocircuit D^* of M and $supp(\mathbf{w})$ is a proper subset of $supp(\mathbf{u})$ not containing i. If $j \notin supp(\mathbf{w})$, then $supp(\mathbf{w}) \subseteq supp(\mathbf{v})$. If $supp(\mathbf{w}) = supp(\mathbf{v})$, then $\mathbf{v} = \mathbf{w} \in \mathcal{R}(A)$; a contradiction. If $supp(\mathbf{w})$ is a proper subset of $supp(\mathbf{v})$, then it follows from Lemma 2.3 that a proper subset of $supp(\mathbf{v})$ is a cocircuit of $M_{i,j}$. This contradicts the assumption that $supp(\mathbf{v})$ is a cocircuit of $M_{i,j}$. We conclude that $j \in supp(\mathbf{w})$. Let $\mathbf{y}$ be the vector in $M_{i,j}$ with $supp(\mathbf{y}) = \{j\}$. If $supp(\mathbf{y} + \mathbf{w})$ is a proper subset of $supp(\mathbf{v})$, then $supp(\mathbf{v})$ is not minimal in $\mathcal{R}(A_{i,j})$; a contradiction. Hence $supp(\mathbf{y} + \mathbf{w}) = supp(\mathbf{v})$. Thus $\mathbf{v} = \mathbf{y} + \mathbf{w}$. Therefore $supp(\mathbf{v}) = D^* - \{j\}$ for a cocircuit D^* of M containing j. Thus each cocircuit of $M_{i,j}$ is described by one of the sets listed in the result and the theorem holds. □

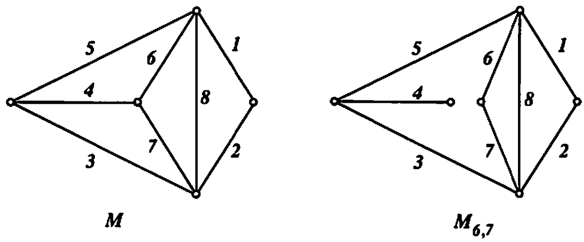

FIGURE 2. $G_{6,7}$ is obtained by splitting edges 6 and 7 from vertex v of G.

Proposition 2.4 and Theorem 2.6 characterize the cocircuits of a splitting matroid $M_{i,j}$ in terms of the cocircuits of M whenever the set $\{i,j\}$ contains a cocircuit of M. The next three results describe the cocircuits of $M_{i,j}$ whenever $\{i,j\}$ does not contain a cocircuit of M. We define a Type I set of M to be a set $C^* - \{i,j\}$ where C^* is a cocircuit of M that has $\{i,j\}$ as a proper subset. Table 1 displays the Type I sets and cocircuits of M along with the cocircuits of $M_{6,7}$ for the graphic matroids M and $M_{6,7}$ shown in Figure 2.

TABLE 1

Cocircuits of M	Type I sets of M	Cocircuits of $M_{6,7}$
$\{1,2\}$	$\{4\}$	$\{1,2\}$
$\{3,4,5\}$	$\{3,5\}$	$\{4\}$
$\{4,6,7\}$		$\{3,5\}$
$\{3,6,7,5\}$		$\{6,7\}$
$\{2,3,7,8\}$		$\{1,3,7,8\}$
$\{1,3,7,8\}$		$\{1,3,6,8\}$
$\{1,5,6,8\}$		$\{2,3,7,8\}$
$\{2,5,6,8\}$		$\{2,3,6,8\}$
$\{1,3,4,6,8\}$		$\{1,5,6,8\}$
$\{2,3,4,6,8\}$		$\{1,5,7,8\}$
$\{2,4,5,7,8\}$		$\{2,5,7,8\}$
$\{1,4,5,7,8\}$		$\{2,5,6,8\}$

Notice that $\{6,7\}$, the Type I sets of M, and the cocircuits of M that do not contain a Type I set are cocircuits of $M_{6,7}$. In addition, for each Type I set X containing exactly one of i and j, the symmetric difference $X \triangle \{i,j\}$ is a cocircuit of $M_{i,j}$. The following theorem establishes that this relationship holds in general.

Theorem 2.7. *Suppose $M = M[A]$ is a binary matroid with ground set $E(M) = \{1, 2, \ldots, n\}$ and $i, j \in E(M)$. Let $M_{i,j} = M[A_{i,j}]$ be a splitting matroid of M. If $\{i, j\}$ does not contain a cocircuit of M, then $\{i, j\}$ and each non-empty set in the union of the following collections of sets is a cocircuit of $M_{i,j}$:*
a) $\{C^* - \{i, j\} \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *contains* $\{i, j\}$ *as a proper subset*$\}$
b) $\{C^* \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *does not contain a Type I set*$\}$
c) $\{C^* \triangle \{i, j\} \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *contains exactly one of* i *and* j *and does not contain a Type I set*$\}$.

Proof. Assume there is no cocircuit of M having $\{i, j\}$ as a proper subset. Suppose also that $\mathbf{v} \in \mathcal{R}(A_{i,j})$ and $supp(\mathbf{v})$ is a proper subset of $\{i, j\}$. Then either $supp(\mathbf{v}) = \{i\}$ or $supp(\mathbf{v}) = \{j\}$. If $supp(\mathbf{v}) = \{i\}$, then as $\mathbf{v} \notin \mathcal{R}(A)$, Lemma 2.2 implies that $\mathbf{v} = \mathbf{u} + \delta_{i,j}$ for some vector $\mathbf{u}$ in $\mathcal{R}(A)$. Hence $supp(\mathbf{u}) = \{j\}$ contradicting the assumption that $\{i, j\}$ does not contain a cocircuit of M. By symmetry, if $supp(\mathbf{v}) = \{j\}$, then $supp(\mathbf{u}) = \{i\}$; a contradiction. Thus $supp(\delta_{i,j}) = \{i, j\}$ is minimal in $\mathcal{R}(A_{i,j})$ and $\{i, j\}$ is a cocircuit of $M_{i,j}$.

If, on the other hand, $\{i, j\}$ is a proper subset of a cocircuit of M, then it follows that $supp(\delta_{i,j}) = \{i, j\}$ is minimal in $\mathcal{R}(A_{i,j})$. Hence $\{i, j\}$ is a cocircuit of $M_{i,j}$ whenever $\{i, j\}$ does not contain a cocircuit of M. Furthermore, Lemma 2.3 implies that for each cocircuit C^* of M containing $\{i, j\}$ as a proper subset, if any, the set $C^* - \{i, j\}$ is a cocircuit of $M_{i,j}$.

Now suppose C^* is a cocircuit of M that does not contain a Type I set. Lemma 2.3 implies that either C^* is a cocircuit of $M_{i,j}$ or C^* is the disjoint union of two cocircuits C_1^* and C_2^* of $M_{i,j}$. Suppose there is a vector $\mathbf{v}$ in $\mathcal{R}(A)$ such that $supp(\mathbf{v}) = C^*$ and there are vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ in $\mathcal{R}(A_{i,j})$ so that $supp(\mathbf{v_1}) = C_1^*$ and $supp(\mathbf{v_2}) = C_2^*$. Lemma 2.2 implies that $\mathbf{v_1} = \mathbf{u_1} + \delta_{i,j}$ and $\mathbf{v_2} = \mathbf{u_2} + \delta_{i,j}$ for some vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ in $\mathcal{R}(A)$. Evidently $supp(\mathbf{u_k}) \subseteq C_k^* \cup \{i, j\}$ for each k in $\{1, 2\}$. Since $\{i, j\} \not\subseteq supp(\mathbf{v})$ and $supp(\mathbf{v})$ is minimal in $\mathcal{R}(A)$, it follows that $supp(\mathbf{u_1}) \not\subseteq supp(\mathbf{v})$ and $supp(\mathbf{u_2}) \not\subseteq supp(\mathbf{v})$. Moreover, as $\mathbf{u_1} + \mathbf{u_2} = \mathbf{v}$, we see that $supp(\mathbf{u_1}) \cap supp(\mathbf{u_2})$ is a nonempty subset of $E(M) - C^*$. In particular, $supp(\mathbf{u_1}) \cap supp(\mathbf{u_2}) \subseteq \{i, j\}$. Hence $|supp(\mathbf{u_1}) \cap supp(\mathbf{u_2})|$ is either 1 or 2.

First assume that $|supp(\mathbf{u_1}) \cap supp(\mathbf{u_2})| = 1$. Suppose $i \in C_1^* = supp(\mathbf{v_1})$. Then $i \notin supp(\mathbf{u_1})$, $j \in supp(\mathbf{u_1})$, $i \in supp(\mathbf{u_2})$, and $j \in supp(\mathbf{u_2})$. Thus $supp(\mathbf{u_2}) = C_2^* \cup \{i, j\}$. Suppose that $supp(\mathbf{u_2})$ is not minimal in $\mathcal{R}(A)$. Then there exists a vector $\mathbf{w}$ in $\mathcal{R}(A)$ so that $supp(\mathbf{w}) \subset supp(\mathbf{u_2})$ and $supp(\mathbf{w})$ is minimal in $\mathcal{R}(A)$. Notice that $C_1^* - \{i\}$ is non-empty since $\{i\}$ is not a cocircuit of $M_{i,j}$. Then $j \in supp(\mathbf{w})$, for if not, $supp(\mathbf{w}) \subset supp(\mathbf{v})$ contradicting the minimality of $supp(\mathbf{v})$ in $\mathcal{R}(A)$. If $i \notin supp(\mathbf{w})$, then as $i \notin supp(\mathbf{u_1})$, we have $supp(\mathbf{w} + \mathbf{u_1}) \subset supp(\mathbf{v})$; a contradiction. Similarly, if $e \in C_2^* - \{i, j\}$ and $e \notin supp(\mathbf{w})$, then

$supp(\mathbf{w}+\mathbf{u_1}) \subset supp(\mathbf{v})$, a contradiction. Thus $supp(\mathbf{u_2})$ is minimal in $\mathcal{R}(A)$ and $supp(\mathbf{u_2}) = C_2^* \cup \{i,j\}$ is a cocircuit of M. Hence the subset C_2^* of C^* is a Type I set. As a result of this contradiction to the assumption that C^* does not contain a Type I set, we conclude that $i \notin C_1^* = supp(\mathbf{v_1})$.

By symmetry of i and j and C_1^* and C_2^*, we conclude that neither i nor j is in C^*. Thus we may assume that $|supp(\mathbf{u_1}) \cap supp(\mathbf{u_2})| = 2$. Hence $supp(\mathbf{u_1}) \cap supp(\mathbf{u_2}) = \{i,j\}$. Now suppose neither $C_1^* \cup \{i,j\}$ nor $C_2^* \cup \{i,j\}$ is a cocircuit of M. In particular, neither $supp(\mathbf{v_1}) = supp(\mathbf{u_1}) \cap C^*$ nor $supp(\mathbf{v_2}) = supp(\mathbf{u_2}) \cap C^*$ is a Type I set of M. Then there exist vectors $\mathbf{w_1}$ and $\mathbf{w_2}$ with minimal supports in $\mathcal{R}(A)$ such that $supp(\mathbf{w_1}) \subset supp(\mathbf{u_1})$ and $supp(\mathbf{w_2}) \subset supp(\mathbf{u_2})$. Moreover, for k in $\{1,2\}$, each $\mathbf{w_k}$ contains at least one element of $\{i,j\}$, for if not, then $supp(\mathbf{w_k}) \subset supp(\mathbf{v})$; a contradiction. In addition, for k in $\{1,2\}$, each $\mathbf{w_k}$ contains at least one element of C_k^*, for if not, then $\{i,j\}$ contains a cocircuit of M; a contradiction. If $\mathbf{w_1}$ contains both i and j, then $supp(\mathbf{u_2}+\mathbf{w_1}) \subset supp(\mathbf{v})$. As a result of this contradiction, we conclude that $\mathbf{w_1}$ contains exactly one element of $\{i,j\}$. Similarly, $\mathbf{w_2}$ contains exactly one element of $\{i,j\}$. Now $C_1^* - supp(\mathbf{w_1}) \neq \emptyset$ for if not, then $\mathbf{w_1}+\mathbf{u_1} \in \mathcal{R}(A)$ and $supp(\mathbf{w_1}+\mathbf{u_1}) \subseteq \{i,j\}$; a contradiction. Similarly, $C_2^* - supp(\mathbf{w_2}) \neq \emptyset$. If $\{i\} = supp(\mathbf{w_1}) \cap supp(\mathbf{w_2})$, then $\mathbf{w_1}+\mathbf{w_2} \in \mathcal{R}(A)$ and $supp(\mathbf{w_1}+\mathbf{w_2})$ is a proper subset of c^*; a contradiction. Hence $\{i\} \neq supp(\mathbf{w_1}) \cap supp(\mathbf{w_2})$. Similarly $\{j\} \neq supp(\mathbf{w_1}) \cap supp(\mathbf{w_2})$. Thus $supp(\mathbf{w_1}) \cap supp(\mathbf{w_2}) = \emptyset$. Suppose without loss of generality that $i \in supp(\mathbf{w_1})$ and $j \in supp(\mathbf{w_2})$. Then $supp(\mathbf{u_1}+\mathbf{w_1}+\mathbf{w_2}) \subset supp(\mathbf{v})$; a contradiction. We conclude that $C_1^* \cup \{i,j\}$ or $C_2^* \cup \{i,j\}$ is a cocircuit of M contradicting the assumption that C^* does not contain a Type I set. We conclude that each cocircuit C^* of M that does not contain a Type I set is a cocircuit of $M_{i,j}$.

Now assume C^* is a cocircuit of M containing exactly one of i and j and does not contain a Type I set. The argument in the previous three paragraphs establishes that C^* is a cocircuit of $M_{i,j}$. Since $\{i,j\}$ is also a cocircuit of $M_{i,j}$, it follows (see [2; Theorem 9.1.2]) that the symmetric difference $C^* \triangle \{i,j\}$ is a cocircuit or a disjoint union of cocircuits of $M_{i,j}$. If $C^* - \{i\} = \{x\}$, then $\{i,x\}$, $\{j,x\}$, and $\{i,j\}$ are cocircuits of $M_{i,j}$. However if $C^* \triangle \{i,j\}$ is a disjoint union of cocircuits of $M_{i,j}$, then $\{j\}$ is a cocircuit of $M_{i,j}$; a contradiction. Hence $|C^* - \{i\}| \geq 2$. Now, if $C^* \triangle \{i,j\}$ is a disjoint union of the cocircuits C_1^* and C_2^* of $M_{i,j}$, then either $C_1^* \triangle \{i,j\}$ or $C_2^* \triangle \{i,j\}$ is a proper subset of C^* containing a cocircuit of $M_{i,j}$. As a result of this contradiction we conclude that $C^* \triangle \{i,j\}$ is a cocircuit of $M_{i,j}$. □

Theorem 2.7 identified three classes of cocircuits other than $\{i,j\}$ of the matroid $M_{i,j}$. However, not all cocircuits of $M_{i,j}$ are described by these classes. For example, the cocircuit $\{1,2,3,4\}$ of the matroid $M_{i,j}$ shown in

Figure 3 is a member of none of the three classes described in Theorem 2.7. The next result introduces another class of cocircuits of $M_{i,j}$ and shows that each cocircuit of $M_{i,j}$ other than $\{i,j\}$ belongs to one of these four classes.

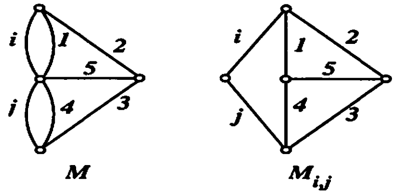

FIGURE 3. The matroids M and $M_{i,j}$.

Theorem 2.8. *Suppose $M = M[A]$ is a binary matroid with ground set $E(M) = \{1, 2, \ldots, n\}$ and $i, j \in E(M)$. Let $M_{i,j} = M[A_{i,j}]$ be a splitting matroid of M. If $\{i,j\}$ does not contain a cocircuit of M, then each cocircuit of $M_{i,j}$ other than $\{i,j\}$ belongs to one of the following collections:*

a) $\{C^* - \{i,j\} \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *contains* $\{i,j\}$ *as a proper subset*$\}$

b) $\{C^* \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *does not contain a Type I set*$\}$

c) $\{C^* \triangle \{i,j\} \mid C^* \in \mathcal{C}^*(M)$ *and* C^* *contains exactly one of* i *and* j *and does not contain a Type I set*$\}$.

d) $\{(C^* \cup D^*) - \{i,j\} \mid C^* \cap D^* = \emptyset$, C^* *is a cocircuit of* M *containing* i, D^* *is a cocircuit of* M *containing* j, *and neither* C^* *nor* D^* *contains a Type I set*$\}$.

Proof. Assume that C^* is a cocircuit of $M_{i,j}$ other than $\{i,j\}$ and $C^* = supp(\mathbf{v})$ for some vector $\mathbf{v} \in \mathcal{R}(A_{i,j})$. If $\mathbf{v} \in \mathcal{R}(A)$, then the set C^* is a cocircuit of M. Suppose C^* contains a Type I set X. Since X and C^* are cocircuits of $M_{i,j}$, it must be that $C^* = X$. But then $C^* \cup \{i,j\}$ is a cocoircuit of M that contains the cocircuit C^* of M as a proper subset; a contradiction. Thus, as described in case b), the set C^* is a cocircuit of M that does not contain a Type I set. Suppose $\mathbf{v} \notin \mathcal{R}(A)$. Then Lemma 2.2 implies that $\mathbf{v} = \mathbf{u} + \delta_{i,j}$ for some vector $\mathbf{u}$ in $\mathcal{R}(A)$. We now consider three cases.

If i and j are not elements of $supp(\mathbf{u})$, then $\{i,j\}$ is a proper subset of $supp(\mathbf{v})$. This is impossible since $\{i,j\}$ and $supp(\mathbf{v})$ are distinct cocircuits of $M_{i,j}$.

Suppose $|supp(\mathbf{u}) \cap \{i,j\}| = 1$. Assume $i \in supp(\mathbf{u})$ and $j \notin supp(\mathbf{u})$. Then $i \notin supp(\mathbf{v})$ and $j \in supp(\mathbf{v})$. Now either $supp(\mathbf{u})$ is a cocircuit of M or $supp(\mathbf{u})$ contains a proper subset, say $supp(\mathbf{w})$, that is a cocircuit of M. If $i \notin supp(\mathbf{w})$, then $supp(\mathbf{w}) \subset supp(\mathbf{v})$, a contradiction. For each element $k \neq i$ in $supp(\mathbf{u})$, if $k \notin supp(\mathbf{w})$, then $supp(\mathbf{w} + \delta_{i,j})$ is a proper subset of $supp(\mathbf{v})$. However $\mathbf{w} + \delta_{i,j} \in \mathcal{R}(A_{i,j})$, contradicting the minimality of $supp(\mathbf{v})$. Hence $supp(\mathbf{u})$ is a cocircuit of M and Lemma 2.3 implies that $supp(\mathbf{u})$ is either a cocircuit of $M_{i,j}$ or a disjoint union of two cocircuits of $M_{i,j}$. However, if $supp(\mathbf{u})$ is a disjoint union of two cocircuits C_1^* and C_2^* of $M_{i,j}$, then $supp(\mathbf{v})$ has either C_1^* or C_2^* as a proper subset; a contradiction. We conclude that $supp(\mathbf{u})$ is a cocircuit of both M and $M_{i,j}$. In particular, $C^* = supp(\mathbf{u})\triangle\{i,j\}$ and $supp(\mathbf{u})$ is a cocircuit of M containing exactly one of i and j and no Type I sets. It follows that case c) holds.

Now suppose $|supp(\mathbf{u}) \cap \{i,j\}| = 2$. Now either $supp(\mathbf{u})$ is a cocircuit of M or $supp(\mathbf{u})$ has a proper subset that is a cocircuit of M. In the first case, since $supp(\mathbf{u})$ is a cocircuit of M containing $\{i,j\}$ and $supp(\mathbf{v}) = supp(\mathbf{u}) - \{i,j\}$, we see that case a) holds. Now assume $supp(\mathbf{u})$ contains a proper subset, say $supp(\mathbf{w})$, that is a cocircuit of M. If $supp(\mathbf{w}) \cap \{i,j\} = \emptyset$, then $supp(\mathbf{w}) \subsetneq supp(\mathbf{v})$. Since $\mathbf{w} \in \mathcal{R}(A)$ and $\mathbf{v} \notin \mathcal{R}(A)$ we conclude that $supp(\mathbf{w})$ must be a proper subset of $supp(\mathbf{v})$. This is a contradiction since $\mathbf{w} \in \mathcal{R}(A) \subseteq \mathcal{R}(A_{i,j})$ and $supp(\mathbf{v})$ is minimal in $\mathcal{R}(A_{i,j})$. Similarly, if $supp(\mathbf{w}) \cap \{i,j\} = \{i,j\}$, then $\mathbf{w} + \mathbf{u} \in \mathcal{R}(A) \subseteq \mathcal{R}(A_{i,j})$ and $supp(\mathbf{w} + \mathbf{u})$ is a proper subset of $supp(\mathbf{v})$; a contradiction. Finally, suppose that $|supp(\mathbf{w}) \cap \{i,j\}| = 1$. Then $supp(\mathbf{w})$ is a cocircuit of M containing exactly one of i and j, say j. Now either $supp(\mathbf{w})$ is a cocircuit of M or $supp(\mathbf{w}) = C_1^* \cup C_2^*$ for disjoint cocircuits C_1^* and C_2^* of $M_{i,j}$. If the latter holds, then the set $\{j\}$ or a proper subset of $supp(\mathbf{v})$ is a cocircuit of $M_{i,j}$; a contradiction. Thus $supp(\mathbf{w})$ is a cocircuit of both M and $M_{i,j}$. Similarly, $supp(\mathbf{u} + \mathbf{w})$ is a cocircuit of both M and $M_{i,j}$ and $supp(\mathbf{u} + \mathbf{w})$ contains j. It follows that case d) holds. □

It is possible that a set described by case d) of Theorem 2.8 is not a cocircuit of $M_{i,j}$. For example, the cocircuits $\{1,2,i\}$ and $\{3,4,j\}$ are disjoint cocircuits of N containing no Type I sets. Moreover, each set contains exactly one element of $\{i,j\}$. However the set $(\{1,2,i\} \cup \{3,4,j\}) - \{i,j\}$ is not a cocircuit of $N_{i,j}$. The next result shows that such a set must be partitioned into cocircuits of the splitting matroid. In particular, these cocircuits belong to one of the four classes of cocircuits described in Theorem 2.8.

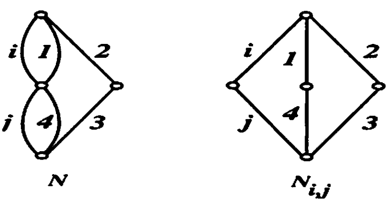

FIGURE 4. The matroids N and $N_{i,j}$.

Theorem 2.9. *Suppose $M = M[A]$ is a binary matroid with ground set $E(M) = \{1, 2, \ldots, n\}$ and $i, j \in E(M)$. Let $M_{i,j} = M[A_{i,j}]$ be a splitting matroid of M and suppose $\{i, j\}$ does not contain a cocircuit of M. Let C^* be a cocircuit of M containing i, and D^* be a cocircuit of M containing j. Suppose $C^* \cap D^* = \emptyset$ and neither C^* nor D^* contains a Type I set. If the set $(C^* \cup D^*) - \{i, j\}$ is not a cocircuit of $M_{i,j}$, then it is partitioned into cocircuits of $M_{i,j}$ of the types described in Theorem 2.8.*

Proof. Suppose $(C^* \cup D^*) - \{i, j\}$ is not a cocircuit of $M_{i,j}$. Since C^* and D^* are cocircuits of M which contain no Type I set, C^* and D^* are cocircuits of $M_{i,j}$. Then, as $\{i, j\}$, C^*, and D^* are cocircuits of $M_{i,j}$, the set $C^* \triangle \{i, j\} \triangle D^* = (C^* \cup D^*) - \{i, j\}$ is partitioned into cocircuits of $M_{i,j}$. It follows from Theorem 2.8 that each of these cocircuits is of one of the types described by cases a)- d) of the theorem.

□

Whenever the set $\{i, j\}$ contains a cocircuit of a matroid M, Proposition 2.4 and Theorem 2.6 give a complete description of the cocircuits of the splitting matroid $M_{i,j}$. In addition, whenever $\{i, j\}$ does not contain a cocircuit of M, the combination of Theorems 2.7, 2.8, and 2.9 establishes the following procedure to determine $C^*(M_{i,j})$ from $C^*(M)$.

Given $C^*(M)$:

1. Identify the Type I sets, if any, of M; that is, identify the sets described by case a) of Theorem 2.8.
2. Identify the sets described by case b) or c) of Theorem 2.8.
3. Form all sets, if any, of the kind described by case d) of Theorem 2.8.
4. Determine the sets, if any, formed in Step 3 that contain none of the sets obtained in Steps 1, 2, or 3 as a proper subset.
5. $C^*(M_{i,j})$ consists of $\{i, j\}$ and the union of the collections of sets developed in Steps 1, 2, and 4.

The reader can easily verify that for the matroids in Figures 3 and 4, this procedures yields $C^*(M_{i,j})$ from $C^*(M)$ and $C^*(N_{i,j})$ from $C^*(N)$.

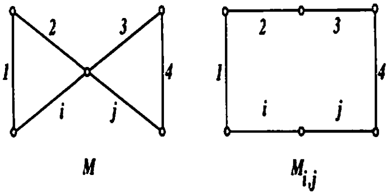

FIGURE 5. The matroids M and $M_{i,j}$.

If a matroid M has no cocircuits which properly contain $\{i, j\}$, then the matroid has no Type I sets and the procedure described above becomes simpler to implement. For example, the matroid M shown in Figure 5 has no Type I sets. In determining the cocircuits of $M_{i,j}$, Step 1 of the procedure contributes no sets while in Step 2, the cocircuits of M and the sets $\{1, j\}$, $\{2, j\}$, $\{3, i\}$, and $\{4, i\}$ are identified as cocircuits of $M_{i,j}$. In Step 3, the sets $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{2, 4\}$ are identified as potential cocircuits of $M_{i,j}$. Since none of the previously identified cocircuits of $M_{i,j}$ is a proper subset of any of $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, or $\{2, 4\}$, these five sets are indeed cocircuits of $M_{i,j}$. Thus the procedure correctly determines that $C^*(M_{i,j})$ consists of all 2-element subsets of $E(M)$.

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