Notes on the (s,t)-Lucas and Lucas Matrix Sequences*

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Abstract

In this article, defining the matrix extensions of the Fibonacci and Lucas numbers we start a new approach to derive formulas for some integer numbers which have appeared , often surprisingly, as answers to intricate problems, in conventional and in recreational Mathematics. Our approach provides a new way of looking at integer sequences from the perspective of matrix algebra, showing how several of these integer sequences relate to each other.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; Mersenne numbers; Fermat numbers.

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1 Introduction

In the present days there is a huge interest of modern science in the application of the generalized Fibonacci and Lucas numbers. Many scholars studied these numbers and their numerous properties. We can start from the generalized k-Fibonacci numbers introduced by Falcón and Plaza [10] such that:

Definition 1 ([*10*])*For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1.$$

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This sequence generalizes, between others, both the classic Fibonacci sequence and the Pell sequence. In [10], Falcón and Plaza showed the relation between the 4-triangle longest-edge (4TLE) partition and the k-Fibonacci numbers, as another example of the relation between geometry and numbers, and many properties of these numbers are deduced directly from elementary matrix algebra. In [11], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [12], the 3-dimensional k-Fibonacci spirals are studied from a geometric point of view. These curves appear naturally from studying the k-th Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci functions.

Definition 2 *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the (s,t)th Fibonacci sequence, say $\{F_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{1}$$

with $F_0(s,t) = 0, \ \ F_1(s,t) = 1$.

In [19], it has been studied the relations between the Bell matrix and the Fibonacci matrix, which provide a unified approach to some lower triangular matrices, such as the Stirling matrices of both kinds, the Lah matrix, and the generalized Pascal matrix. To make the results more general, the discussion is also extended to the (s,t)th Fibonacci numbers and the corresponding matrix. Moreover, based on the matrix representations, various identities are derived.

Definition 3 *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the (s,t)th Lucas sequence, say $\{L_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$L_{n+1}(s,t) = sL_n(s,t) + tL_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{2}$$

with $L_0(s,t) = 2, \ \ L_1(s,t) = s$.

The following table summarizes special cases of $F_n(s,t)$ and $L_n(s,t)$:

(s,t)	F_n	L_n
$(1,1)$	Fibonacci numbers	Lucas numbers
$(2,1)$	Pell numbers	Pell-Lucas numbers
$(1,2)$	Jacobsthal numbers	Jacobsthal-Lucas numbers
$(3,-2)$	Mersenne numbers	Fermat numbers

Jacobsthal and Jacobsthal-Lucas numbers were investigated earlier by Horadam [9]. (See also a systematic investigation by Raina and Srivastava [15], dealing with an interesting class of numbers associated with the familiar Lucas numbers.) and then by the recent works by Filipponi [7], Pintór and Srivastava [14], and Chu and Vicenti [1].

In the sequel we will write simply F_n, f_n, p_n, j_n, m_n, L_n, l_n, q_n, $\hat{j}_n$, and r_n instead of $F_n(s,t)$, $F_n(1,1)$, $F_n(2,1)$, $F_n(1,2)$, $F_n(3,-2)$, $L_n(s,t)$, $L_n(1,1)$, $L_n(2,1)$, $L_n(1,2)$, $L_n(3,-2)$ respectively.

In this note, we define a new matrix generalization of the Fibonacci and Lucas numbers, and using essentially a matrix approach we show some properties of this matrix sequence. Moreover, based on the matrix representations, various identities are derived for the (s,t)th Fibonacci and Lucas numbers.

2 (s,t)-Fibonacci and Lucas matrix sequences

In [2] the (s,t)-Fibonacci matrix sequence $\{\mathcal{F}_n\}_{n=0}^{\infty}$ were defined such that:

Definition 4 ([2])*For any integer numbers $s>0$ and $t\neq 0$ with $s^2+4t>0$; the (s,t)th Fibonacci matrix sequence, say $\{\mathcal{F}_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\mathcal{F}_{n+1}(s,t)=s\mathcal{F}_n(s,t)+t\mathcal{F}_{n-1}(s,t)\quad for\ n\geq 1, \tag{3}$$

with $\mathcal{F}_0(s,t)=I$, $\mathcal{F}_1(s,t)=\begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix}$, where I is the 2×2 unit matrix.

And then it was showed some properties for the sum of the terms of this sequence, obtained by summing up the first n terms $\mathcal{F}_n$ and related with the (s,t)th Fibonacci numbers. Also, in that paper the generating functions for the (s,t)-Fibonacci matrix sequences have been given.

Lemma 5 ([2])*For any integer $n\geq 1$ holds:*

$$\mathcal{F}_n=\begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix}. \tag{4}$$

Lemma 6 ([2]) *$\mathcal{F}_{m+n}=\mathcal{F}_m\mathcal{F}_n$ for any integers $m,n\geq 0$.*

In Definitions 5, a new matrix generalization of the Lucas numbers is introduced. It should be noted that the recurrence formula of this sequence depends on two integral parameters.

Definition 7 *For any integer numbers $s>0$ and $t\neq 0$ with $s^2+4t>0$; the (s,t)th Lucas matrix sequence, say $\{\mathcal{L}_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\mathcal{L}_{n+1}(s,t)=s\mathcal{L}_n(s,t)+t\mathcal{L}_{n-1}(s,t)\quad for\ n\geq 1, \tag{5}$$

with $\mathcal{L}_0(s,t)=\begin{pmatrix} s & 2 \\ 2t & -s \end{pmatrix}$, $\mathcal{L}_1(s,t)=\begin{pmatrix} s^2+2t & s \\ st & 2t \end{pmatrix}$

From the recurrence relation in (5), we obtain, for $n\geq 0$,

$$\mathcal{L}_n=\begin{pmatrix} L_{n+1} & L_n \\ tL_n & tL_{n-1} \end{pmatrix}.$$

In the sequel we will write simply $\mathcal{F}_n$ and $\mathcal{L}_n$ instead of $\mathcal{F}_n(s,t)$ and $\mathcal{L}_n(s,t)$, respectively.

2.1 Explicit formulas for the general term of the (s,t)-Fibonacci matrix and Fibonacci sequences

Binet's formulas are well known in the Fibonacci and Lucas numbers theory [8]. In our case, Binet's formula allows us to express the (s,t)-Fibonacci and Lucas matrix sequences and the (s,t)-Fibonacci and Lucas numbers in function of the roots α and β of the following characteristic equation, associated to the recurrence relation (1):

$$x^2 = sx + t. \tag{6}$$

Theorem 8 *For any integer* $n \geq 1$ *holds:*

$$\mathcal{F}_n = \begin{pmatrix} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & \frac{\alpha^n-\beta^n}{\alpha-\beta} \\ t\frac{\alpha^n-\beta^n}{\alpha-\beta} & t\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} \end{pmatrix}. \tag{7}$$

Proof. A *standart eigenvalue/eigenvector calculation tells us that* $\mathcal{F}_1$ has eigenvalues

$$\alpha = \frac{s+\sqrt{s^2+4t}}{2} \text{ and } \beta = \frac{s-\sqrt{s^2+4t}}{2},$$

with corresponding eigenvectors

$$u_1 = \begin{pmatrix} \frac{\alpha}{t} \\ 1 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} \frac{\beta}{t} \\ 1 \end{pmatrix}.$$

Note that, since $0 < s$, then

$$\beta < 0 < \alpha \text{ and } |\beta| < |\alpha|,$$

$$\alpha + \beta = s \text{ and } \alpha\beta = -t,$$

$$\alpha - \beta = \sqrt{s^2+4t}.$$

Thus, $\mathcal{F}_1$ is diagonalizable, and we can use the facts about diagonalizable matrices to obtain

$$\begin{aligned} \mathcal{F}_1 &= \begin{pmatrix} \frac{\alpha}{t} & \frac{\beta}{t} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \frac{\alpha}{t} & \frac{\beta}{t} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{t}{\alpha-\beta} \begin{pmatrix} \frac{\alpha}{t} & \frac{\beta}{t} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & -\frac{\beta}{t} \\ -1 & \frac{\alpha}{t} \end{pmatrix}. \end{aligned}$$

From Lemma 6 the result is obtained. ∎

From (7), we get the Binet's form for (s,t)- Fibonacci matrix sequence such that

$$\mathcal{F}_n = \left(\frac{\mathcal{F}_1 - \beta\mathcal{F}_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{\mathcal{F}_1 - \alpha\mathcal{F}_0}{\alpha-\beta}\right)\beta^n, \; n \geq 0, \tag{8}$$

or

$$\mathcal{F}_n = \mathcal{F}_1\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) + t\mathcal{F}_0\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right), \; n \geq 1. \tag{9}$$

2.1.1 The limit of the quotient of two consecutive (s,t)-Fibonacci terms and its application

An useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation.

Corollary 9 *The nth (s,t)-Fibonacci number is given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{10}$$

where α and β are the roots of the characteristic equation (6)*, and $\alpha > \beta$.*

Proof. From Lemma 5 and Theorem 8, since the term a_{12} is at the same time F_n we get the result. ■

Particular cases are:

- If $s = t = 1$, for the classic Fibonacci sequence, we obtain:

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

- If $s = 2$ and $t = 1$, for the classic Pell sequence we have:

$$p_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

- If $s = 1$ and $t = 2$, for the classic Jacobsthal sequence we get:

$$j_n = \frac{2^n - (-1)^n}{3}.$$

- If $s = 3$ and $t = -2$, for the Mersenne sequence we obtain:

$$m_n = 2^n - 1.$$

Proposition 10

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \alpha \tag{11}$$

Proof. By using Eq. (10)

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^{n-1} - \beta^{n-1}} = \alpha \lim_{n\to\infty} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^n \frac{\alpha}{\beta}},$$

and taking into account that $\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^n = 0$ since $|\beta| < \alpha$, Eq.(11) is obtained. ■

The metallic ratios or metallic means, following the nomenclature introduced in [16] and [17] are the positive solution of the characteristic equation $r^2 = sr + t$. In [6], it has been given various infinite sums connected to the theory of Fibonacci and Lucas numbers, and showed how they can be used to construct the dimensionalities of heterotic superstrings as well as the $\varepsilon^{(\infty)}$ theory [4, 5].

Our result here will be to show that the dimension of $\varepsilon^{(\infty)}$ can be easily derived from the relation between the $(s, 1)$-Fibonacci F_n and Lucas L_n sequences in the form of a continues fraction.

To show this, we need the following lemma.

Proposition 11 *The $(s, 1)$-Fibonacci F_n and Lucas L_n sequences satisfy the recurrence relation $F_{k+n} = L_n F_k - (-1)^n F_{k-n}$ for $n \geq 1$.*

Proof. To prove this proposition we will use the mathematical induction method. For $n = 1$, the proposition leads to $F_{k+1} = sF_k + F_{k-1}$. Hence, the proposition is true for $n = 1$. Now we have the show that if the statement holds when $n = N$, then it also holds when $n = N + 1$. This can be done as follows. ■

Assume that the statement is true for $n = N$, i.e.,

$$F_{k+N} = L_N F_k - (-1)^N F_{k-N}.$$

Then,

$$\begin{aligned} F_{k+N+1} &= sF_{k+N} + F_{k+N-1} \\ &= sL_N F_k - s(-1)^N F_{k-N} + L_{N-1} F_k - (-1)^{N-1} F_{k-N+1} \\ &= (sL_N + L_{N-1}) F_k + (-1)^{N+2} (F_{k-N+1} - sF_{k-N}) \\ &= L_{N+1} F_k - (-1)^{N+1} F_{k-(N+1)}. \end{aligned}$$

Now, we start from the relationship

$$F_{n+m} = L_m F_n - (-1)^m F_{n-m}.$$

Setting $n = km$,

$$\frac{F_{(k+1)m}}{F_{km}} = L_m - \frac{(-1)^m}{F_{km}/F_{(k-1)m}}.$$

Developing the last ratio in a continued fraction, then one finds

$$\frac{F_{(k+1)m}}{F_{km}} = L_m - \frac{(-1)^m}{L_m - \frac{(-1)^m}{L_m - \frac{(-1)^m}{L_m \cdots}}}.$$

Letting $k \to \infty$ then from (11) one finds that

$$\lim_{k \to \infty} \frac{F_{(k+1)m}}{F_{km}} = \alpha^m = \left(\frac{s + \sqrt{s^2 + 4}}{2} \right)^m.$$

Setting $s = 1$ and $m = 3$, for the classical Fibonacci and Lucas numbers, the well known expectation value of the dimension of $\varepsilon^{(\infty)}$ [6] is found namely

$$\begin{aligned}
\sim \quad \langle n \rangle &= \left(\frac{1+\sqrt{5}}{2}\right)^m \Bigg|_{m=3} \\
&= \phi^3 \\
&= 4 + \frac{1}{\phi^3} \\
&= 4 + \frac{1}{4 + \frac{1}{4 + \frac{1}{4+\dots}}} = 4.236067977.
\end{aligned}$$

2.2 Explicit formulas for the general term of the (s,t)-Lucas matrix and Lucas sequences

Lemma 12 *For $n \geq 0$ holds:*

$$\mathcal{L}_{n+1} = \mathcal{L}_1 \mathcal{F}_n. \tag{12}$$

Proof. We use the second principle of finite induction on n to prove this lemma. When $n = 0$, since $\mathcal{F}_0 = I$ the result is true. Let $n = 1$. Then the lemma yields

$$\mathcal{L}_2 = \mathcal{L}_1 \mathcal{F}_1,$$

which defines the matrix $\mathcal{L}_2$. Now assume that $\mathcal{L}_{n+1} = \mathcal{L}_1 \mathcal{F}_n$ for $n \leq N$. Then

$$\begin{aligned}
\mathcal{L}_1 \mathcal{F}_{N+1} &= \mathcal{L}_1 \mathcal{F}_N . \mathcal{F}_1 \\
&= \mathcal{L}_{N+1} \mathcal{F}_1 \\
&= \begin{pmatrix} L_{n+2} & L_{n+1} \\ tL_{n+1} & tL_n \end{pmatrix} \begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix} \\
&= \mathcal{L}_{N+2}.
\end{aligned}$$

Thus it is true for every nonnegative integer n. ∎

From (8), (9), and (12) we get the Binet's form for (s,t)- Lucas matrix sequence such that

$$\mathcal{L}_{n+1} = \left(\frac{\mathcal{L}_2 - \beta\mathcal{L}_1}{\alpha - \beta}\right)\alpha^n - \left(\frac{\mathcal{L}_2 - \alpha\mathcal{L}_1}{\alpha - \beta}\right)\beta^n, \; n \geq 0, \tag{13}$$

or

$$\mathcal{L}_{n+1} = \mathcal{L}_2 \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + t\mathcal{L}_1 \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right), \; n \geq 1. \tag{14}$$

Corollary 13 *The nth (s,t)-Lucas number is given by*

$$L_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation (6), *and $\alpha > \beta$.*

Proof. From (12), we write

$$\mathcal{L}_{n+1} = \mathcal{L}_1 \mathcal{F}_n, \text{ for } n \geq 0.$$

Since the terms a_{22} of both sides are equal, the equality

$$\begin{aligned} L_n &= sF_n + 2tF_{n-1} \\ &= s\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + 2t\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) \\ &= \frac{\left(\frac{\alpha s+2t}{\alpha}\right)\alpha^n - \left(\frac{\beta s+2t}{\beta}\right)\beta^n}{\alpha - \beta} \\ &= \frac{\left(\frac{\alpha^2+t}{\alpha}\right)\alpha^n - \left(\frac{\beta^2+t}{\beta}\right)\beta^n}{\alpha - \beta} \end{aligned}$$

is obtained. Finally, since

$$\alpha^2 + t = \alpha^2 - \alpha\beta \quad \text{and} \quad \beta^2 + t = \beta^2 - \alpha\beta,$$

the result

$$\begin{aligned} L_n &= \frac{\left(\frac{\alpha^2-\alpha\beta}{\alpha}\right)\alpha^n - \left(\frac{\beta^2-\alpha\beta}{\beta}\right)\beta^n}{\alpha - \beta} \\ &= \alpha^n + \beta^n \end{aligned}$$

is obtained. ■

Particular cases are:

- If $s = t = 1$, for the classic Lucas sequence, we obtain:

$$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

- If $s = 2$ and $t = 1$, for the classic Pell-Lucas sequence we have:

$$q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n.$$

- If $s = 1$ and $t = 2$, for the classic Jacobsthal-Lucas sequence we get:

$$\hat{\jmath}_n = 2^n + (-1)^n.$$

- If $s = 3$ and $t = -2$, for the Fermat sequence we obtain:

$$r_n = 2^n + 1.$$

2.3 The value for the sum of the first nth (s,t)-Fibonacci and Lucas matrix sequences with weights x^{-i}

Let x be a non-null real number. Next Theorem gives us the value for the sum of the first (s,t)th Fibonacci matrices with weights x^{-k}.

Proposition 14 *For each non-vanishing real number x :*

$$\sum_{k=0}^{n}\frac{\mathcal{F}_k}{x^k}=-\frac{1}{x^n\left(x^2-sx-t\right)}\left[x\mathcal{F}_{n+1}+t\mathcal{F}_n\right]+\frac{1}{x^2-sx-t}\left[x\mathcal{F}_1+\left(x^2-sx\right)\mathcal{F}_0\right]. \tag{15}$$

Proof. Since

$$\sum_{k=0}^{n}\alpha^k=\frac{1-\alpha^{n+1}}{1-\alpha},$$

(analogously for β) and $(x-\alpha)(x-\beta)=x^2-sx-t$, from (8) we have

$$\begin{aligned}
\sum_{k=0}^{n}\frac{\mathcal{F}_k}{x^k} &= \left(\frac{\mathcal{F}_1-\beta\mathcal{F}_0}{\alpha-\beta}\right)\sum_{k=0}^{n}\left(\frac{\alpha}{x}\right)^k-\left(\frac{\mathcal{F}_1-\alpha\mathcal{F}_0}{\alpha-\beta}\right)\sum_{k=0}^{n}\left(\frac{\beta}{x}\right)^k\\
&= \left(\frac{\mathcal{F}_1-\beta\mathcal{F}_0}{\alpha-\beta}\right)\left(\frac{1-\left(\frac{\alpha}{x}\right)^{n+1}}{1-\left(\frac{\alpha}{x}\right)}\right)-\left(\frac{\mathcal{F}_1-\alpha\mathcal{F}_0}{\alpha-\beta}\right)\left(\frac{1-\left(\frac{\beta}{x}\right)^{n+1}}{1-\left(\frac{\beta}{x}\right)}\right)\\
&= \frac{1}{x^n}\left\{\left(\frac{\mathcal{F}_1-\beta\mathcal{F}_0}{\alpha-\beta}\right)\left(\frac{x^{n+1}-\alpha^{n+1}}{x-\alpha}\right)\right.\\
&\quad\left.-\left(\frac{\mathcal{F}_1-\alpha\mathcal{F}_0}{\alpha-\beta}\right)\left(\frac{x^{n+1}-\beta^{n+1}}{x-\beta}\right)\right\}\\
&= \frac{1}{x^n\left(x^2-sx-t\right)}\left\{\left(\frac{\mathcal{F}_1-\beta\mathcal{F}_0}{\alpha-\beta}\right)\left(x^{n+1}-\alpha^{n+1}\right)(x-\beta)\right.\\
&\quad\left.-\left(\frac{\mathcal{F}_1-\alpha\mathcal{F}_0}{\alpha-\beta}\right)\left(x^{n+1}-\beta^{n+1}\right)(x-\alpha)\right\}.
\end{aligned}$$

By considering (8) and $\alpha\beta=-t$ and, after some algebra, we get the result. ■

Corollary 15 *For each non-vanishing real number x :*

$$\sum_{k=0}^{n}\frac{\mathcal{L}_{k+1}}{x^k}=-\frac{1}{x^n\left(x^2-sx-t\right)}\left[x\mathcal{L}_{n+2}+t\mathcal{L}_{n+1}\right]+\frac{1}{x^2-sx-t}\left[x\mathcal{L}_2+\left(x^2-sx\right)\mathcal{L}_1\right]. \tag{16}$$

Now, we will obtain a closed expression for $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\mathcal{F}_k}{x^k}$.

Corollary 16 *For each real number x, such that $x>\frac{s+\sqrt{s^2+4t}}{2}$:*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\mathcal{F}_k}{x^k}=\sum_{k=0}^{\infty}\frac{\mathcal{F}_k}{x^k}=\frac{1}{x^2-sx-t}\left[x\mathcal{F}_1+\left(x^2-sx\right)\mathcal{F}_0\right]. \tag{17}$$

Proof. The proof is based in the so-called Binet's formula for the nth (s,t)-Fibonacci matrix sequence. Since $\lim_{n\to\infty}\left(\frac{\alpha}{x}\right)^n = 0$ and $\lim_{n\to\infty}\left(\frac{\beta}{x}\right)^n = 0$, from (8) and (15) we obtain the result. ■

Particular cases are:

• If $s = t = 1$, for the classic Fibonacci sequence, we obtain:

$$\sum_{k=0}^{\infty} \frac{f_{k+1}}{x^k} = \frac{x^2}{x^2 - x - 1} \text{ and } \sum_{k=0}^{\infty} \frac{f_k}{x^k} = \frac{x}{x^2 - x - 1}.$$

• If $s = 2$ and $t = 1$, for the classic Pell sequence we have:

$$\sum_{k=0}^{\infty} \frac{p_{k+1}}{x^k} = \frac{x^2}{x^2 - 2x - 1} \text{ and } \sum_{k=0}^{\infty} \frac{p_k}{x^k} = \frac{x}{x^2 - 2x - 1}.$$

• If $s = 1$ and $t = 2$, for the classic Jacobsthal sequence we get:

$$\sum_{k=0}^{\infty} \frac{j_{k+1}}{x^k} = \frac{x^2}{x^2 - x - 2} \text{ and } \sum_{k=0}^{\infty} \frac{j_k}{x^k} = \frac{x}{x^2 - x - 2}.$$

• If $s = 3$ and $t = -2$, for the Mersenne sequence we obtain:

$$\sum_{k=0}^{\infty} \frac{m_{k+1}}{x^k} = \frac{x^2}{x^2 - 3x + 2} \text{ and } \sum_{k=0}^{\infty} \frac{m_k}{x^k} = \frac{x}{x^2 - 3x + 2}.$$

Corollary 17 *For each real number x, such that $x > \frac{s+\sqrt{s^2+4t}}{2}$:*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{\mathcal{L}_{k+1}}{x^k} = \sum_{k=0}^{\infty} \frac{\mathcal{L}_{k+1}}{x^k} = \frac{1}{x^2 - sx - t}\left[x\mathcal{L}_2 + \left(x^2 - sx\right)\mathcal{L}_1\right].$$

Particular cases are:

• If $s = t = 1$, for the classic Lucas sequence, we obtain:

$$\sum_{k=0}^{\infty} \frac{l_{k+2}}{x^k} = \frac{3x^2 + x}{x^2 - x - 1}, \sum_{k=0}^{\infty} \frac{l_{k+1}}{x^k} = \frac{x^2 + 2x}{x^2 - x - 1}, \text{ and } \sum_{k=0}^{\infty} \frac{l_k}{x^k} = \frac{x(2x - 1)}{x^2 - x - 1}.$$

• If $s = 2$ and $t = 1$, for the classic Pell-Lucas sequence we have:

$$\sum_{k=0}^{\infty} \frac{q_{k+2}}{x^k} = \frac{6x^2 + 2x}{x^2 - 2x - 1}, \sum_{k=0}^{\infty} \frac{q_{k+1}}{x^k} = \frac{2x(x + 1)}{x^2 - 2x - 1}, \text{ and } \sum_{k=0}^{\infty} \frac{q_k}{x^k} = \frac{2x(x - 1)}{x^2 - 2x - 1}.$$

• If $s = 1$ and $t = 2$, for the classic Jacobsthal-Lucas sequence we get:

$$\sum_{k=0}^{\infty} \frac{\hat{j}_{k+2}}{x^k} = \frac{5x^2 + 2x}{x^2 - x - 2}, \sum_{k=0}^{\infty} \frac{\hat{j}_{k+1}}{x^k} = \frac{x^2 + 4x}{x^2 - x - 2}, \text{ and } \sum_{k=0}^{\infty} \frac{\hat{j}_k}{x^k} = \frac{x(2x - 1)}{x^2 - x - 2}.$$

• If $s = 3$ and $t = -2$, for the Fermat sequence we obtain:

$$\sum_{k=0}^{\infty} \frac{r_{k+2}}{x^k} = \frac{5x^2 - 6x}{x^2 - 3x + 2}, \sum_{k=0}^{\infty} \frac{r_{k+1}}{x^k} = \frac{3x^2 - 4x}{x^2 - 3x + 2}, \text{ and } \sum_{k=0}^{\infty} \frac{r_k}{x^k} = \frac{x(2x - 3)}{x^2 - 3x + 2}.$$

3 More on the (s,t)-Fibonacci and Lucas sequences

In various recent publications, the connection between the heterotic string theory and the $\varepsilon^{(\infty)}$ space-time theory was discussed, and the role of the classical Fibonacci rations were briefly outlined [3, 4, 13].

Corollary 18 *Let* $x > \frac{s+\sqrt{s^2+4t}}{2}$ *and* $\Omega = \frac{1}{x^2-sx-t}$. *Then, for* (s,t)*-Fibonacci numbers,*

$$\sum_{m=3}^{\infty} \left(F_{m-2}/x^{m-1}\right) = \Omega,$$

$$\sum_{m=1}^{\infty} \left(F_m/x^m\right) = x\Omega,$$

$$\sum_{m=1}^{\infty} \left(F_m/x^{m-1}\right) = x^2\Omega.$$

Proof. Note that $\sum_{k=0}^{\infty} \frac{\mathcal{F}_k}{x^k} = \sum_{k=0}^{\infty} \frac{1}{x^k} \begin{pmatrix} F_{k+1} & F_k \\ tF_k & tF_{k-1} \end{pmatrix}$ and since, the terms a_{12} of both sides are equal, from (16) the formulas are obtained. ■

If $s = t = 1$ and $x = 2$, for the classic Fibonacci sequence, particular cases are:

• The one dimensionality of the strings could be interpreted as

$$\sum_{m=3}^{\infty} \left(f_{m-2}/2^{m-1}\right) = 1 \text{ [6]}.$$

• In [6] this could be interpreted the base 2 as the dimensionality $\left\langle d_c^{(2)} \right\rangle$ or the string world sheet dimensionality and find that

$$\sum_{m=1}^{\infty} \left(f_m/2^m\right) = 2.$$

• The intersection or union of two world sheets which define the topological dimension $n = 4$ is then given by

$$\sum_{m=1}^{\infty} \left(F_m/x^{m-1}\right) = 4 \text{ [6]}.$$

3.1 Some identities for the (s,t)-Fibonacci and Lucas sequences using the (s,t)-Fibonacci and Lucas matrix sequences

In this section, we shall prove some interesting properties of the (s,t)-Fibonacci and Lucas sequences which may be easily deduced from the product of some terms of (s,t)-Fibonacci and Lucas matrix sequences. The first property is called convolution product:

Proposition 19 *For $n, m \in \mathbb{Z}^+$,*

$$F_{n+m} = F_{n+1}F_m + tF_nF_{m-1}. \tag{18}$$

Proof. Considering the term a_{12} of the product $\mathcal{F}_n \times \mathcal{F}_m$, which is equal to the term a_{12} of the matrix $\mathcal{F}_{n+m}$ we get the result. ■

Eq.(18) may be particularized in many ways. For example, if $m = n$ we get

$$\begin{aligned} F_{2n} &= F_{n+1}F_n + tF_nF_{n-1} \\ &= (F_{n+1} + tF_{n-1})\,F_n \\ &= (F_{n+1} + tF_{n-1})\left(\frac{F_{n+1} - tF_{n-1}}{s}\right) \\ &= \frac{1}{s}\left(F_{n+1}^2 - t^2F_{n-1}^2\right). \end{aligned} \tag{19}$$

On the other hand, if $m = n + 1$ in Eq.(18) we obtain

$$F_{2n+1} = F_{n+1}^2 + tF_n^2. \tag{20}$$

By doing $m = 2n$ in Eq.(18) we have

$$\begin{aligned} F_{3n} &= F_{n+1}F_{2n} + tF_nF_{2n-1} \\ &= \frac{1}{s}F_{n+1}\left(F_{n+1}^2 - t^2F_{n-1}^2\right) + tF_n\left(F_n^2 + tF_{n-1}^2\right) \\ &= \frac{1}{s}F_{n+1}^3 - \frac{t^2}{s}F_{n+1}F_{n-1}^2 + tF_n^3 + t^2\left(\frac{F_{n+1} - tF_{n-1}}{s}\right)F_{n-1}^2 \\ &= \frac{1}{s}\left(F_{n+1}^3 + stF_n^3 - t^3F_{n-1}^2\right). \end{aligned} \tag{21}$$

Particular cases are:

• If $s = t = 1$, for the classic Fibonacci sequence, from $(18 - 21)$ we obtain:

$$f_{n+m} = f_{n+1}f_m + f_nf_{m-1} \text{ (Honsberger formula [18])},$$

$$f_{2n} = f_{n+1}^2 - f_{n-1}^2,$$

$$f_{2n+1} = f_{n+1}^2 + f_n^2,$$

$$f_{3n} = f_{n+1}^3 + f_n^3 - f_{n-1}^2,$$

respectively.

• If $s = 2$ and $t = 1$, for the classic Pell sequence, from $(18 - 21)$ we have:

$$p_{n+m} = p_{n+1}p_m + p_np_{m-1},$$

$$p_{2n} = \frac{1}{2}\left(p_{n+1}^2 - p_{n-1}^2\right),$$

$$p_{2n+1} = p_{n+1}^2 + p_n^2,$$

$$p_{3n} = \frac{1}{2}\left(p_{n+1}^3 + 2p_n^3 - p_{n-1}^2\right),$$

respectively.

• If $s = 1$ and $t = 2$, for the classic Jacobsthal sequence, from $(18 - 21)$ we get:

$$j_{n+m} = j_{n+1}j_m + 2j_n j_{m-1}$$

$$j_{2n} = j_{n+1}^2 - 4j_{n-1}^2,$$

$$j_{2n+1} = j_{n+1}^2 + 2F_n^2,$$

$$j_{3n} = j_{n+1}^3 + 2j_n^3 - 8j_{n-1}^2,$$

respectively.

• If $s = 3$ and $t = -2$, for the Mersenne sequence, from $(18 - 21)$ we obtain:

$$m_{n+k} = m_{n+1}m_k - 2m_n m_{k-1}$$

$$m_{2n} = \frac{1}{3}\left(m_{n+1}^2 - 4m_{n-1}^2\right),$$

$$m_{2n+1} = m_{n+1}^2 - 2m_n^2,$$

$$m_{3n} = \frac{1}{3}\left(m_{n+1}^3 - 6m_n^3 + 8m_{n-1}^2\right),$$

respectively.

Proposition 20 *For $n, m \in \mathbb{Z}^+$,*

$$L_{n+m+1} = L_{n+2}F_m + tL_{n+1}F_{m-1}. \tag{22}$$

Proof. Since $\mathcal{F}_{n+m} = \mathcal{F}_n\mathcal{F}_m$, $n, m \geq 0$, and $\mathcal{L}_{n+1} = \mathcal{L}_1\mathcal{F}_n$, $n \geq 0$, we get $\mathcal{L}_{n+m+1} = \mathcal{L}_{n+1}\mathcal{F}_m$, $n, m \geq 0$. Since the terms a_{12} of the both sides are equal, the formula is obtained. ■

Particular cases are:

$$l_{n+m+1} = l_{n+2}f_m + l_{n+1}f_{m-1},$$

$$q_{n+m+1} = q_{n+2}p_m + q_{n+1}p_{m-1},$$

$$\hat{j}_{n+m+1} = \hat{j}_{n+2}j_m + 2\hat{j}_{n+1}j_{m-1},$$

$$r_{n+k+1} = r_{n+2}m_k - 2r_{n+1}m_{k-1}.$$

Remark 21 *Notice that, if in the matrix products $\mathcal{F}_{n+m} = \mathcal{F}_n\mathcal{F}_m$ and $\mathcal{L}_{n+m+1} = \mathcal{L}_{n+1}\mathcal{F}_m$ we would have considered the term a_{11}, a_{21}, and a_{22} instead of the term a_{12}, we would have obtained the another equations for the (s,t)-Fibonacci and Lucas sequences.*

4 Conclusions

In this note, the new matrix generalization of the Fibonacci and Lucas sequences have been introduced and studied. Many of the properties are proved by simple matrix algebra. Using (s,t)-Fibonacci and Lucas matrix sequences, many mathematical formulas, which allows us to express in a compact form the (s,t)-Fibonacci and Lucas sequences, have been given.

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