On Generalizing Motzkin Numbers using k-Trees*

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Abstract

We use k-trees to generalize the sequence of Motzkin numbers and show that Baxter's generalization of Temperley-Lieb operators is a special case of our generalization of Motzkin numbers. We also obtain a recursive summation formula for the terms of 3-Motzkin numbers and investigate some asymptotic properties of the terms of k-Motzkin numbers.

Keywords: Motzkin Numbers; k-Trees; Temperley-Lieb Operators; Generating Functions.

1 Introduction

Th. Motzkin in his paper [9] entitled "Relations between hypersurfaces and combinatorial formula for partitions of polygon" looked at the number of divisions without crossings of nl points on a circle into n l-tuples and partitions of a convex n-gon by non-intersecting diagonals into polygons of l sides, where l is not restricted to a single value. In the former case, for $l = 1$ or 2 one obtains the sequence M_n whose first few entries are $1, 1, 2, 4, 9, 21, 51, \cdots$. Donaghey and Shapiro [5] refer to this sequence of numbers as Motzkin numbers and provided a survey of combinatorial settings enumerated by the Motzkin numbers along with algebraic relations between Motzkin and Catalan numbers. Problem 6.38 in Richard Stanley's book [11] lists thirteen combinatorial objects enumerated by the Motzkin numbers.

There has been a renewed interest in Motzkin numbers in recent years [1, 2, 3, 8, 15] and our attempt to provide a generalization of the Motzkin numbers in this paper is partly motivated by these developments. One of the combinatorial settings enumerated by the Motzkin numbers [4, 5, 11] is the rooted ordered trees in which every vertex has outdegree of at most 2. We use this interpretation of the Motzkin numbers to

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obtain k-Motzkin numbers ($k \geq 2$) using k-trees, which we introduced in [7] as a generalization of ordered trees. Our generalization seems to have many applications and an evidence of this is the ease with which we obtained Baxter's generalization of Temperley-Lieb operators [13] as its special case. Detailed discussion of this observation and our interpretation of Tempereley-Lieb operators as partitions of $[2n]$ with crossings is given in Section 3.

2 Generalized Motkin Numbers

A k-tree [7] is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or internal edge.
For example, if we use three 3-cycles, we get the following twelve 3-trees:

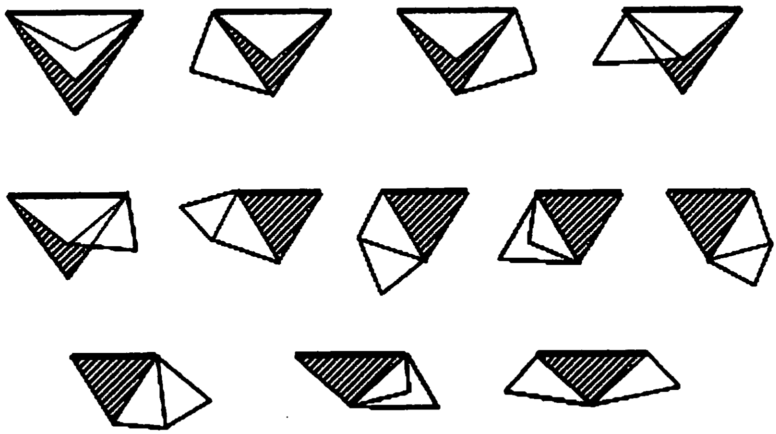

Figure 1: The twelve 3-trees consisting of three 3-cycles.

Note: k-trees generalize ordered trees (rooted plane trees) in the sense that ordered trees are 2-trees in which every edge between two vertices is drawn as a 2-cycle.

If K is any nonempty subset of $\{2, 3, 4, \cdots\}$, then a K-tree is obtained in a similar way using k-cycles with $k \in K$. We have shown in [7] that the number of K-trees consisting of n_i k_i-cycles, $i = 1, 2, \cdots, m$ is

$$C_n^K = \frac{1}{\sum n_i k_i + 1} \binom{\sum n_i k_i + 1}{n} \binom{n}{n_1, \ldots, n_m} \tag{1}$$

where $n = n_1 + n_2 + \ldots + n_m$.

If all the cycles used in the construction are of the same size, say k, (1) reduces to

$$C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

and these sequences of numbers are known as Fuss-Catalan numbers [6] or generalized Catalan numbers [7, 11]. Let M_n^k be the number of k-trees consisting of n k-cycles in which every edge has out degree of at most two. We refer to these sequences of numbers as k-Motzkin numbers or generalized Motzkin numbers.

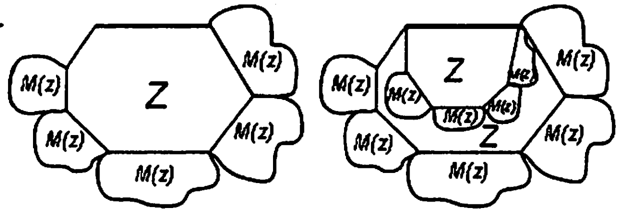

Figure 2: Recursive construction of k-Motzkin trees.

It is easy to see from Figure 2 that the generating function of k-Motzkin numbers

$$M(z) = \sum_{n=0}^{\infty} M_n^k z^n$$

satisfies a functional equation

$$M(z) = 1 + zM^{k-1}(z) + z^2 M^{2(k-1)}(z). \tag{2}$$

Writing

$$u(z) = zM^{k-1}(z)$$

we see that

$$M(z) = 1 + u(z) + u^2(z)$$

and

$$u(z) = z(1 + u + u^2)^{k-1}.$$

Now, letting

$$f(u) = 1 + u + u^2 \text{ and } \phi(u) = f(u)^{k-1}$$

and applying the Lagrange Inversion Formula (see [[14], p.167]), we obtain:

$$
\begin{aligned}
M_n^k = [z^n]\{f(u(z))\} &= \frac{1}{n}[u^{n-1}]\{f'(u)\phi^n(u)\} \\
&= \frac{1}{n}[u^{n-1}]\{(1+2u)((1+u+u^2)^{(k-1)n})\} \\
&= \frac{1}{n}[u^{n-1}]\{(1+u+u^2)^{(k-1)n}\} + \frac{2}{n}[u^{n-2}]\{(1+u+u^2)^{(k-1)n}\} \\
&= \frac{1}{n}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{(k-1)n}{i}\binom{(k-1)n-i}{n-1-2i} \\
&+ \frac{2}{n}\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{(k-1)n}{i}\binom{(k-1)n-i}{n-2-2i}.
\end{aligned}
$$

The following table shows generalized Motzkin numbers for various values of k and n.

k/n	0	1	2	3	4	5	6	7
2	1	1	2	4	9	21	51	127
3	1	1	3	11	46	207	979	4797
4	1	1	4	21	127	833	5763	41401
5	1	1	5	34	268	2299	20838	196326
6	1	1	6	50	485	5130	57391	667777

It is easy to see from the table that for $k = 2$ we obtain the regular Motzkin numbers. The only 3-tree among the 12 3-trees in Figure 1 which has an outdegree more than 2 is the one in which all the three 3-cycles are glued to the distinguished edge. Hence $M_3^3 = 11$ and this can also be confirmed very easily from the table.

After generating these sequences of numbers, we looked up in Sloan's Online Encyclopedia of Integer Sequences (OEIS) and discovered that 3-Motzkin numbers are the same as the sequence A006605 in OEIS [10]. The entries of the sequence A006605 enumerate the number of modes of connections of $2n$ points as proposed by R. Baxter in [13].

3 Temperley-Lieb Operators

Temperley and Lieb [12, 13] represent wave functions arising under their operator calculus by connecting $2n$ points (spin variables), $x_i, 1 \le i \le 2n$, in n disjoint pairs such that x_s, x_t are not connected for $i < s < j < t$ whenever $x_i, x_j, i < j$ are connected. [Connective Relation or Planarity Condition.]

If C_n is the number of such modes of connections, it is well known that

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \text{ the } n^{th} \text{ Catalan number.}$$

In addition to taking points in disjoint pairs as described in the Planarity Condition, R. Baxter proposed a generalization of the Temperley-Lieb operators in [13] by allowing taking points from $X_{2n} = \{1, 2, 3, \ldots, 2n\}$ in groups of four so that whenever i and k are connected and j and l are connected for $i < j < k < l$, then s and t are connected $(s < t)$ only if both s and t are in one of the four disjoint subsets S_1, S_2, S_3, and S_4 of X_{2n}, where $S_1 = \{x \in X_{2n} | x < i \text{ or } x > l\}$, $S_2 = \{x \in X_{2n} | i < x < j\}$, $S_3 = \{x \in X_{2n} | j < x < k\}$, and $S_4 = \{x \in X_{2n} | k < x < l\}$.

If b_n is the number of modes of connections of X_{2n} which are now permissible, then

$$b_1 = 1, b_2 = 3, b_3 = 11, \ldots,$$

and these agree with the first few entries of M_n^3. The 11 permissible modes of connection for $n = 3$ are shown below in Figure 3.

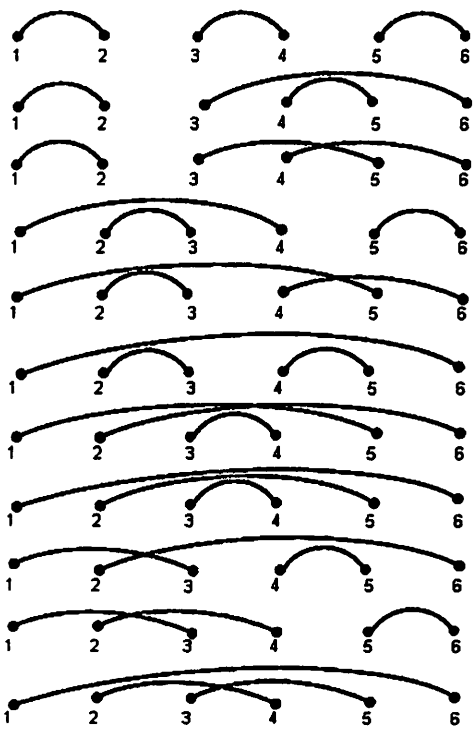

Figure 3: The 11 permissible modes of connections for $n = 3$.

Theorem 1. *There is a one-to-one correspondence between permissible modes of connections of X_{2n} and 3-Motzkin numbers.*

Proof. Given a 3-Motzkin tree with n 3-cycles, label the edges $1, 2, \cdots, 2n$ (excluding the distinguished egde) using postorder traversal as shown below in Figure 4.

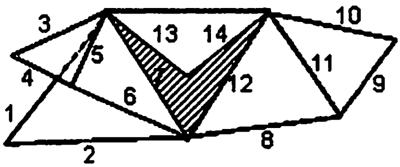

Figure 4: An example of a labeled 3-tree.

Then use the following two simple rules to carry out the mapping.
Rule-1: If two cycles share a common edge, then group the edges depending on whether they are to the left or right of the common edge and allow a crossing as shown below in Figure 5.

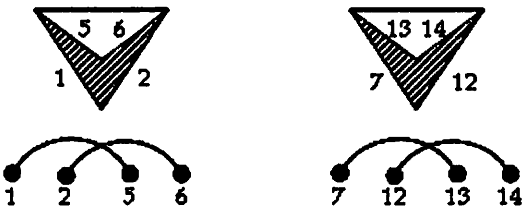

Figure 5: Mappings of two cycles with a common edge.

Rule-2: If a single 3-cycle is glued to an edge, then we connect the left and right edge labels by a chord as shown below in Figure 6.

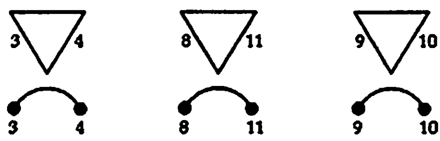

Figure 6: Mappings of single cycles

Putting together the results obtained using Rules 1 and 2, we obtain Baxter's generalization of Temperley-Lieb operators such as the one shown below in Figure 7 that corresponds to the 3-tree in Figure 4.

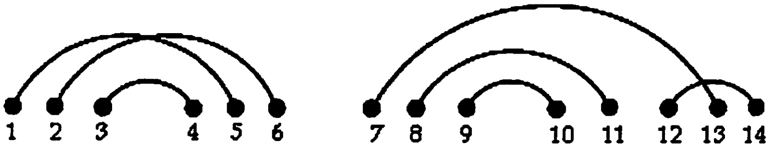

Figure 7: An example of Baxter's generalization of Temperley-Lieb operators.

To obtain an inverse map, start with a Baxter's generalization of Temperley-Lieb operators such as the one given in Figure 7 above, and determine all the corresponding subtrees consisting of a single cycle or two cycles as shown below in Figure 8.

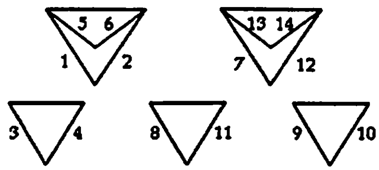

Figure 8: Subtrees consisting of a single cycle or two cycles.

Then, connect the distinguished edge of a subtree with highest label i to a subtree consisting of the edge labeled $i+1$. Repeat this process until you obtain a connected $k-tree$ containing all the n cycles as shown in Figure 9.

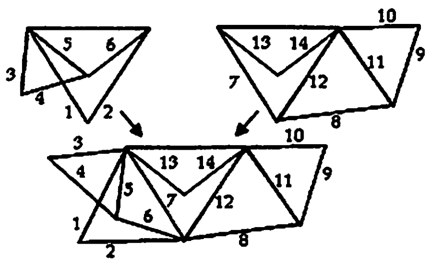

Figure 9: A labeled 3-tree corresponding to the example in Figure 7.

□

4 A Recursive Relation for 3-Motzkin Numbers

In this section, we obtain a recursive relation to compute 3-Motzkin numbers using Baxter's idea of allowing one crossing in a group of four points along with non-crossing pairs.

Given $2n$ points labeled $1, 2, 3, \cdots, 2n$, assume 1 is grouped or connected to j in a permissible mode of connection.

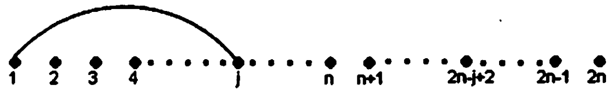

Figure 10: A case in which 1 is connected to j.

Let B_j be the number of permissible modes of connections in which 1 is connected to j.

By symmetry, we see that

$$B_j = B_{2n+2-j}.$$

Figure 11: A case in which 1 is connected to $2n - j + 2$.

There are two cases depending on whether j is even or odd.

Case I: If 1 is connected to j and j is even, then the chord divides the remaining $2n - 2$ points into two groups each with even number of points. Since only one crossing is allowed, no number between 2 and $j - 1$ crosses the chord connecting 1 and j.

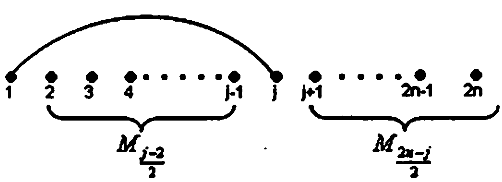

Figure 12: A case in which j is even.

Case II: If 1 is connected to j and j is odd, there will be an odd number of points between 1 and j. Hence, there is a point between 2 and $j-1$ that connects to a number between $j+1$ and $2n$ crossing the chord connecting 1 and j.

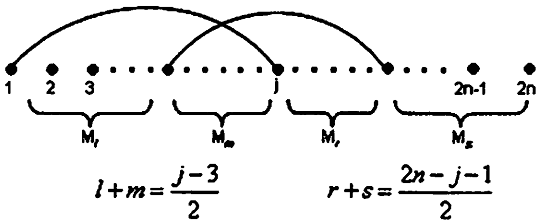

Figure 13: A case in which j is odd.

Thus, we obtain the following result:

Theorem 2. *If M_n^3 denotes the nth 3-Motzkin number and B_j is the number of permissible modes of connections in which 1 is connected to j, then*

$$M_n^3 = 2\left(\sum_{j=2}^{n} B_j\right) + B_{n+1}$$

where

$$B_j = \begin{cases} M_{\frac{j-2}{2}} M_{\frac{2n-j}{2}} & : \; j \text{ even} \\ & : \\ (\sum M_l M_m)(\sum M_r M_s) & : \; j \text{ odd} \end{cases}$$

For example,

$$\begin{aligned} M_3^3 &= 2(B_2 + B_3) + B_4 \\ &= 2(M_0 M_2 + (M_0 M_0)(M_0 M_1 + M_0 M_1)) + M_1 M_1 \\ &= 2(3 + 1(1+1)) + 1 \\ &= 11. \end{aligned}$$

5 Investigation of Asymptotic Properties

Wen-jin Woan [15] considered the family of paths with unit NE, E, or SE steps (Up, Level, or Down) without restriction and obtained the following three term recurrence relation for the regular Motzkin numbers:

$$(n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2}.$$

It is a routine algebraic manipulation to see from Woan's relation that

$$3 - \frac{6}{n+3} < \frac{M_{n+1}}{M_n} < 3 - \frac{4}{n+3}$$

and hence

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = 3.$$

This result can also be obtained by looking at the smallest singularity of the generating function of the Motzkin numbers [1]. Having generalized the Motzkin numbers using k-trees, it is natural to consider the limit

$$\lim_{n\to\infty} \frac{M^k_{n+1}}{M^k_n}$$

and expect the answer to be some function of k.

Here, we show that when $k = 3$, $\lim_{n\to\infty} \frac{M^3_{n+1}}{M^3_n}$ exists and is an irrational number $\frac{(70+26\sqrt{13})}{27} \approx 6.064604\cdots$.

We know from (2) that the generating function of M^3_n satisfies the functional relation

$$M(z) = 1 + zM^2(z) + z^2M^4(z).$$

Using Zeilberger's Maple Program (SCHUTZENBERGER) [16] that converts functional equations into recurrence relations we see that M^3_n satisfies a third order recurrence relation:

$$\begin{gathered}(180(2n+3)(2n+1)(n+1))M^3_n + \\ (2(2n+3)(139n^2+327n+128))M^3_{n+1} + \\ (-4903n - 275n^3 - 2000n^2 - 4066)M^3_{n+2} + \\ (3(3n+10)(n+3)(3n+11))M^3_{n+3} = 0.\end{gathered}$$

Using mathematical induction, one can show that $\frac{M^3_{n+1}}{M^3_n}$ is increasing and $5 < \frac{M^3_{n+1}}{M^3_n} < 7$ for $n > 8$. Hence, $\lim_{n\to\infty} \frac{M^3_{n+1}}{M^3_n}$ exists. Assuming that $\lim_{n\to\infty} \frac{M^3_{n+1}}{M^3_n}$ equals some number x, the above recurrence relation reduces to the cubic polynomial equation

$$27x^3 - 275x^2 + 556x + 720 = 0.$$

This polynomial equation has three roots, namely, $x = 5$, $x = \frac{70\pm 26\sqrt{13}}{27}$. Since $\lim_{n\to\infty} \frac{M^3_{n+1}}{M^3_n}$ is clearly positive and greater than 5, we conclude that

$$\lim_{n\to\infty} \frac{M^3_{n+1}}{M^3_n} = \frac{(70+26\sqrt{13})}{27}.$$

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