Realizability of Connected, Separable, p-Point, q-Line Graphs with Prescribed Minimum Degree and Line Connectivity

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Abstract

It is widely recognized that certain graph-theoretic extremal questions play a major role in the study of communication network vulnerability. These extremal problems are special cases of questions concerning the realizability of graph invariants. We define a CS(p, q, λ, δ) graph as a connected, separable graph having p points, q lines, line connectivity λ and minimum degree δ. In this notation, if the "CS" is omitted the graph is not necessarily connected and separable. An arbitrary quadruple of integers (a, b, c, d) is called CS(p, q, λ, δ) realizable if there is a CS(p, q, λ, δ) graph with $p = a$, $q = b$, $\lambda = c$ and $\delta = d$. Necessary and sufficient conditions for a quadruple to be CS(p, q, λ, δ) realizable are derived. In recent papers, the author gave necessary and sufficient conditions for (p, q, κ, Δ), (p, q, λ, Δ), (p, q, δ, Δ), (p, q, λ, δ) and (p, q, κ, δ) realizability, where Δ denotes the maximum degree for all points in a graph and κ denotes the point connectivity of a graph. Boesch and Suffel gave the solutions for (p, q, κ), (p, q, λ), (p, q, δ), (p, Δ, δ, λ) and (p, Δ, δ, κ) realizability in earlier manuscripts.

1. Introduction

In this work, we consider an undirected graph $G = (V, X)$ with a finite point set V and a set X whose elements, called lines, are two point subsets of V. The number of points $|V|$ is denoted by p, and the number of lines $|X|$ is called q or q(G). This paper uses the notation and terminology of Harary [14]; however a few basic concepts are now reproduced.

The line connectivity of a graph G (denoted by λ or λ(G)) is the minimum number of lines whose removal results in a disconnected graph. A graph is called trivial if it has just one point. The point connectivity (denoted by κ or κ(G)) is the minimum number of points whose removal results in a disconnected or trivial graph. The number of lines connected to a point v of G is the degree of that point, denoted $d_v(G)$ or d_v. The minimum degree is denoted by δ or δ(G) while the maximum degree is denoted by Δ. A regular graph has $\delta = \Delta$. The notation K_p denotes a p point graph with $\delta = p - 1$ and such a graph is called complete. A set of λ lines whose removal disconnects G is called a minimum line disconnecting set. A set of κ points whose removal disconnects G, or makes G trivial is called a minimum point disconnecting set. A point whose removal increases the number of components in G is called a cutpoint. A graph is called separable if it contains one or more cutpoints. The graph obtained from C_p (the cycle on p points) by adding lines between all pairs of points that are distance at least two but not greater than A apart is denoted by C_p^A.

It is widely recognized that certain graph-theoretic extremal questions play a major role in the study of communication network vulnerability [1-13,16]. Harary [15] found the maximum point connectivity among all graphs with a given number of points and a given number of lines. These extremal problems are special cases of questions concerning the realizability of graph invariants. We define a CS(p, q, λ, δ) graph as a connected, separable graph having p points, q lines, line connectivity λ and minimum degree δ. In this notation, if the "CS" is omitted the graph is not necessarily connected and separable. An arbitrary quadruple of integers (a, b, c, d) is called CS(p, q, λ, δ) realizable if there is a CS(p, q, λ, δ) graph with $p = a$, $q = b$, $\lambda = c$ and $\delta = d$. Necessary and sufficient conditions for a quadruple to be CS(p, q, λ, δ) realizable (or, more briefly, realizable) are derived. The author derived necessary and sufficient conditions for (p, q, κ, Δ), (p, q, λ, Δ), (p, q, δ, Δ), (p, q, λ, δ) and (p, q, κ, δ) realizability in recent papers [9-12]. In [1-3] the conditions for (p, q, κ), (p, q, λ), (p, q, δ), (p, Δ, δ, λ) and (p, Δ, δ, κ) realizability were given by Boesch and Suffel.

2. Preliminaries

We start by reviewing some known results that are pertinent to the realizability question.

Lemma 1 [2]: If a graph is not complete, and $p \geq 2$, then $p \geq 2\delta + 2 - \kappa$.

Lemma 2 [8]: If $\delta \geq \lfloor p/2 \rfloor$, then $\lambda = \delta$.

Lemma 3 [14]: If $2 \leq \delta \leq p - 1$, then there is a graph on p points with $q(G) = \lceil \frac{1}{2} p \delta \rceil$ and $\lambda = \delta = \kappa$. If we denote this graph (which is usually called the Harary graph) by H, then $H = C_p^{\lambda/2}$ if λ is even. However, if λ is odd then H is the graph formed by adding diameters to $C_p^{(\lambda - 1)/2}$.

Lemma 4 [12]: If $\lambda < \delta$, then $q \leq \lambda + \frac{1}{2}\delta(\delta + 1) + \frac{1}{2}(p - \delta - 1)(p - \delta - 2)$.

Lemma 5 [12]: If $q = \lceil \frac{1}{2} p \delta \rceil$ and λ is odd, then δ is odd.

Lemma 6 [11]: For all graphs,

$$q \leq \tfrac{1}{2}\delta(\delta + 1) + \tfrac{1}{2}(p - \delta - 1)(p - \delta - 2) + \kappa(p - \delta - 1).$$

Lemma 7 [11]: For all graphs,

$$q \geq \left\lceil \tfrac{1}{2} p \delta + \tfrac{1}{2} \max\left(0, (p - \delta - 1)(2\delta + 2 - p) - \kappa(\kappa - 1)\right)\right\rceil.$$

Of course, if a graph is connected and separable then $\kappa = 1$. Next we establish some new results.

Lemma 8: If $\kappa = 1$ for a graph $G \neq K_2$, then $q \geq \lambda + \lceil \frac{1}{2} \delta(p-1) \rceil$.

The result follows from the fact that G has a point of degree at least 2λ.

Lemma 9: There is no graph with $\kappa = 1$, λ odd, δ even and $q = \lambda + \frac{1}{2} \delta(p-1)$.

Proof: Let G be such a graph. Since $G \neq K_2$, there exists a point (denote it by v) whose removal disconnects G into at least two components. Thus the point set of $G - \{v\}$ may be partitioned into two subsets T and U such that no lines of $G - \{v\}$ join T and U, and T spans a component of $G - \{v\}$. As $q = \lambda + \frac{1}{2} \delta(p-1)$, we have $d_v(G) = 2\lambda$ and the other points of G have degree δ. Therefore we note $\sum_{i \in T} d_i(G - \{v\}) = |T| \delta - \lambda$, which is odd. Since T spans a component of $G - \{v\}$, this is impossible and the result is proven.

Lemma 10: There is no graph with $\kappa = 1$, λ and δ both odd, $p = 2\delta + 3$, $q = \lambda + \frac{1}{2} \delta(p-1)$ and $\lambda < \delta$.

To prove this lemma, proceed as in the proof of the previous lemma. (Note that here $\lambda < \delta$ implies $|T| = \delta + 1$.)

Lemma 11: If $\kappa = 1$ and $p = 2\delta + 2$, then $q \geq \lceil \lambda/2 \rceil + \frac{1}{2} p \delta$.

Proof: Let G be a graph with $\kappa = 1$ and $p = 2\delta + 2$. Let T, U and v be defined as they were in the proof of Lemma 9. We note that $|T| \geq \delta$ and $|U| \geq \delta$. Consequently, either $|T| = \delta$ or $|U| = \delta$. Suppose without loss of generality that $|T| = \delta$. Therefore, v must be adjacent to every point in T and $d_v(G) \geq \delta + \lambda$. Thus $q \geq \lceil \lambda/2 \rceil + \frac{1}{2} p \delta$.

3. The CS(p, q, λ, δ) realizability theorem

Theorem. A quadruple of non-negative integers (p, q, λ, δ) is realizable as a connected, separable graph if and only if exactly one of the following conditions holds:

(I) $1 = \delta = \lambda$, $p \geq 2$ and $p - 1 \leq q \leq 1 + \frac{1}{2}(p-1)(p-2)$.

(II) $2 \leq \delta$ and $1 \leq \lambda \leq \delta$.

(A) $p \geq 2\delta + 3$.

(a) $\max(\lambda + \lceil \frac{1}{2}\delta(p-1)\rceil, \lceil \frac{1}{2} p\delta \rceil) + 1 \leq q \leq$

$\frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta)(p-\delta-1)$; and $\lambda = \delta$ if

$q > \lambda + \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2)$.

(b) $q = \max(\lambda + \lceil \frac{1}{2}\delta(p-1)\rceil, \lceil \frac{1}{2} p\delta \rceil)$.

(1) λ is even.

(2) λ is odd, δ is odd; and if $\delta/2 \leq \lambda < \delta$, then $p > 2\delta + 3$.

(B) $2\delta + 1 \leq p \leq 2\delta + 2, \lceil \frac{1}{2} p\delta + \lambda/2 \rceil \leq q \leq$

$\frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta)(p-\delta-1)$; and $\lambda = \delta$ if either

i) $q > \lambda + \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p-\delta-1)(p-\delta-2)$, or ii) $p = 2\delta + 1$.

Proof: As noted previously, all connected, separable graphs have κ = 1. The conditions in (I) follow from the fact that $p - 1 \leq q$ in all connected graphs and other well known facts concerning graphs (e.g. $q \leq \delta + \frac{1}{2}(p-1)(p-2)$). The conditions in (II) are a result of Lemmas 1, 2 and 4 through 11 and basic graph theory (e.g. substitute κ = 1 into Lemma 6 to get the upper bound for q and substitute κ = 1 and $p = 2\delta + 1$ (and therefore λ = δ) into Lemma 7 to get the lower bound for q when $p = 2\delta + 1$).

We now provide constructions to prove sufficiency.

Case 1. Suppose that either i) $1 \leq \lambda \leq \delta$, $p \geq 2\delta + 2$, $\delta \geq 2$,

$$\lceil \tfrac{1}{2} p\delta \rceil + \lambda \leq q \leq \tfrac{1}{2}\delta(\delta+1) + \tfrac{1}{2}(p-\delta)(p-\delta-1);$$

and if $q > \lambda + \frac{1}{2}\delta(\delta + 1) + \frac{1}{2}(p - \delta - 1)(p - \delta - 2)$ then $\lambda = \delta$, or ii) $\delta \geq 2$, $p = 2\delta + 1$, $\lambda = \delta$ and

$$q = \lceil \tfrac{1}{2} p\,\delta + \lambda/2 \rceil = \tfrac{1}{2}\delta(\delta + 1) + \tfrac{1}{2}(p - \delta)(p - \delta - 1) = \delta(\delta + 1).$$

First we consider condition i) when $q \leq \lambda + \frac{1}{2}\delta(\delta + 1) + \frac{1}{2}(p - \delta - 1)(p - \delta - 2)$. Let H_1 denote the Harary graph on $p - \delta - 1$ points with $\lceil \frac{1}{2}\delta(p - \delta - 1)\rceil$ lines and $\lambda(H_1) = \delta(H_1) = \kappa(H_1) = \delta$. Take the union of H_1 and $K_{\delta+1}$ to form a single graph G. Denote a point in $K_{\delta+1}$ by b. Add λ lines to G, each incident to a point in H_1 and to b. Adding lines to H_1 until we have our desired number of lines completes our construction. It is easily verified that the resulting graph has the desired properties, including $\kappa = 1$.

If condition i) holds with $q > \lambda + \frac{1}{2}\delta(\delta + 1) + \frac{1}{2}(p - \delta - 1)(p - \delta - 2)$, or if condition ii) holds, a similar construction is used with $K_{p-\delta-1}$ in place of H_1. Here however, additional lines incident to b and to a point in $K_{p-\delta-1}$ are added to achieve the desired number of lines. (If condition ii holds, $p - \delta - 1 = \delta$ and b must be made adjacent to every point in $K_{p-\delta-1}$.) Since $\lambda = \delta$, our graph has the appropriate properties.

Case 2. Suppose that $1 \leq \lambda \leq \delta$, $p \geq 2\delta + 3$, $\delta \geq 2$,

$$\max\left(\lambda + \lceil \tfrac{1}{2}\delta(p - 1)\rceil, \lceil \tfrac{1}{2} p\,\delta\rceil\right) \leq q < \lceil \tfrac{1}{2} p\,\delta\rceil + \lambda,$$

and either λ is even or $q \neq \max\left(\lambda + \lceil \frac{1}{2}\delta(p - 1)\rceil, \lceil \frac{1}{2} p\,\delta\rceil\right)$.

Subcase 2-i. If $2\lambda \geq \delta$ we realize that $\max\left(\lambda + \lceil \frac{1}{2}\delta(p - 1)\rceil, \lceil \frac{1}{2} p\,\delta\rceil\right) = \lambda + \lceil \frac{1}{2}\delta(p - 1)\rceil$. Consider a set containing p points. Denote $\delta + 1$ of these points by A, denote $p - \delta - 2$ of the remaining points by B and denote the re- maining point by b. On B we construct the Harary graph on $p - \delta - 2$ points with $\lceil \frac{1}{2}\delta(p - \delta - 2)\rceil$ lines and line and point connectivities of δ. Next construct a complete graph on A. Note that B contains a path containing λ points. Now add λ lines joining b to λ points of B which lie on a path, and add λ lines joining b to points of A. If λ is even delete a matching of size $\lambda/2$ from the subgraph of $K_{\delta+1}$ spanned by the endpoints of the λ lines and a matching of size $\lambda/2$ from the path of B previously designated. Then all points of $A \cup B$ have degree δ (except possibly one point, which may have degree $\delta + 1$). At this point of the construction our graph contains $\lambda + \lceil \frac{1}{2}\delta(p - 1)\rceil$ lines. To finish our construction, we replace $q - \lambda - \lceil \frac{1}{2}\delta(p - 1)\rceil$ of the deleted lines. If λ is odd, $q \neq \lambda + \lceil \frac{1}{2}\delta(p - 1)\rceil$ and we proceed in a like manner, except there will be two or three points in $A \cup B$ with degree $\delta + 1$. Our graph has line connectivity λ because each deleted line can be replaced by a path containing b. The graph also has the other desired properties.

Subcase 2-ii. We now assume that $2\lambda < \delta$ and either δ is even or p is odd. Thus $\max(\lambda + \lceil \frac{1}{2} p(\delta - 1) \rceil, \lceil \frac{1}{2} p\delta \rceil) = \lceil \frac{1}{2} p\delta \rceil$. Let A denote a set of δ points, B denote a set of $p - \delta - 2$ points and c and b each denote a single point. On B, construct the same graph we did in subcase 2-i and on A construct the Harary graph on δ points with $\lceil \frac{1}{2} \delta(\delta - 2) \rceil$ lines and line and point connectivities of $\delta - 2$. Assume λ is even. Next, add λ lines joining b to B and delete lines in B as was done in subcase 2-i. Both b and c are each made adjacent to every point in A. In this subcase $\lambda < \delta$, thus λ of the points in A can be partitioned into non-adjacent pairs. We now add $\lambda/2$ lines (denoted by E) making each of these non-adjacent pairs adjacent. To finish the construction delete λ lines, each of which is incident to b, in such a way that our graph is regular with degree δ, or possibly has one point with degree $\delta + 1$. Since each pair of lines deleted in the last step can be replaced by a line in E, it is apparent that the graph has line connectivity λ, as well as the other appropriate properties. The case when λ is odd is handled as it was in subcase 2-i.

Subcase 2-iii. When $2\lambda < \delta$, p is even and δ is odd, a construction similar to the one in subcase 2-ii is used. We merely note the differences here. Now B contains $p - \delta - 3$ points (note $p \geq 2\delta + 4$) and A will have a Harary graph that is regular of degree $\delta - 3$ constructed on it. In subcase 2-ii, there were two points (b and c) that were made adjacent to every point in A, in subcase 2-iii three points (b and two others) will be made adjacent to every point in A. Aside from these differences, the construction used here is the same as the one in subcase 2-ii.

Case 3. Suppose that $1 \leq \lambda \leq \delta$, $p \geq 2\delta + 3$, $\delta \geq 2$; λ and δ are both odd and $q = \max(\lambda + \lceil \frac{1}{2} \delta(p - 1) \rceil, \lceil \frac{1}{2} p\,\delta \rceil)$.

Subcase 3-i. In this subcase, we consider when $\delta/2 \leq \lambda < \delta$ and if p is odd, then $p \geq 2\delta + 5$. If p is even, a construction similar to the one in subcase 2-i when λ was odd is used. The only difference here is that the Harary graph constructed on B has a point with degree $\delta + 1$, so it is now possible to build our final graph in such a way that it will have precisely one point with degree $\delta + 1$ in A, and every point in B will have degree δ. (To do this, pick a path containing $\lambda + 1 \leq \delta$ points in B with one of its endpoints having degree $\delta + 1$ and proceed as in subcase 2-i.) Note that one of the deleted lines in B cannot be replaced with a path containing b. However, since the Harary graph constructed on B had line connectivity δ and $\lambda < \delta$, the resulting graph will have the desired properties. If p is odd, we proceed in a similar manner except for the following differences. A will now contain $\delta + 2$ points and B will contain $p - \delta - 3$ points (note $p \geq 2\delta + 5$). A Harary graph with minimum degree δ is constructed on A. Note the Harary graphs built on A and B each have a point with degree $\delta + 1$. Here a path will also be chosen in A, much like the path chosen in B. Continuing as in the previous construction, every point in $A \cup B$ will have degree δ in the final graph and we are finished with subcase 3-i.

Subcase 3-ii. We now consider the possibility that $\delta/2 \leq \lambda = \delta$. Let H_2 denote the Harary graph on $p - \delta$ points with minimum degree δ. Take the union of H_2 and K_δ to form a single graph. Denote a point in H_2 by b. Making b adjacent to every point in K_δ completes the construction.

Subcase 3-iii. If $\delta/2 > \lambda$ we use a construction similar to the one used in subcase 2-ii. Note that here δ is odd and the Harary graph constructed on A has a point with degree $\delta + 1$. But since λ is odd, the points in A will all have degree δ in the final graph. If p is odd, the Harary graph constructed on B is regular of degree δ and one point in B, in the final graph, will have degree $\delta + 1$. However, if p is odd, we pick our path in B as was done in subcase 3-i and all points in B will eventually have degree δ. Since $\lambda < \delta$, the graph has the appropriate properties.

According to our main theorem, under the assumptions made in case 3, $\delta/2 \leq \lambda < \delta$ implies $p > 2\delta + 3$. Thus all possibilities within case 3 have been considered.

Case 4. Suppose that $1 \leq \lambda \leq \delta$, $p = 2\delta + 2$, $\delta \geq 2$ and $\frac{1}{2} p\delta + \lceil \lambda/2 \rceil \leq q < \frac{1}{2} p\delta + \lambda$. We start by taking the union of K_δ, $K_{\delta+1}$ and a single point (denoted by b) to form a single graph. Next make b adjacent to every point in K_δ. We also make b adjacent to λ points in $K_{\delta+1}$ and delete lines in $K_{\delta+1}$ in such a way that every point in $K_{\delta+1}$ has degree δ (if λ is even) or one point in $K_{\delta+1}$ has degree $\delta + 1$ (if λ is odd). Finally, replace the desired number of deleted lines. Since the deleted lines can be replaced by paths containing b, our graph has the necessary properties.

Case 5. Suppose that $1 = \delta = \lambda$, $p \geq 2$ and $p - 1 \leq q \leq 1 + \frac{1}{2}(p - 1)(p - 2)$. Take a path on p points and denoted one of the endpoints by b. The construction is completed by adding $q - (p - 1)$ lines, none of which are incident to b.

This completes our constructions and we now show sufficiency. If we assume $p \geq 2\delta + 3$ and $\delta \geq 2$ then cases 1, 2 and 3 show the sufficiency of the conditions of the theorem. Similarly, if we assume $p = 2\delta + 2$ and $\delta \geq 2$ then cases 1 and 4 are adequate. Case 1 shows the conditions of the theorem are sufficient if we also have $p = 2\delta + 1$ and $\delta \geq 2$. Case 5 proves sufficiency for $\delta < 2$ and our proof is completed.

Conclusion

The CS(p, q, λ, δ) realizability theorem in this paper solves several extremal problems for connected, separable graphs. If any three of the parameters p, q, λ and δ are given we can find the range of values for the unknown parameter. Let max (λ | p,q,δ) denote the maximum value of λ among all connected, separable (p, q, δ) graphs and min (λ | p,q,δ) denote the minimum value of λ among all connected, separable (p, q, δ) graphs. We now consider the problem of finding max (λ | p,q,δ). Note that Lemma 1 says that $p \geq 2\delta + 1$ (since $\kappa = 1$) only if the graph is not complete, thus $p = 2$ must be considered here. Solving for λ in the inequalities in Lemmas 8 and 11 gives us $\lambda \leq q - \lceil \frac{1}{2}\delta(p-1) \rceil$ and $\lambda \leq 2q - p\delta$, respectively. To complete the list of relevant upper bounds of λ, note that $\lambda \leq \delta$. As was stated in case 1 of the constructions, if $p = 2\delta + 1$ then $\lambda = \delta$ and $q = \delta(\delta + 1)$. Therefore, $p = 2\delta + 1$ implies $q - \lceil \frac{1}{2}\delta(p-1) \rceil = 2q - p\delta = \delta$. The resulting solution for max (λ | p,q,δ) is given below. The solution for min (λ | p,q,δ) is also given below without proof, which is straightforward.

$$\max(\lambda \mid p,q,\delta) = \begin{cases} 1, & \text{if } p = 2 \\ \min(\delta, q - \lceil \frac{1}{2}\delta(p-1) \rceil, 2q - p\delta), & \text{if } 2\delta + 1 \leq p \leq 2\delta + 2 \\ \min(\delta, q - \lceil \frac{1}{2}\delta(p-1) \rceil), & \text{if } p \geq 2\delta + 3. \end{cases}$$

$$\min(\lambda \mid p,q,\delta) = \begin{cases} \delta, & \text{if } p \leq 2\delta + 1 \text{ or } q > \delta + \frac{1}{2}\delta(\delta+1) + \frac{1}{2}(p - \delta - 1)(p - \delta - 2) \\ 1, & \text{otherwise.} \end{cases}$$

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