Ternary Strings with No Consecutive 1's

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Abstract

For $n \in \mathbf{N}$, let a_n count the number of ternary strings of length n that contain no consecutive $1's$. We find that $a_n = (\frac{1}{2} + \frac{\sqrt{3}}{3})(1+\sqrt{3})^n + (\frac{1}{2} - \frac{\sqrt{3}}{3})(1-\sqrt{3})^n$. For a given $n \geq 0$, we then determine the following for these a_n ternary strings: (1) the number of $0's$, $1's$, and $2's$; (2) the number of runs; (3) the number of rises, levels, and descents; and (4) the sum obtained when these strings are considered as base 3 integers. Following this, we consider the special case for those ternary strings (among the a_n strings we first considered) that are palindromes, and determine formulas comparable to those in (1) - (4) above for this special case.

Key Phrases: Ternary Strings, Fibonacci Numbers, Lucas Numbers, Runs, Palindromes, Base 3

1. Introduction - Determining a_n.

In the paper by R. A. Deininger [3], an application dealing with sewage treatment provided an instance where the alternate Fibonacci numbers arose. The coverage of this result, as given on pp. 30 - 31 of the text by T. Koshy [7], also demonstrated that the number of ternary strings of length n, which avoid the substring '21', is F_{2n+2}, where F_n denotes the n-th Fibonacci number. [The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.] In the paper by R. P. Grimaldi [5], these strings were studied with regard to the same kinds of properties mentioned above in the Abstract. Further, as mentioned in that paper, the results therein, with appropriate changes, could be derived for any substring 'xy', where x, $y \in \{0, 1, 2\}$ with $x \neq y$. However, for the substring 'xx', where $x \in \{0, 1, 2\}$, the derivations proved to be substantially different and this is what has led to the present paper.

As we mentioned in the Abstract, here we let a_n count the number of ternary strings (using the alphabet $\{0, 1, 2\}$) that contain no consecutive $1's$. Then

$$a_n = 2\,a_{n-1} + 2\,a_{n-2}\,, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 3,$$

where $2\,a_{n-1}$ accounts for the strings of length n that end in 0 or 2, and $2\,a_{n-2}$ for the strings of length n that end in 01 or 21.

Following the methods for solving recurrence relations, as given in Chapter 7 of R. Brualdi [2] or Chapter 10 of R. Grimaldi [4], the characteristic equation for the preceding recurrence relation is given as

$$x^2 - 2\,x - 2 = 0,$$

and from this we obtain the characteristic roots $1+\sqrt{3}$ and $1-\sqrt{3}$. Consequently, $a_n = c_1\,(1+\sqrt{3})^n + c_2\,(1-\sqrt{3})^n$. With $a_0 = 1$ and $a_1 = 3$, it follows that $c_1 = (\frac{1}{2} + \frac{\sqrt{3}}{3})$ and $c_2 = (\frac{1}{2} - \frac{\sqrt{3}}{3})$, so

$$a_n = (\frac{1}{2} + \frac{\sqrt{3}}{3})\,(1+\sqrt{3})^n + (\frac{1}{2} - \frac{\sqrt{3}}{3})\,(1-\sqrt{3})^n, \ n \geq 0.$$

[Note that the results obtained here and throughout the remainder of this paper generalize (with appropriate changes) for consecutive $0's$ and consecutive $2's$. The first 20 terms for this sequence are

1, 3, 8, 22, 60, 164, 448, 1224, 3344, 9136, 24960,
68192, 186304, 508992, 1390592, 3799168, 10379520,
28357376, 77473792, 211662336.

This appears as sequence A028859 on the website maintained by N. J. A. Sloane [8], where it is noted that a_n counts the number of ternary strings of length n, with no consecutive $0's$. In addition, a_n also counts the number of walks of length $n+1$ between two specific vertices u, v of the three-cycle C_3, where a loop has been added at each of u and v.]

2. Zeros, Ones, and Twos in the a_n Strings

For $n \geq 0$, let z_n count the number of $0's$ that occur among the a_n ternary strings of length n which do not contain any consecutive $1's$. Likewise, let w_n and t_n count the number of $1's$ and $2's$, respectively. We find, for instance, that

$$\begin{array}{llll} z_0 = 0 & z_1 = 1 & z_2 = 6 & z_3 = 25 \\ w_0 = 0 & w_1 = 1 & w_2 = 4 & w_3 = 16 \\ t_0 = 0 & t_1 = 1 & t_2 = 6 & t_3 = 25 \end{array}$$

For each of these a_n ternary strings, there is a corresponding string obtained by replacing each 0 with a 2 and each 2 with a 0. As a result, it follows that $z_n = t_n$, for all $n \geq 0$ (as suggested by the given display).

For $n \geq 1$ we find that

$$z_{n+1} = z_n + a_n + z_n + z_{n-1} + a_{n-1} + z_{n-1},$$

where (i) the first two summands account for the strings of length $n+1$ that end in 0 - here we consider all of the z_n $0's$ that occur among the a_n strings and then add a 0 to the end of each of these a_n strings; (ii) the third summand accounts for the strings of length $n+1$ that end in 2; (iii) the fourth and fifth summands deal with the $0's$ for the strings of length $n-1$ that end in 01; and (iv) the last summand accounts for the $0's$ that occur among the strings of length $n-1$ that end in 21. The solution to the recurrence relation

$$\begin{aligned} z_{n+1} &= 2\, z_n + 2\, z_{n-1} + a_n + a_{n-1} = 2\, z_n + 2\, z_{n-1} + \\ & (\frac{1}{2} + \frac{\sqrt{3}}{3})(1+\sqrt{3})^n + (\frac{1}{2} - \frac{\sqrt{3}}{3})(1-\sqrt{3})^n + \\ & (\frac{1}{2} + \frac{\sqrt{3}}{3})(1+\sqrt{3})^{n-1} + (\frac{1}{2} - \frac{\sqrt{3}}{3})(1-\sqrt{3})^{n-1}, \end{aligned}$$

where $n \geq 1$, has the form

$$z_n = c_1\, (1+\sqrt{3})^n + c_2\, (1-\sqrt{3})^n + A\, n\, (1+\sqrt{3})^n + B\, n\, (1-\sqrt{3})^n.$$

To determine that part of the particular solution that involves A, we substitute $A\, n\, (1+\sqrt{3})^n$ for z_n in the relation. This gives us

$$\begin{aligned} A\, (n+1)(1+\sqrt{3})^{n+1} &= 2\, A\, n(1+\sqrt{3})^n + 2\, A\, (n-1)(1+\sqrt{3})^{n-1} + \\ & (\frac{1}{2} + \frac{\sqrt{3}}{3})(1+\sqrt{3})^n + (\frac{1}{2} + \frac{\sqrt{3}}{3})(1+\sqrt{3})^{n-1}, \end{aligned}$$

from which it follows that $A = (5 + 3\,\sqrt{3})/24$. Similar calculations show that $B = (5 - 3\,\sqrt{3})/24$. Consequently, $z_n = c_1\, (1+\sqrt{3})^n + c_2\, (1-\sqrt{3})^n + [(5+3\,\sqrt{3})/24]\, n\, (1+\sqrt{3})^n + [(5-3\,\sqrt{3})/24]\, n\, (1-\sqrt{3})^n$. From the initial conditions $z_0 = 0$ and $z_1 = 1$, we find that $c_1 = -\sqrt{3}/36$ and $c_2 = \sqrt{3}/36$. So for $n \geq 0$,

$$\begin{aligned} z_n &= (\frac{-\sqrt{3}}{36})\, (1+\sqrt{3})^n + (\frac{\sqrt{3}}{36})\, (1-\sqrt{3})^n + (\frac{5+3\,\sqrt{3}}{24})\, n\, (1+\sqrt{3})^n \\ & + (\frac{5-3\,\sqrt{3}}{24})\, n\, (1-\sqrt{3})^n. \end{aligned}$$

The first 20 terms for this sequence are

0, 1, 6, 25, 92, 316, 1040, 3324, 10400, 32016, 97312, 292752, 873280, 2586560, 7614976, 22302656, 65025024, 188834048, 546455040, 1576409344.

With $t_n = z_n$, for $n \geq 0$, we have

$$\begin{aligned} w_n &= n\, a_n - 2\, z_n \\ &= \left(\frac{\sqrt{3}}{18}\right) [(1+\sqrt{3})^n - (1-\sqrt{3})^n] + \\ &\quad \left(\frac{n}{12}\right) [(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1}]. \end{aligned}$$

In this case the first 20 terms are

0, 1, 4, 16, 56, 188, 608, 1920, 5952, 18192, 54976, 164608, 489088, 1443776, 4238336, 12382208, 36022272, 104407296, 301618176, 868765696.

3. Runs in the a_n Strings

Given a ternary string, a *run* in such a string is a consecutive list of identical entries (that is, consecutive $0's$, $1's$, or $2's$) that are preceded and followed by different entries or no entries at all. For example, there are four runs in the ternary string 0001002222 – namely, 000, 1, 00, and 2222.

For $n \geq 1$, in this section we determine r_n, the total number of runs that occur among the a_n ternary strings of length n that contain no consecutive $1's$. To do so we let (i) $a_{n,i}$ count those ternary strings (among the a_n strings) that end in i, for $i = 0,\ 1,\ 2$ and (ii) $r_{n,i}$ count the number of runs among the r_n runs, where now the ternary string ends in i, for $i = 0,\ 1,\ 2$.

Initially we learn that

$r_{1,0} = 1$	$r_{2,0} = 5$	$r_{3,0} = 19$	$r_{4,0} = 68$
$r_{1,1} = 1$	$r_{2,1} = 4$	$r_{3,1} = 16$	$r_{4,1} = 54$
$r_{1,2} = 1$	$r_{2,2} = 5$	$r_{3,2} = 19$	$r_{4,2} = 68$
$r_1 = 3$	$r_2 = 14$	$r_3 = 54$	$r_4 = 190.$

We find that (i) for all $n \geq 1$, $r_{n,0} = r_{n,2}$; and, (ii) for all $n \geq 3$, $a_{n-1,0} = a_{n-2}$. Further,

$$r_{n+1} = 3\ r_{n,0} + 2\ a_{n,0} + 2\ r_{n,1} + 2\ a_{n,1} + 3\ r_{n,2} + 2\ a_{n,2}.$$

Here the first two summands account for the strings of length $n+1$ that end in 00, 01, or 02 - these each account for $r_{n,0}$ runs and, for each of the $a_{n,0}$ strings of length n that end in 0, we get two additional runs - one upon appending 1 and one upon appending 2. The third and fourth summands account for the runs that arise for the strings of length $n+1$ that end in 10 or 12 (but not 11). Lastly, the fifth and sixth summands deal with the runs for the strings of length $n+1$ that end in 20, 21, or 22.

This now leads us to the following:

$$\begin{aligned} r_{n+1} &= 3r_n + 2a_n - r_{n,1} \\ &= 3\ r_n + 2\ a_n - [r_n - 2\ r_{n,0}] = 2\ r_n + 2\ a_n + 2\ r_{n,0} \\ &= 2\ r_n + 2\ a_n + 2\ [r_{n-1} + a_{n-1,1} + a_{n-1,2}] \\ &= 2\ r_n + 2\ r_{n-1} + 2\ a_n + 2\ [a_{n-1} - a_{n-1,0}] \\ &= 2\ r_n + 2\ r_{n-1} + 2\ a_n + 2\ a_{n-1} - 2\ a_{n-2}. \end{aligned}$$

So $r_{n+1} = 2\ r_n + 2\ r_{n-1} + 2\ [(\frac{1}{2} + \frac{\sqrt{3}}{3})\ (1+\sqrt{3})^n + (\frac{1}{2} - \frac{\sqrt{3}}{3})\ (1-\sqrt{3})^n] +$ $2\ [(\frac{1}{2} + \frac{\sqrt{3}}{3})\ (1+\sqrt{3})^{n-1} + (\frac{1}{2} - \frac{\sqrt{3}}{3})\ (1-\sqrt{3})^{n-1}] -$ $2\ [(\frac{1}{2} + \frac{\sqrt{3}}{3})\ (1+\sqrt{3})^{n-2} + (\frac{1}{2} - \frac{\sqrt{3}}{3})\ (1-\sqrt{3})^{n-2}]$ and the solution for this recurrence relation has the form

$$r_n = c_1(1+\sqrt{3})^n + c_2\ (1-\sqrt{3})^n + A\ n\ (1+\sqrt{3})^n + B\ n\ (1-\sqrt{3})^n\ ,\ n \geq 1.$$

Upon substituting $A\ n\ (1+\sqrt{3})^n + B\ n\ (1-\sqrt{3})^n$ into the recurrence relation we find that $A = (4+3\sqrt{3})/12$ and that $B = (4-3\sqrt{3})/12$. Consequently, $r_n = c_1(1+\sqrt{3})^n + c_2\ (1-\sqrt{3})^n + [(4+3\sqrt{3})/12]\ n\ (1+\sqrt{3})^n + [(4-3\sqrt{3})/12]\ n\ (1-\sqrt{3})^n\ ,\ n \geq 1$. From $r_1 = 3$ and $r_2 = 14$, it follows that $c_1 = [(9+2\sqrt{3})/36]$ and $c_2 = [(9-2\sqrt{3})/36]$. Consequently, for $n \geq 1$,

$$\begin{aligned} r_n = &[(9+2\sqrt{3})/36]\ (1+\sqrt{3})^n + [(9-2\sqrt{3})/36]\ (1-\sqrt{3})^n + \\ &[(4+3\sqrt{3})/12]\ n\ (1+\sqrt{3})^n + [(4-3\sqrt{3})/12]\ n\ (1-\sqrt{3})^n. \end{aligned}$$

Here the first 20 terms of the sequence are

3, 14, 54, 190, 636, 2056, 6488, 20104, 61424, 185568, 555488, 1650144, 4870336, 14295168, 41757568, 121467008, 352025344, 1016860160, 2928674304, 8412626432.

4. Rises, Levels, and Descents in the a_n Strings

Within a ternary string of length n, where $n \geq 2$, a substring of the form 01, 02, or 12 is called a *rise*. Similarly, each of the substrings 10, 20, or 21, is called a *descent*. Lastly, substrings of the form 00, 11, or 22, are called *levels*, although here we will not find the level 11. We shall let ri_n, s_n, and l_n count the number of rises, descents, and levels, respectively, among the a_n ternary strings of length n that contain no consecutive $1's$. [These concepts are studied for compositions of $1's$ and $2's$ in the article by K. Alladi and V. E. Hoggatt, Jr. [1]. Rises, levels, and descents are also counted for binary strings with no odd runs of zeros in the article by R. Grimaldi and S. Heubach [6].]

We find here that,

$$\begin{array}{llll} ri_1 = 0 & ri_2 = 3 & ri_3 = 16 & ri_4 = 65 \\ s_1 = 0 & s_2 = 3 & s_3 = 16 & s_4 = 65 \\ l_1 = 0 & l_2 = 2 & l_3 = 12 & l_4 = 50. \end{array}$$

To determine ri_n, for $n \geq 1$, we define $ri_{n,j} =$ the number of rises that occur among the ternary strings of length n that have no consecutive $1's$ and end in j, where $j = 0$, 1, or 2. We see that for $n \geq 2$, $ri_{n-1,0} = ri_{n-2}$, and for $n \geq 3$, $ri_{n-1,2} = ri_{n-2}+a_{n-2}-a_{n-2,2} = ri_{n-2}+a_{n-2}-a_{n-3}$. In addition, for $n \geq 1$, $a_{n,2} = a_{n,0} = a_{n-1}$, while for $n \geq 2$, $a_{n,1} = 2\,a_{n-2}$, since here we append either 01 or 21 to each of the a_{n-2} strings of length $n-2$. Consequently, for $n \geq 3$,

$$\begin{aligned}
ri_n &= ri_{n,0} + ri_{n,1} + ri_{n,2} \\
&= ri_{n-1,0} + ri_{n-1,1} + ri_{n-1,2} + ri_{n-1,0} + a_{n-1,0} + ri_{n-1,2} + \\
&\quad ri_{n-1,0} + a_{n-1,0} + ri_{n-1,1} + a_{n-1,1} + ri_{n-1,2} \\
&= 3\, ri_{n-1,0} + 2\, ri_{n-1,1} + 3\, ri_{n-1,2} + 2\, a_{n-1,0} + a_{n-1,1} \\
&= 3\, ri_{n-1} - ri_{n-1,1} + 2\, a_{n-2} + 2\, a_{n-3} \\
&= 3\, ri_{n-1} - [ri_{n-1} - ri_{n-1,0} - ri_{n-1,2}] + 2\, a_{n-2} + 2\, a_{n-3} \\
&= 3\, ri_{n-1} - [ri_{n-1} - ri_{n-2} - ri_{n-2} - a_{n-2} + a_{n-3}] + \\
&\quad 2\, a_{n-2} + 2\, a_{n-3} \\
&= 2\, ri_{n-1} + 2\, ri_{n-2} + 3\, a_{n-2} + a_{n-3} \\
&= 2\, ri_{n-1} + 2\, ri_{n-2} + 3\, [(\frac{1}{2} + \frac{\sqrt{3}}{3})\, (1+\sqrt{3})^{n-2} + \\
&\quad (\frac{1}{2} - \frac{\sqrt{3}}{3})\, (1-\sqrt{3})^{n-2}] + [(\frac{1}{2} + \frac{\sqrt{3}}{3})\, (1+\sqrt{3})^{n-3} + \\
&\quad (\frac{1}{2} - \frac{\sqrt{3}}{3})\, (1-\sqrt{3})^{n-3}].
\end{aligned}$$

The solution to the preceding recurrence relation has the form

$$ri_n = c_1\, (1+\sqrt{3})^n + c_2\, (1-\sqrt{3})^n + A\, n\, (1+\sqrt{3})^n + B\, n\, (1-\sqrt{3})^n.$$

Substituting the particular part of the solution into the given recurrence relation we find that $A = \frac{1}{6} + \frac{\sqrt{3}}{8}$ and that $B = \frac{1}{6} - \frac{\sqrt{3}}{8}$. So $ri_n = c_1\, (1+\sqrt{3})^n + c_2\, (1-\sqrt{3})^n + (\frac{1}{6} + \frac{\sqrt{3}}{8})\, n\, (1+\sqrt{3})^n + (\frac{1}{6} - \frac{\sqrt{3}}{8})\, n\, (1-\sqrt{3})^n$. From $ri_1 = 0$ and $ri_2 = 3$, it follows that $c_1 = -\frac{1}{8} - \frac{5\sqrt{3}}{36}$ and that $c_2 = -\frac{1}{8} + \frac{5\sqrt{3}}{36}$. Consequently, for $n \geq 1$,

$$\begin{aligned}
ri_n &= (-\frac{1}{8} - \frac{5\sqrt{3}}{36})\, (1+\sqrt{3})^n + (-\frac{1}{8} + \frac{5\sqrt{3}}{36})\, (1-\sqrt{3})^n + \\
&\quad (\frac{1}{6} + \frac{\sqrt{3}}{8})\, n\, (1+\sqrt{3})^n + (\frac{1}{6} - \frac{\sqrt{3}}{8})\, n\, (1-\sqrt{3})^n.
\end{aligned}$$

The first 20 terms for this sequence are given as follows:

0, 3, 16, 65, 236, 804, 2632, 8380, 26144, 80304, 26144,
80304, 243648, 731920, 2180672, 6452288, 18979200,
55543744, 161833984, 469693184, 1358505984, 3917177088.

By symmetry $ri_n = s_n$, for $n \geq 1$. And since $ri_n + s_n + l_n = (n-1)\, a_n$, it follows that, for $n \geq 1$,

$$\begin{aligned}
l_n &= (n-1)\, a_n - 2\, ri_n \\
&= (n-1)\, [(\tfrac{1}{2} + \tfrac{\sqrt{3}}{3})\,(1+\sqrt{3})^n + (\tfrac{1}{2} - \tfrac{\sqrt{3}}{3})\,(1-\sqrt{3})^n] \\
&\quad -2\, [(-\tfrac{1}{8} - \tfrac{5\sqrt{3}}{36})\,(1+\sqrt{3})^n + (-\tfrac{1}{8} + \tfrac{5\sqrt{3}}{36})\,(1-\sqrt{3})^n + \\
&\quad (\tfrac{1}{6} + \tfrac{\sqrt{3}}{8})\, n\,(1+\sqrt{3})^n + (\tfrac{1}{6} - \tfrac{\sqrt{3}}{8})\, n\,(1-\sqrt{3})^n] \\
&= \left[n\left(\frac{2+\sqrt{3}}{12}\right) - \left(\frac{9+2\sqrt{3}}{36}\right)\right](1+\sqrt{3})^n + \\
&\quad \left[n\left(\frac{2-\sqrt{3}}{12}\right) - \left(\frac{9-2\sqrt{3}}{36}\right)\right](1-\sqrt{3})^n.
\end{aligned}$$

The first 20 terms for this sequence are

0, 2, 12, 50, 184, 632, 2080, 6648, 20800, 64032, 194624,
585504, 1746560, 5173120, 15229952, 44605312,
130050048, 377668096, 1092910080, 3152818688.

5. The Value of the Sum of the a_n Strings Considered as Base 3 Integers

For $n \geq 1$, let v_n equal the value of the sum of the a_n ternary strings of length n (that contain no consecutive $1's$) when considered as base 3 integers. Then $v_1 = 3$, $v_2 = 32$, $v_3 = 286$, and $v_4 = 2400$. In general we find that for $n \geq 3$,

$$v_n = 2 \cdot 3 \cdot v_{n-1} + 2\, a_{n-1} + 2 \cdot 9 \cdot v_{n-2} + 8\, a_{n-2},$$

where (i) the factor 2 in the first summand is due to the fact that there are two cases depending on whether we append 0 or 2 on the right; (ii) the factor 3 in the first summand accounts for shifting the strings of length $n-1$ one position to the left – thus multiplying the value of such a string by 3; (iii) the summand $2\, a_{n-1}$ is needed for we are adding a_{n-1} $2's$ (and a_{n-1} $0's$); (iv) the factor 2 in the third summand is due to the fact that we once again have two cases to consider – depending on whether we append 01 or 21 on the right; (v) the factor 9 in the third summand accounts for shifting the strings of length $n-2$ two positions to the left – thus multiplying the value of such a string by 9; and (vi) the factor 8 in the last summand is needed because for each of the a_{n-2} strings we have added either $(01)_3$ or $(21)_3$, and this, in base 10, is $1+7=8$.

The above recurrence relation can be written as

$$\begin{aligned} v_n &= 6\, v_{n-1} + 18\, v_{n-2} + 2\, a_{n-1} + 8\, a_{n-2} \\ &= 6v_{n-1} + 18v_{n-2} + \\ &\quad 2[(\frac{1}{2}+\frac{\sqrt{3}}{3})(1+\sqrt{3})^{n-1} + (\frac{1}{2}-\frac{\sqrt{3}}{3})(1-\sqrt{3})^{n-1}] + \\ &\quad 8\,[(\frac{1}{2}+\frac{\sqrt{3}}{3})\,(1+\sqrt{3})^{n-2} + (\frac{1}{2}-\frac{\sqrt{3}}{3})\,(1-\sqrt{3})^{n-2}], \end{aligned}$$

where the characteristic roots for the homogeneous part of the solution are the roots of the equation $x^2 - 6\,x - 18 = 0$ – namely, $3 \pm 3\sqrt{3}$. Consequently, the solution has the form

$$v_n = c_1\,(3+3\sqrt{3})^n + c_2\,(3-3\sqrt{3})^n + A\,(1+\sqrt{3})^n + B\,(1-\sqrt{3})^n.$$

Substituting $A\,(1+\sqrt{3})^n + B\,(1-\sqrt{3})^n$ into the above recurrence relation for v_n we find that $A = -\frac{1}{4} - \frac{\sqrt{3}}{6}$ and $B = -\frac{1}{4} + \frac{\sqrt{3}}{6}$. Then from $v_1 = 3$ and $v_2 = 32$ we learn that $c_1 = \frac{1}{4} + \frac{\sqrt{3}}{6}$ and $c_2 = \frac{1}{4} - \frac{\sqrt{3}}{6}$. Consequently, for $n \geq 1$,

$$\begin{aligned} v_n &= (\frac{1}{4}+\frac{\sqrt{3}}{6})\,(3+3\sqrt{3})^n + (\frac{1}{4}-\frac{\sqrt{3}}{6})\,(3-3\sqrt{3})^n + \\ &\quad (-\frac{1}{4}-\frac{\sqrt{3}}{6})\,(1+\sqrt{3})^n + (-\frac{1}{4}+\frac{\sqrt{3}}{6})\,(1-\sqrt{3})^n \\ &= (3^n-1)\,[(\frac{1}{4}+\frac{\sqrt{3}}{6})\,(1+\sqrt{3})^n + (\frac{1}{4}-\frac{\sqrt{3}}{6})\,(1-\sqrt{3})^n]. \end{aligned}$$

The first 20 terms for this sequence are

3, 32, 286, 2400, 19844, 163072, 1337832, 10968320, 89907376, 736919040, 6039970016, 49504698880, 405748571712, 3325578518528, 27256952255104, 223402145587200, 1831038065267456, 15007467151925248, 123003488468172288, 1008155340586803200.

6. Palindromes among the a_n Strings

In this section we determine, for $n \geq 0$, the number p_n of palindromes that occur among the a_n ternary strings of length n that contain no consecutive $1's$. We recall that a palindrome occurs when the string and its reversal are one and the same. Further, for these p_n palindromes, we also determine the formulas for $\widetilde{z_n}$, $\widetilde{w_n}$, $\widetilde{t_n}$, $\widetilde{r_n}$, $\widetilde{ri_n}$, $\widetilde{s_n}$, $\widetilde{l_n}$, and $\widetilde{v_n}$ in Sections 7-10. These correspond to the results we obtained for a_n in Sections 2-5.

We find that $p_1 = 3$, $p_2 = 2$, $p_3 = 8$, $p_4 = 6$, $p_5 = 22$, and $p_6 = 16$. Defining $p_0 = 1$, we learn that for $n \geq 4$,

$$p_n = 2\, p_{n-2} + 2\, p_{n-4}.$$

Here the multiplier 2 for the first summand accounts for the two cases where we place 0 or 2 at the start of a palindrome of length $n-2$ and then append the same symbol at the end. The multiplier 2 for the second summand takes into account the palindromes of length $n-4$ where we place 10 (or 12) at the start of the palindrome and then append 01 (or 21) at the end. There are two cases to consider.

The even case: $p_{2n} = 2\, p_{2n-2} + 2\, p_{2n-4}$, $n \geq 2$. If we let $g_n = p_{2n}$, we have $g_n = 2\, g_{n-1} + 2\, g_{n-2}$, $n \geq 2$, $g_0 = p_0 = 1$, $g_1 = p_2 = 2$. The solution for this recurrence relation gives us $g_n = c_1\,(1+\sqrt{3})^n + c_2\,(1-\sqrt{3})^n$, $n \geq 0$. From the initial conditions we learn that $c_1 = (\frac{1}{2} + \frac{\sqrt{3}}{6})$ and that $c_2 = (\frac{1}{2} - \frac{\sqrt{3}}{6})$. So $g_n = (\frac{1}{2} + \frac{\sqrt{3}}{6})\,(1+\sqrt{3})^n + (\frac{1}{2} - \frac{\sqrt{3}}{6})\,(1-\sqrt{3})^n$, $n \geq 0$, and for n even we have

$$p_n = (\frac{1}{2} + \frac{\sqrt{3}}{6})\,(1+\sqrt{3})^{(n/2)} + (\frac{1}{2} - \frac{\sqrt{3}}{6})\,(1-\sqrt{3})^{(n/2)},\ n \geq 0.$$

Alternately, these palindromes of length $2n$ can be constructed (i) by taking one of the $a_{n,0}$ strings of length n (ending in 0) and appending its reverse on the right; or (ii) by taking one of the $a_{n,2}$ strings of length n (ending in 2) and appending its reverse on the right. This gives us $p_{2n} = a_{n,0} + a_{n,2} = 2\, a_{n,0} = 2\, a_{n-1} = 2\,[(\frac{1}{2}+\frac{\sqrt{3}}{3})\,(1+\sqrt{3})^{n-1} + (\frac{1}{2}-\frac{\sqrt{3}}{3})\,(1-\sqrt{3})^{n-1}]$. Since $2\,(\frac{1}{2}+\frac{\sqrt{3}}{3})\,(1+\sqrt{3})^{-1} = (1+\frac{2\sqrt{3}}{3})\,(1+\sqrt{3})^{-1}\,(\frac{1-\sqrt{3}}{1-\sqrt{3}}) = (\frac{1}{2}+\frac{\sqrt{3}}{6})$ and $2\,(\frac{1}{2}-\frac{\sqrt{3}}{3})\,(1-\sqrt{3})^{-1} = (1-\frac{2\sqrt{3}}{3})\,(1-\sqrt{3})^{-1}\,(\frac{1+\sqrt{3}}{1+\sqrt{3}}) = (\frac{1}{2}-\frac{\sqrt{3}}{6})$, we again obtain our original result.

The first 20 terms for this sequence are

1, 2, 6, 16, 44, 120, 328, 896, 2448, 6688, 18272, 49920, 136384, 372608, 1017984, 2781184, 7598336, 20759040, 56714752, 154947584.

This appears as sequence A002605 on the website maintained by N. J. A. Sloane [8],

The odd case: $p_{2n+1} = 2\ p_{2n-1} + 2\ p_{2n-3},\ n \geq 2$. Now we let $h_n = p_{2n+1}$ with $h_0 = p_1 = 3$ and $h_1 = p_3 = 8$. The solution to the resulting recurrence relation gives us $h_n = d_1\ (1+\sqrt{3})^n + d_2\ (1-\sqrt{3})^n,\ n \geq 0$. From the initial conditions it follows that $d_1 = (\frac{3}{2} + \frac{5\sqrt{3}}{6})$ and that $d_2 = (\frac{3}{2} - \frac{5\sqrt{3}}{6})$. So $h_n = (\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^n + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^n,\ n \geq 0$, and for n odd we have

$$p_n = (\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{(n-1)/2},\ n \geq 1.$$

This result can also be obtained as follows. Consider the central, that is the $n+1$-st, term of a palindrome of length $2n+1$. If this term is a 0 (or 2) we place one of the a_n strings of length n on the left of the 0 (or 2) and the reverse of the string on the right of the 0 (or 2). This gives us the $2\ a_n$ palindromes of length $2n+1$ with 0 (or 2) at the center. If, however, the central term is 1 we place one of the $a_{n,0}$ strings of length n ending in 0 (or one of the $a_{n,2}$ strings of length n ending in 2) on the left of this central 1 and the reverse of the string on the right. So the number of palindromes of length $2n+1$ with 1 at the center is $a_{n,0} + a_{n,2} = 2\ a_{n,0} = 2\ a_{n-1}$. Consequently, $p_{2n+1} = 2\,a_n + 2\,a_{n-1} = a_{n+1} = (\frac{1}{2} + \frac{\sqrt{3}}{3})\ (1+\sqrt{3})^{n+1} + (\frac{1}{2} - \frac{\sqrt{3}}{3})\ (1-\sqrt{3})^{n+1}$. Since $(\frac{1}{2} + \frac{\sqrt{3}}{3})\ (1+\sqrt{3}) = (\frac{3}{2} + \frac{5\sqrt{3}}{6})$ and $(\frac{1}{2} - \frac{\sqrt{3}}{3})\ (1-\sqrt{3}) = (\frac{3}{2} - \frac{5\sqrt{3}}{6})$, this provides us with the same result as above.

The first 20 terms for this sequence are

3, 8, 22, 60, 164, 448, 1224, 3344, 9136, 24960, 68192, 186304, 508992, 1390592, 3799168, 10379520, 28357376, 77473792, 211662336, 578272256.

This appears as sequence A028859 on the website maintained by N. J. A. Sloane [8]. It is the first sequence we encountered with the first term - namely, 1 - deleted.

7. Zeros, Ones, and Twos in the p_n Palindromes

Now let us consider $\widetilde{z_n}$, $\widetilde{w_n}$, and $\tilde{t}_n$, the number of $0's$, $1's$, and $2's$, respectively, that occur among the p_n palindromes. Here we see that

$$\begin{array}{llllll} \widetilde{z_1}=1 & \widetilde{z_2}=2 & \widetilde{z_3}=9 & \widetilde{z_4}=10 & \widetilde{z_5}=42 & \widetilde{z_6}=40 \\ \widetilde{w_1}=1 & \widetilde{w_2}=0 & \widetilde{w_3}=6 & \widetilde{w_4}=\ 4 & \widetilde{w_5}=26 & \widetilde{w_6}=16 \\ \tilde{t}_1=1 & \tilde{t}_2=2 & \tilde{t}_3=9 & \tilde{t}_4=10 & \tilde{t}_5=42 & \tilde{t}_6=40. \end{array}$$

For $n \geq 5$ we find that $\widetilde{z_n} = 2\,\widetilde{z_{n-2}} + 2\,p_{n-2} + 2\,\widetilde{z_{n-4}} + 2\,p_{n-4}$ where (1) the 2 for the first summand accounts for when we start with a palindrome p of length $n-2$ and then place 0 (or 2) at the start and end of p; (2) the 2 for the second summand is for the 0 placed at the start and end of each palindrome when we go from the case for $n-2$ to that of n; (3) the 2 for the third summand accounts for when we start with a palindrome p' of length $n-4$ and then place 10 (or 12) at the start of p' and append 01 (or 21) at the end; and (4) the 2 for the fourth summand is for when we place 10 at the start and 01 at the end of each palindrome as we go from the case for $n-4$ to that of n. As in the previous section there are two cases.

The even case: $\widetilde{z_{2n}} = 2\,\widetilde{z_{2n-2}} + 2\,p_{2n-2} + 2\,\widetilde{z_{2n-4}} + 2\,p_{2n-4}$, $n \geq 2$, if we define $\widetilde{z_0} = 0$. If we let $g_n = \widetilde{z_{2n}}$, for $n \geq 2$, then we find that

$$\begin{aligned} g_n &= 2\,g_{n-1} + 2\,p_{2n-2} + 2\,g_{n-2} + 2\,p_{2n-4} \\ &= 2\,g_{n-1} + 2\,g_{n-2} \\ &\quad +2\,[(\frac{1}{2}+\frac{\sqrt{3}}{6})\,(1+\sqrt{3})^{n-1} + (\frac{1}{2}-\frac{\sqrt{3}}{6})\,(1-\sqrt{3})^{n-1}] \\ &\quad +2\,[(\frac{1}{2}+\frac{\sqrt{3}}{6})\,(1+\sqrt{3})^{n-2} + (\frac{1}{2}-\frac{\sqrt{3}}{6})\,(1-\sqrt{3})^{n-2}]. \end{aligned}$$

Here the solution has the form $g_n = c_1\,(1+\sqrt{3})^n + c_2\,(1-\sqrt{3})^n + A\,n\,(1+\sqrt{3})^n + B\,n\,(1-\sqrt{3})^n$. Upon substituting the particular part of the solution into the given recurrence relation we learn that $A = (\frac{1}{3}+\frac{\sqrt{3}}{6})$ and that $B = (\frac{1}{3}-\frac{\sqrt{3}}{6})$. From $g_0 = \widetilde{z_0} = 0$ and $g_1 = \widetilde{z_2} = 2$ it follows that $c_1 = \frac{\sqrt{3}}{18}$ and $c_2 = -\frac{\sqrt{3}}{18}$. So for $n \geq 0$, $\widetilde{z_{2n}} = g_n = (\frac{\sqrt{3}}{18})\,(1+\sqrt{3})^n + (-\frac{\sqrt{3}}{18})\,(1-\sqrt{3})^n + (\frac{1}{3}+\frac{\sqrt{3}}{6})\ \ n\,(1+\sqrt{3})^n + (\frac{1}{3}-\frac{\sqrt{3}}{6})\ \ n\,(1-\sqrt{3})^n$. Consequently, for n even,

$$\begin{aligned} \widetilde{z_n} &= (\frac{\sqrt{3}}{18})\,(1+\sqrt{3})^{(n/2)} + (-\frac{\sqrt{3}}{18})\,(1-\sqrt{3})^{(n/2)} + \\ &\quad (\frac{1}{3}+\frac{\sqrt{3}}{6})\,(n/2)\,(1+\sqrt{3})^{(n/2)} + (\frac{1}{3}-\frac{\sqrt{3}}{6})\,(n/2)\,(1-\sqrt{3})^{(n/2)}. \end{aligned}$$

The first 20 terms for this sequence are

0, 2, 10, 40, 144, 488, 1592, 5056, 15744, 48288, 146336, 439168, 1307392, 3865728, 11364224, 33241088, 96808960, 280859136, 812050944, 2340767744.

The odd case: $\widetilde{z_{2n+1}} = 2\ \widetilde{z_{2n-1}} + 2\ p_{2n-1} + 2\ \widetilde{z_{2n-3}} + 2\ p_{2n-3},\ n \geq 2$. Here we let $h_n = \widetilde{z_{2n+1}}$, for $n \geq 0$, and find that $h_n = 2\ h_{n-1} + 2\ h_{n-2} + 2\ p_{2n-1} + 2\ p_{2n-3}$. Working, as in the even case, with $h_0 = \widetilde{z_1} = 1$ and $h_1 = \widetilde{z_3} = 9$, we learn that for $n \geq 0$,

$$\begin{aligned} h_n &= \widetilde{z_{2n+1}} = (\frac{1}{2} + \frac{5\sqrt{3}}{18})\ (1+\sqrt{3})^n + (\frac{1}{2} - \frac{5\sqrt{3}}{18})\ (1-\sqrt{3})^n \\ &+ (\frac{7}{6} + \frac{2\sqrt{3}}{3})\ n\ (1+\sqrt{3})^n + (\frac{7}{6} - \frac{2\sqrt{3}}{3})\ n\ (1-\sqrt{3})^n. \end{aligned}$$

Consequently, for n odd,

$$\begin{aligned} \widetilde{z_n} &= (\frac{1}{2} + \frac{5\sqrt{3}}{18})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{1}{2} - \frac{5\sqrt{3}}{18})\ (1-\sqrt{3})^{(n-1)/2} + \\ &(\frac{7}{6} + \frac{2\sqrt{3}}{3})\ [(n-1)/2]\ (1+\sqrt{3})^{(n-1)/2} + \\ &(\frac{7}{6} - \frac{2\sqrt{3}}{3})\ [(n-1)/2]\ (1-\sqrt{3})^{(n-1)/2}. \end{aligned}$$

In this case the first 20 terms in the sequence are

1, 9, 42, 162, 572, 1916, 6200, 19576, 60688, 185488, 560544, 1678368, 4986816, 14720960, 43214720, 126250880, 367288576, 1064552704, 3075344896, 8858067456.

Since each 0 can be replaced by a 2 and vice versa, we have $\widetilde{t_n} = \widetilde{z_n}$ for $n \geq 1$, and with $n\ p_n = \widetilde{z_n} + \widetilde{w_n} + \widetilde{t_n}$, it follows that $\widetilde{w_n} = n\ p_n - 2\ \widetilde{z_n}$. So

$$\widetilde{w_n} = \begin{cases} (\frac{n}{6} - \frac{\sqrt{3}}{9})\ (1+\sqrt{3})^{(n/2)} + (\frac{n}{6} + \frac{\sqrt{3}}{9})\ (1-\sqrt{3})^{(n/2)}\ , \\ \qquad n \text{ even, } n \geq 2. \\ [(\frac{n}{3})\ (1+\frac{\sqrt{3}}{2}) + (\frac{1}{6} + \frac{\sqrt{3}}{9})]\ (1+\sqrt{3})^{(n-1)/2} + \\ [(\frac{n}{3})\ (1-\frac{\sqrt{3}}{2}) + (\frac{1}{6} - \frac{\sqrt{3}}{9})]\ (1-\sqrt{3})^{(n-1)/2}\ ,\ n \text{ odd, } n \geq 1. \end{cases}$$

The first 20 terms for the sequence when n is even are

0, 4, 16, 64, 224, 752, 2432, 7680, 23808, 72768, 219904, 658432, 1956352, 5775104, 16953344, 49528832, 144089088, 417629184, 1206472704, 3475062784.

When n is odd the first 20 terms for the sequence are

1, 6, 26, 96, 332, 1096, 3512, 11008, 33936, 103264, 310944, 928256, 2751168, 8104064, 23746432, 69263360, 201216256, 582477312, 1680816640, 4836483072.

8. Runs in the p_n Palindromes

At this stage we direct our attention to $\widetilde{r_n}$, the number of runs that occur among these p_n palindromes of length n. We find that $\widetilde{r_1} = 3$, $\widetilde{r_2} = 2$, $\widetilde{r_3} = 20$, $\widetilde{r_4} = 14$. To determine $\widetilde{r_n}$, for $n \geq 1$ we let (i) $p_{n,j}$ count the number of palindromes of length n that (begin and) end in j, where $j = 0,\ 1,\ 2$, and (ii) $\widetilde{r_{n,j}}$ count the number of runs among the $p_{n,j}$ palindromes. We first observe that for $n \geq 1$, $\widetilde{r_{n,0}} = \widetilde{r_{n,2}}$ and $p_{n,0} = p_{n-2}$.

For $n \geq 3$ we find that

$$\widetilde{r_n} = 3\ \widetilde{r_{n-2}} - \widetilde{r_{n-2,1}} + 4\ p_{n-2},$$

where (i) the first summand accounts for when we place an x at the beginning and end of each palindrome of length $n-2$ - where $x = 0,\ 1$, or 2; (ii) the second summand removes the runs that result when the substring 11 arises in step (i); and (iii) the third summand $4\ p_{n-2}$ equals $4\ p_{n-2,0} + 4\ p_{n-2,1} + 4\ p_{n-2,2}$, where, for instance, the 4 in front of $p_{n-2,0}$ accounts for the four new runs that are created when we place 1 (or 2) at the beginning and end of each palindrome of length $n-2$ that begins and ends in 0.

In order to solve this recurrence relation we consider two cases. This time we consider the odd case first and rewrite the above recurrence relation as

$$\widetilde{r_{2n+1}} = 3\ \widetilde{r_{2n-1}} - \widetilde{r_{2n-1,1}} + 4\ p_{2n-1}.$$

This leads us to the following:

$$\begin{aligned}
\widetilde{r_{2n+1}} &= 3\ \widetilde{r_{2n-1}} - \widetilde{r_{2n-1,1}} + 4\ p_{2n-1} \\
&= 3\widetilde{r_{2n-1}} + 4p_{2n-1} - [\widetilde{r_{2n-1}} - 2\widetilde{r_{2n-1,0}}] \\
&= 2\ \widetilde{r_{2n-1}} + 4\ p_{2n-1} + 2\ \widetilde{r_{2n-1,0}} \\
&= 2\ \widetilde{r_{2n-1}} + 4\ p_{2n-1} + 2\ [\widetilde{r_{2n-3}} + 2\ p_{2n-3,1} + 2\ p_{2n-3,2}] \\
&= 2\ \widetilde{r_{2n-1}} + 2\ \widetilde{r_{2n-3}} + 4\ p_{2n-1} + \ 4\ [p_{2n-3,1} + \ p_{2n-3,2}] \\
&= 2\ \widetilde{r_{2n-1}} + 2\ \widetilde{r_{2n-3}} + 4\ p_{2n-1} + \ 4\ [p_{2n-3} - \ p_{2n-3,0}] \\
&= 2\ \widetilde{r_{2n-1}} + 2\ \widetilde{r_{2n-3}} + 4\ p_{2n-1} + \ 4\ p_{2n-3} - 4\ p_{2n-5}.
\end{aligned}$$

Assigning $x_n = \widetilde{r_{2n+1}}$ we obtain the recurrence relation

$$\begin{aligned} x_n &= 2x_{n-1} + 2x_{n-2} \\ &+4[(\frac{3}{2}+\frac{5\sqrt{3}}{6})(1+\sqrt{3})^{n-1} + (\frac{3}{2}-\frac{5\sqrt{3}}{6})(1-\sqrt{3})^{n-1}] \\ &+4\ [(\frac{3}{2}+\frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{n-2} + (\frac{3}{2}-\frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{n-2}] \\ &-4\ [(\frac{3}{2}+\frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{n-3} + (\frac{3}{2}-\frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{n-3}]. \end{aligned}$$

The solution for this recurrence relation has the form $x_n = c_1\ (1+\sqrt{3})^n + c_2\ (1-\sqrt{3})^n + \ A\ n\ (1+\sqrt{3})^n + B\ n\ (1-\sqrt{3})^n$, $n \geq 0$. Substituting the part of x_n that involves A into the corresponding part of the recurrence relation we find that

$$\begin{aligned} A\ n\ (1+\sqrt{3})^n &= 2\ A\ (n-1)\ (1+\sqrt{3})^{n-1} + 2\ A\ (n-2)\ (1+\sqrt{3})^{n-2} \\ &+4\ (\frac{3}{2}+\frac{5\sqrt{3}}{6})\ \cdot \\ &[(1+\sqrt{3})^{n-1} + (1+\sqrt{3})^{n-2} - (1+\sqrt{3})^{n-3}], \end{aligned}$$

and learn that $A = \frac{13+7\sqrt{3}}{6}$. A similar calculation gives us $B = \frac{13-7\sqrt{3}}{6}$. From $x_0 = \widetilde{r_1} = 3$ and $x_1 = \widetilde{r_3} = 20$, we have $c_1 = \frac{3}{2} + \frac{17\sqrt{3}}{18}$ and $c_2 = \frac{3}{2} - \frac{17\sqrt{3}}{18}$. So $x_n = (\frac{3}{2} + \frac{17\sqrt{3}}{18})\ (1+\sqrt{3})^n + (\frac{3}{2} - \frac{17\sqrt{3}}{18})\ (1-\sqrt{3})^n + (\frac{13+7\sqrt{3}}{6})\ n\ (1+\sqrt{3})^n + (\frac{13-7\sqrt{3}}{6})\ n\ (1-\sqrt{3})^n$, and it follows that

$$\begin{aligned} \widetilde{r_n} &= (\frac{3}{2}+\frac{17\sqrt{3}}{18})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{3}{2}-\frac{17\sqrt{3}}{18})\ (1-\sqrt{3})^{(n-1)/2} + \\ &(\frac{13+7\sqrt{3}}{6})\ [(n-1)/2]\ (1+\sqrt{3})^{(n-1)/2} + \\ &(\frac{13-7\sqrt{3}}{6})\ [(n\ -1)/2]\ (1-\sqrt{3})^{(n-1)/2}, \end{aligned}$$

for $n \geq 1$, n odd. The first 20 terms for this sequence are

3, 20, 86, 320, 1108, 3664, 11752, 36864, 113712, 346176, 1042784, 3113984, 9231680, 27199744, 79715968, 232554496, 675693312, 1956246528, 5645686272, 16246980608.

For the even case our original recurrence relation takes the form

$$\widetilde{r_{2n}} = 3\ \widetilde{r_{2n-2}} - \widetilde{r_{2n-2,1}} + 4\ p_{2n-2}.$$

This now leads us to the following:

$$\begin{aligned}
\widetilde{r_{2n}} &= 3\,\widetilde{r_{2n-2}} - \widetilde{r_{2n-2,1}} + 4\,p_{2n-2} \\
&= 3\,\widetilde{r_{2n-2}} + 4\,p_{2n-2} - [\,\widetilde{r_{2n-2}} - 2\,\widetilde{r_{2n-2,0}}] \\
&= 2\,\widetilde{r_{2n-2}} + 4\,p_{2n-2} + 2\,\widetilde{r_{2n-2,0}} \\
&= 2\,\widetilde{r_{2n-2}} + 4\,p_{2n-2} + 2[\widetilde{r_{2n-4}} + 2\,p_{2n-4,1} + 2\,p_{2n-4,2}] \\
&= 2\,\widetilde{r_{2n-2}} + 2\,\widetilde{r_{2n-4}} + 4\,p_{2n-2} + 4[\,p_{2n-4,1} + p_{2n-4,2}] \\
&= 2\,\widetilde{r_{2n-2}} + 2\,\widetilde{r_{2n-4}} + 4\,p_{2n-2} + 4[\,p_{2n-4} - p_{2n-4,0}] \\
&= 2\,\widetilde{r_{2n-2}} + 2\,\widetilde{r_{2n-4}} + 4\,p_{2n-2} + 4\,p_{2n-4} - 4\,p_{2n-6}.
\end{aligned}$$

Assigning $y_n = \widetilde{r_{2n}}$ we see that

$$\begin{aligned}
y_n = \; & 2\,y_{n-1} + 2\,y_{n-2} + \\
& 4\,[(\frac{1}{2} + \frac{\sqrt{3}}{6})(1+\sqrt{3})^{n-1} + (\frac{1}{2} - \frac{\sqrt{3}}{6})(1-\sqrt{3})^{n-1}] + \\
& 4\,[(\frac{1}{2} + \frac{\sqrt{3}}{6})\,(1+\sqrt{3})^{n-2} + (\frac{1}{2} - \frac{\sqrt{3}}{6})\,(1-\sqrt{3})^{n-2}] \\
& -4\,[(\frac{1}{2} + \frac{\sqrt{3}}{6})\,(1+\sqrt{3})^{n-3} + (\frac{1}{2} - \frac{\sqrt{3}}{6})\,(1-\sqrt{3})^{n-3}].
\end{aligned}$$

The solution here has the same form as x_n above. Now when we substitute the part that involves A into the corresponding part of the recurrence relation we find that $A = \frac{5+\sqrt{3}}{6}$. In a similar manner we find that $B = \frac{5-\sqrt{3}}{6}$. From $y_1 = \tilde{r_2} = 2$ and $y_2 = \tilde{r_4} = 14$ it follows that $c_1 = -\frac{1}{2} + \frac{\sqrt{3}}{18}$ and $c_2 = -\frac{1}{2} - \frac{\sqrt{3}}{18}$, so $y_n = (-\frac{1}{2} + \frac{\sqrt{3}}{18})\,(1+\sqrt{3})^n + (-\frac{1}{2} - \frac{\sqrt{3}}{18})\,(1-\sqrt{3})^n + (\frac{5+\sqrt{3}}{6})\,n\,(1+\sqrt{3})^n + (\frac{5-\sqrt{3}}{6})\,n\,(1-\sqrt{3})^n$. Consequently, for n even, $n \geq 2$,

$$\begin{aligned}
\tilde{r_n} = \; & (-\frac{1}{2} + \frac{\sqrt{3}}{18})\,(1+\sqrt{3})^{n/2} + \,(-\frac{1}{2} - \frac{\sqrt{3}}{18})\,(1-\sqrt{3})^{n/2} + \\
& (\frac{5+\sqrt{3}}{6})\,(\frac{n}{2})\,(1+\sqrt{3})^{n/2} + (\frac{5-\sqrt{3}}{6})\,(\frac{n}{2})\,(1-\sqrt{3})^{n/2}.
\end{aligned}$$

In this case the first 20 terms are

2, 14, 60, 228, 792, 2632, 8464, 26608, 82208, 250592, 755648, 2258496, 6700416, 19754112, 57925888, 169066240, 491430400, 1423298048, 4108958720, 11828126720.

9. Rises, Levels, and Descents in the p_n Palindromes

We turn now to the issue of rises, descents, and levels for the p_n palindromes. These numbers are denoted by $\widetilde{ri_n}$, $\widetilde{s_n}$, and $\widetilde{l_n}$, respectively.

For $n \geq 5$ we find that

$$\widetilde{l_n} = (\widetilde{l_{n-2}} + 2\, p_{n-2,0}) + (\widetilde{l_{n-2}} + 2\, p_{n-2,2}) + (\widetilde{l_{n-2}} - \widetilde{l_{n-4}})$$

where (1) the term $(\widetilde{l_{n-2}}+2\, p_{n-2,0})$ accounts for when we place a 0 at the start and end of a palindrome of length $n-2$; (2) the term $(\widetilde{l_{n-2}}+2\, p_{n-2,2})$ accounts for when we place a 2 at the start and end of a palindrome of length $n-2$; and (3) the term $(\widetilde{l_{n-2}} - \widetilde{l_{n-4}})$ accounts for when we place a 1 at the start and end of a palindrome of length $n-2$ and remove the levels for the palindromes of length $n-2$ that start and end with a 1 – there are $\widetilde{l_{n-4}}$ such levels (since there are no repeated 1's).

For the odd case we rewrite the above recurrence relation as

$$\widetilde{l_{2n+1}} = (\widetilde{l_{2n-1}} + 2\, p_{2n-1,0}) + (\widetilde{l_{2n-1}} + 2\, p_{2n-1,2}) + (\widetilde{l_{2n-1}} - \widetilde{l_{2n-3}}).$$

Since $p_{2n-1,0} = p_{2n-1,2} = p_{2n-3}$, it follows that

$$\begin{aligned} \widetilde{l_{2n+1}} &= 3\, \widetilde{l_{2n-1}} - \widetilde{l_{2n-3}} + 4\, p_{2n-3} \\ &= 3\, \widetilde{l_{2n-1}} - \widetilde{l_{2n-3}} + 4\, (\frac{3}{2} + \frac{5\sqrt{3}}{6})\, (1+\sqrt{3})^{n-2} + \\ &\quad 4\, (\frac{3}{2} - \frac{5\sqrt{3}}{6})\, (1-\sqrt{3})^{n-2}. \end{aligned}$$

If we let $x_n = \widetilde{l_{2n+1}}$, the preceding recurrence relation becomes

$$x_n = 3\, x_{n-1} - x_{n-2} + 4\, (\frac{3}{2} + \frac{5\sqrt{3}}{6})\, (1+\sqrt{3})^{n-2} + 4\, (\frac{3}{2} - \frac{5\sqrt{3}}{6})\, (1-\sqrt{3})^{n-2}.$$

Here the solution has the form $x_n = c_1\, (\frac{3+\sqrt{5}}{2})^n + c_2\, (\frac{3-\sqrt{5}}{2})^n + A\, (1+\sqrt{3})^n + B\, (1-\sqrt{3})^n$. [Note: $(\frac{3+\sqrt{5}}{2}) = \alpha^2$ and $(\frac{3-\sqrt{5}}{2}) = \beta^2$, where $\alpha = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\beta = \frac{1-\sqrt{5}}{2}$. The Fibonacci numbers are given recursively by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$; for $n \geq 0$, $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. The Lucas numbers are defined by the same recursive formula but here the initial conditions are $L_0 = 2$, $L_1 = 1$; for $n \geq 0$, $L_n = \alpha^n + \beta^n$.] Substituting $A\, (1+\sqrt{3})^n$ for x_n in the recurrence relation $x_n = 3\, x_{n-1} - x_{n-2} + 4\, (\frac{3}{2} + \frac{5\sqrt{3}}{6})\, (1+\sqrt{3})^{n-2}$, we learn that $A = \frac{66+38\sqrt{3}}{3}$. A similar calculation gives us $B = \frac{66-38\sqrt{3}}{3}$. From $x_0 = \widetilde{l_1} = 0$ and

$x_1 = \widetilde{l_3} = 4$ we then learn that $c_1 = -22 - 10\sqrt{5}$ and $c_2 = -22 + 10\sqrt{5}$. Consequently, for n odd,

$$\begin{aligned}
\widetilde{l_n} &= (-22-10\sqrt{5})(\frac{3+\sqrt{5}}{2})^{(n-1)/2} + (-22+10\sqrt{5})(\frac{3-\sqrt{5}}{2})^{(n-1)/2} + \\
&\quad (\frac{66+38\sqrt{3}}{3})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{66-38\sqrt{3}}{3})\ (1-\sqrt{3})^{(n-1)/2} \\
&= (-22-10\sqrt{5})\ \alpha^{n-1} + (-22+10\sqrt{5})\ \beta^{n-1} + \\
&\quad (\frac{66+38\sqrt{3}}{3})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{66-38\sqrt{3}}{3})\ (1-\sqrt{3})^{(n-1)/2} \\
&= -22\ L_{n-1} - 50\ F_{n-1} + (\frac{66+38\sqrt{3}}{3})\ (1+\sqrt{3})^{(n-1)/2} + \\
&\quad (\frac{66-38\sqrt{3}}{3})\ (1-\sqrt{3})^{(n-1)/2}.
\end{aligned}$$

The first 20 terms for this sequence are

0, 4, 24, 100, 364, 1232, 3988, 12524, 38480, 116292,
346940, 1024368, 2998932, 8717644, 25189968, 72414628,
207250588, 590855216, 1678744564, 4755273644.

For all $n \geq 1$, from the symmetry in a palindrome it follows that $\widetilde{ri_n} = \widetilde{s_n}$; also, $(n-1)\ p_n = \widetilde{l_n} + \widetilde{ri_n} + \widetilde{s_n} = \widetilde{l_n} + 2\ \widetilde{ri_n}$, so $\widetilde{ri_n} = \widetilde{s_n} = (\frac{1}{2})[(n-1)\ p_n - \widetilde{l_n}]$. Therefore, for n odd, we have

$$\begin{aligned}
\widetilde{ri_n} &= \widetilde{s_n} = 11\ L_{n-1} + 25\ F_{n-1} + \\
&\quad [(n-1)\ (\frac{3}{4} + \frac{5\sqrt{3}}{12}) - (\frac{33+19\sqrt{3}}{3})]\ (1+\sqrt{3})^{(n-1)/2} + \\
&\quad [(n-1)\ (\frac{3}{4} - \frac{5\sqrt{3}}{12}) - (\frac{33-19\sqrt{3}}{3})]\ (1-\sqrt{3})^{(n-1)/2}.
\end{aligned}$$

The first 20 terms for this sequence are

0, 6, 32, 130, 474, 1624, 5350, 17146, 53848, 166494, 508450,
1537160, 4608438, 13718874, 40593368, 119485486,
350092722, 1021626856, 2970549766, 8609536042.

For the even case our initial recurrence relation takes the form $\widetilde{l_{2n}} = 3\ \widetilde{l_{2n-2}} - \widetilde{l_{2n-4}} + 4\ p_{2n-4}$. Assigning $y_n = \widetilde{l_{2n}}$, we obtain

$$y_n = 3\ y_{n-1} - y_{n-2} + 4\ [(\frac{1}{2} + \frac{\sqrt{3}}{6})\ (1+\sqrt{3})^{n-2} + (\frac{1}{2} - \frac{\sqrt{3}}{6})\ (1-\sqrt{3})^{n-2}].$$

Using the same techniques we've seen throughout this paper, now with the initial conditions $y_1 = \widetilde{l_2} = 2$, $y_2 = \widetilde{l_4} = 10$, we learn that, for n even,

$$\begin{aligned}\widetilde{l_n} &= -6\ L_n - 12\ F_n + \\ &\left(\frac{18+10\sqrt{3}}{3}\right)(1+\sqrt{3})^{n/2} + \left(\frac{18-10\sqrt{3}}{3}\right)(1-\sqrt{3})^{n/2},\end{aligned}$$

and

$$\begin{aligned}\widetilde{ri_n} &= \widetilde{s_n} = 3\ L_n + 6\ F_n + \\ &\left[(n-1)\left(\frac{1}{4}+\frac{\sqrt{3}}{12}\right) - \left(\frac{9+5\sqrt{3}}{3}\right)\right](1+\sqrt{3})^{n/2} + \\ &\left[(n-1)\left(\frac{1}{4}-\frac{\sqrt{3}}{12}\right) - \left(\frac{9-5\sqrt{3}}{3}\right)\right](1-\sqrt{3})^{n/2}.\end{aligned}$$

The first 20 terms for the first sequence are

2, 10, 36, 122, 394, 1236, 3794, 11458, 34164, 100826, 295066, 857460, 2476994, 7119058, 20370612, 58064714, 164948266, 467173428, 1319608178, 3718510114.

For the second sequence the first 20 terms are

0, 4, 22, 93, 343, 1186, 3927, 12631, 39766, 123171, 376627, 1139686, 3419103, 10183255, 30141862, 88741851, 260050027, 758921446, 2206726215, 6395576047.

10. The Value of the Sum of the p_n Palindromes Considered as Base 3 Integers

Lastly, for $n \geq 1$, we determine $\widetilde{v_n}$, the sum of the p_n palindromes as base 3 integers. We start by observing that $\widetilde{v_1} = 3$, $\widetilde{v_2} = 8$, $\widetilde{v_3} = 104$, $\widetilde{v_4} = 240$, $\widetilde{v_5} = 2662$, and $\widetilde{v_6} = 5824$. For $n \geq 1$, we define $\widetilde{v_{n,j}}$ as the sum of the palindromes of length n that (begin and) end with j, for $j = 0,\ 1,\ 2$ – again, considered as base 3 integers. We start with the case for n odd.

For $n \geq 2$, we find that

$$\begin{aligned}\widetilde{v_{2n+1}} &= 3\ \widetilde{v_{2n-1}} + [3\ \widetilde{v_{2n-1}} + 2\ p_{2n-1} + 2 \cdot 3^{2n}\ p_{2n-1}] + \\ &[3(\widetilde{v_{2n-1}} - \widetilde{v_{2n-1,1}}) + 3^{2n}(p_{2n-1} - p_{2n-1,1}) + \\ &(p_{2n-1} - p_{2n-1,1})].\end{aligned}$$

Since $p_{2n-1} - p_{2n-1,1} = p_{2n-1,0} + p_{2n-1,2} = 2\ p_{2n-1,0} = 2\ p_{2n-3}$ and $\widetilde{v_{2n-1}} - \widetilde{v_{2n-1,1}} = \widetilde{v_{2n-1,0}} + \widetilde{v_{2n-1,2}} = 3\ \widetilde{v_{2n-3}} + 3\ \widetilde{v_{2n-3}} + 2\ p_{2n-3} + 2 \cdot 3^{2n-2}\ p_{2n-3}$, the preceding recurrence relation becomes

$$
\begin{aligned}
\widetilde{v_{2n+1}} &= 6\ \widetilde{v_{2n-1}} + 18\ \widetilde{v_{2n-3}} + (2 + 2 \cdot 3^{2n})\ p_{2n-1} + \\
&\quad (8 + 2 \cdot 3^{2n} + 6 \cdot 3^{2n-2})\ p_{2n-3} \\
&= 6\ \widetilde{v_{2n-1}} + 18\ \widetilde{v_{2n-3}} + (2 + 18 \cdot 3^{2n-2}) \cdot \\
&\quad [(\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{n-1} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{n-1}] + \\
&\quad (8 + 216 \cdot 3^{2n-4}) \cdot \\
&\quad [(\frac{3}{2} + \frac{5\sqrt{3}}{6})(1+\sqrt{3})^{n-2} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})(1-\sqrt{3})^{n-2}].
\end{aligned}
$$

If we let $g_n = \widetilde{v_{2n+1}}$, we find that

$$
\begin{aligned}
g_n &= 6\ g_{n-1} + 18\ g_{n-2} + 2\ [(\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{n-1} + \\
&\quad (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{n-1}] + \\
&\quad 18\ \ [(\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (9\ (1+\sqrt{3}))^{n-1} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (9\ (1-\sqrt{3}))^{n-1}] + \\
&\quad 8\ [(\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (1+\sqrt{3})^{n-2} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (1-\sqrt{3})^{n-2}] + \\
&\quad 216\ [(\frac{3}{2} + \frac{5\sqrt{3}}{6})\ (9\ (1+\sqrt{3}))^{n-2} + (\frac{3}{2} - \frac{5\sqrt{3}}{6})\ (9\ (1-\sqrt{3}))^{n-2}].
\end{aligned}
$$

Here the solution has the form

$$
\begin{aligned}
g_n &= c_1\ [3\ (1+\sqrt{3})]^n + c_2\ [3\ (1-\sqrt{3})]^n + A(1+\sqrt{3})^n + B(1-\sqrt{3})^n \\
&\quad + C[9\ (1+\sqrt{3})]^n + D[9\ (1-\sqrt{3})]^n.
\end{aligned}
$$

Upon substituting the particular part of the solution for g_n into the recurrence relation we learn that $A = \frac{-9-5\sqrt{3}}{12}$, $B = \frac{-9+5\sqrt{3}}{12}$, $C = \frac{9+5\sqrt{3}}{4}$, and $D = \frac{9-5\sqrt{3}}{4}$. From $g_0 = \widetilde{v_1} = 3$ and $g_1 = \widetilde{v_3} = 104$ we learn that $c_1 = c_2 = 0$. So for $n \geq 0$, $\widetilde{v_{2n+1}} = g_n = (\frac{-9-5\sqrt{3}}{12})\ (1+\sqrt{3})^n + (\frac{-9+5\sqrt{3}}{12})\ (1-\sqrt{3})^n + (\frac{9+5\sqrt{3}}{4})\ [9\ (1+\sqrt{3})]^n + (\frac{9-5\sqrt{3}}{4})\ [9\ (1-\sqrt{3})]^n$. Consequently, for $n \geq 1$, n odd,

$$\begin{aligned}
\widetilde{v_n} &= (\frac{-9-5\sqrt{3}}{12})\ (1+\sqrt{3})^{(n-1)/2} + (\frac{-9+5\sqrt{3}}{12})\ (1-\sqrt{3})^{(n-1)/2} + \\
&\quad (\frac{9+5\sqrt{3}}{4})\ [9\ (1+\sqrt{3})]^{(n-1)/2} + (\frac{9-5\sqrt{3}}{4})\ [9\ (1-\sqrt{3})]^{(n-1)/2} \\
&= (\frac{9+5\sqrt{3}}{12})\ (3^n-1)\ (1+\sqrt{3})^{(n-1)/2} + \\
&\quad (\frac{9-5\sqrt{3}}{12})\ (3^n-1)\ (1-\sqrt{3})^{(n-1)/2}.
\end{aligned}$$

The first 20 terms in this sequence are

3, 104, 2662, 65580, 1613924, 39680704, 975725064, 23991370832, 589912260016, 14505023095680, 356656202775392, 8769625393999552, 215631561948551232, 5302047428920825856, 130369166756292009088, 3205576685098571592960, 78820185346293981115136, 1938066759219261315352576, 47654071692046331505378816, 1171740105450354308075138048, 28811281512217883270783017984.

To determine the comparable formula for $\widetilde{v_2}$, $\widetilde{v_4}$, $\widetilde{v_6}$, ..., we now consider the recurrence relation

$$\begin{aligned}
\widetilde{v_{2n}} &= 3\ \widetilde{v_{2n-2}} + (3\ \widetilde{v_{2n-2}} + 2\ p_{2n-2} + 2\cdot 3\ p_{2n-2}) + \\
&\quad [3(\widetilde{v_{2n-2}} - \widetilde{v_{2n-2,1}}) + (p_{2n-2} - p_{2n-2,1}) + \\
&\quad 3^{2n-1}(p_{2n-2} - p_{2n-2,1})].
\end{aligned}$$

Since $\widetilde{v_{2n-2}} - \widetilde{v_{2n-2,1}} = \widetilde{v_{2n-2,0}} + \widetilde{v_{2n-2,2}} = 3\ \widetilde{v_{2n-4}} + 3\ \widetilde{v_{2n-4}} + 2\ p_{2n-4} + 2\cdot 3^{2n-3}\ p_{2n-4}$ and $p_{2n-2} - p_{2n-2,1} = p_{2n-2,0} + p_{2n-2,2} = 2\ p_{2n-2,0} = 2\ p_{2n-4}$, the preceding recurrence relation may be rewritten as

$$\begin{aligned}
\widetilde{v_{2n}} &= 6\ \widetilde{v_{2n-2}} + 18\ \widetilde{v_{2n-4}} + 2\ (1+3^{2n-1})\ p_{2n-2} \\
&\quad +2\ (3+3^{2n-2})\ p_{2n-4} + 2\ (1+3^{2n-1})\ p_{2n-4}.
\end{aligned}$$

If we let $h_n = \widetilde{v_{2n}}$, then for $n \geq 1$ we have

$$\begin{aligned} h_n &= 6\,h_{n-1} + 18\,h_{n-2} + (1+\frac{\sqrt{3}}{3})\,(1+\sqrt{3})^{n-1} + \\ &\quad (1+\frac{\sqrt{3}}{3})\,(3)\,[9\,(1+\sqrt{3})]^{n-1} + (1-\frac{\sqrt{3}}{3})\,(1-\sqrt{3})^{n-1} + \\ &\quad (1-\frac{\sqrt{3}}{3})\,(3)\,[9\,(1-\sqrt{3})]^{n-1} + (4+\frac{4\sqrt{3}}{3})\,(1+\sqrt{3})^{n-2} + \\ &\quad (4+\frac{4\sqrt{3}}{3})\,(9)\,[9\,(1+\sqrt{3})]^{n-2} + (4-\frac{4\sqrt{3}}{3})\,(1-\sqrt{3})^{n-2} + \\ &\quad (4-\frac{4\sqrt{3}}{3})\,(9)\,[9\,(1-\sqrt{3})]^{n-2}. \end{aligned}$$

In this case the solution has the form

$$\begin{aligned} h_n &= c_1\,(3+3\sqrt{3})^n + c_2\,(3-3\sqrt{3})^n + A\,(1+\sqrt{3})^n + B\,(1-\sqrt{3})^n + \\ &\quad C\,[9\,(1+\sqrt{3})]^n + D\,[9\,(1-\sqrt{3})]^n. \end{aligned}$$

Upon substituting the particular part of the solution into the recurrence relation we find that $A = -\frac{1}{4} - \frac{\sqrt{3}}{12}$, $B = -\frac{1}{4} + \frac{\sqrt{3}}{12}$, $C = \frac{1}{4} + \frac{\sqrt{3}}{12}$, and $D = \frac{1}{4} - \frac{\sqrt{3}}{12}$. Consequently, $h_n = c_1\,(3+3\sqrt{3})^n + c_2\,(3-3\sqrt{3})^n + (-\frac{1}{4} - \frac{\sqrt{3}}{12})\,(1+\sqrt{3})^n + (-\frac{1}{4} + \frac{\sqrt{3}}{12})\,(1-\sqrt{3})^n + (\frac{1}{4} + \frac{\sqrt{3}}{12})\,[9\,(1+\sqrt{3})]^n + (\frac{1}{4} - \frac{\sqrt{3}}{12})\,[9\,(1-\sqrt{3})]^n$. From $h_1 = \widetilde{v_2} = 8$ and $h_2 = \widetilde{v_4} = 240$, we find, as above, that $c_1 = c_2 = 0$, so $\widetilde{v_{2n}} = h_n = (-\frac{1}{4} - \frac{\sqrt{3}}{12})\,(1+\sqrt{3})^n + (-\frac{1}{4} + \frac{\sqrt{3}}{12})\,(1-\sqrt{3})^n + (\frac{1}{4} + \frac{\sqrt{3}}{12})\,[9\,(1+\sqrt{3})]^n + (\frac{1}{4} - \frac{\sqrt{3}}{12})\,[9\,(1-\sqrt{3})]^n$. Consequently, for n even, $n \geq 2$,

$$\begin{aligned} \widetilde{v_n} &= (-\frac{1}{4} - \frac{\sqrt{3}}{12})\,(1+\sqrt{3})^{n/2} + (-\frac{1}{4} + \frac{\sqrt{3}}{12})\,(1-\sqrt{3})^{n/2} + \\ &\quad (\frac{1}{4} + \frac{\sqrt{3}}{12})\,[9\,(1+\sqrt{3})]^{n/2} + (\frac{1}{4} - \frac{\sqrt{3}}{12})\,[9\,(1-\sqrt{3})]^{n/2} \\ &= (\frac{1}{4} + \frac{\sqrt{3}}{12})\,[3^n - 1]\,(1+\sqrt{3})^{n/2} + (\frac{1}{4} - \frac{\sqrt{3}}{12})\,[3^n - 1]\,(1-\sqrt{3})^{n/2}. \end{aligned}$$

The first 20 terms for this last sequence are

8, 240, 5824, 144320, 3542880, 87156160, 2142769664, 52689185280, 1295534111872, 31855262278400, 783271247815680, 19259434951644160, 473559771280819712, 11644104345235000320, 286310561161760751616, 7039935004839867269120, 173101140995323135887360, 4256290008699875717509120, 104655604997840113043357696, 2573319871370501906793676800.

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12. References

1. Alladi, K., and Hoggatt, Jr., V. E. Compositions with Ones and Twos. *The Fibonacci Quarterly*, Volume 13, Issue 3, October, 1975, Pp. 233-239.

2. Brualdi, Richard A. *Introductory Combinatorics*, fourth edition. Upper Saddle River, New Jersey: Pearson Prentice Hall, 2004.

3. Deininger, Rolf A. Fibonacci Numbers and Water Pollution Control. *The Fibonacci Quarterly*, Volume 10, Issue 3, April, 1972, Pp. 299 - 300, 302.

4. Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics*, fifth edition. Boston, Massachusetts: Pearson Addison Wesley, 2004.

5. Grimaldi, Ralph P. Ternary Strings that Avoid the Substring '21'. *Congressus Numerantium*, Volume 170, 2004, Pp. 33-49.

6. Grimaldi, Ralph P., and Heubach, Silvia. Binary Strings Without Odd Runs of Zeros. *Ars Combinatoria*, Volume 75, April, 2005, Pp. 241-255.

7. Koshy, Thomas. *Fibonacci and Lucas Numbers with Applications.* New York, New York: John Wiley & Sons, Inc., 2001.

8. Sloane, Neil J. A., *On-Line Encyclopedia of Integer Sequences (Look Up)*, www.research.att.com/~njas/sequences