Generalized weakly connected domination in graphs *

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Abstract

A weakly connected dominating set W of a graph G is a dominating set such that the subgraph consisting of $V(G)$ and all edges incident on vertices in W is connected. In this paper, we generalize it to $[r, R]$-dominating set which means a distance r-dominating set that can be connected by adding paths with length within R. We present an algorithm for finding $[r, R]$-dominating set with performance ratio not exceeding $\ln \Delta_r + \lceil \frac{2r+1}{R} \rceil - 1$, where Δ_r is the maximum number of vertices that are at distance at most r from a vertex in the graph. The bound for size of minimum $[r, R]$-dominating set is also obtained.

Keywords: approximation algorithm; connected dominating set; weakly connected dominating set; distance domination; graph theory

1 Introduction

In this paper, we introduce a new variation on domination in graphs. The motivation for this research grew from the trend of applying weakly connected dominating set to clustering mobile ad hoc networks recently [2] .

The proliferation of wireless communicating devices has created a wealth of opportunities for the field of mobile computing. However, in the extreme case of self-organizing networks, such as mobile ad hoc networks, it is challenging to guarantee efficient communications. In these infrastructure-less networks, only nodes that are sufficiently close to each other can communicate and, as mobile nodes roam at will, the network topology changes

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arbitrarily and rapidly.

Clustering mobile nodes locally is an effective way to hierarchically organize the structure. A natural method for forming clusters is based on the idea connected domination in graphs [3]. Unfortunately, the connectivity requirement causes the number of clusters to be rather large. Recently, several authors [2] suggested ways to reduce the size of this number by relaxing the requirement of connectedness to weakly-connectedness. In this paper, we theoretically introduce a new variation relaxing the requirement of connectedness of dominating set in a more general way and analyze its relation with other domination parameters.

A dominating set of a graph G is a set of vertices $D \in V(G)$ such that every vertex of G either belongs to D or is adjacent with a vertex of D in G. Define the minimum cardinality of all dominating sets of G as the domination number of G and denote this by $\gamma(G)$. The problem of determining $\gamma(G)$ for a given graph, G, is one of the core NP-hard problems in graph theory (see [7]).

It's natural to generalize the concept of classical dominating set to distance dominating set: a *distance* $r-$ *dominating set* of a graph G is a set of vertices $D \in V(G)$ such that every vertex of G either belongs to D or at most r from a vertex in D, $\gamma_r(G)$ denotes the minimum cardinality of all distance r-dominating set. Determining $\gamma_r(G)$ is also NP-hard[1].

A connected dominating set C of a graph G is a dominating set such that the subgraph induced by the vertices of C in G is connected. Define the minimum cardinality of all connected dominating sets of G as the connected domination number of G and denote it by $\gamma_c(G)$. The connected distance domination number $\gamma_r^c(G)$ is defined to be the minimum cardinality of all connected distance r-dominating sets of G. The problem of determining $\gamma_r^c(G)$ for general graphs is also NP-hard and (see [10]).

Grossman [8] introduced another NP-hard variant of the minimum dominating set problem, i.e., the problem of finding a minimum weakly connected dominating set. A weakly connected dominating set W of a graph G is a dominating set such that the subgraph consisting of $V(G)$ and all edges incident on vertices in W is connected. Define the minimum cardinality of all weakly connected dominating sets of G as the weakly connected domination number of G and denote this by $\gamma_w(G)$.

The relations among these three parameters are discussed in [6], and we can consider $\gamma_w(G)$ as lying between $\gamma(G)$ and $\gamma_c(G)$. That is: if we want to connect the vertices in W by adding paths with endpoints in W, the requirement of path length is no more than 2, while the similar requirement of path length for connecting C and D is 1 and 3, respectively.

We can rewrite the definition of weakly connected dominating set more formally as follows:

Definition 1 *W is a weakly connected dominating set of G if the following conditions are satisfied:*

1. *W is a dominating set of G;*

2. *The union graph of all the paths with length no more than 2 and with endpoints in W is connected.*

For connected distance r-dominating set, what can be done to weaken the requirement of connectedness? We give a possible approach by introducing the concept of "$[r, R]$-dominating set" :

Definition 2 *W is an $[r, R]$-dominating set of G if the following conditions are satisfied:*

1. *W is a distance r-dominating set of G;*

2. *The union graph of all the paths with length no more than R and with endpoints in W is connected.*

We use $\gamma_{r,R}(G)$ to denote the minimum cardinality of all [r,R]-dominating set. Especially, $\gamma_{1,1}(G)$, $\gamma_{1,2}(G)$, $\gamma_{1,3}(G)$ correspond to the ordinary connected domination number $\gamma_c(G)$, weakly domination number $\gamma_w(G)$, and classical domination number $\gamma(G)$, respectively. Observe that it is certain to connect the vertices in a distance r-dominating set by adding paths with length no more than $2r + 1$, we restrict R to be within $2r + 1$ in this paper without notification.

The remaining of this paper is organized as follows. In Section 2, we present our algorithm for approximating $[r, R]$-dominating sets and analyze its performance. Section 3 shows relations between $\gamma_{r,R}(G)$ and some other domination related parameters. We also establish lower and upper bounds on $\gamma_{r,R}(G)$ in this section. Finally we conclude this paper in Section 4.

2 Algorithmic results

Since the problem of determining $\gamma_r(G)$ is NP-hard, the computing of $\gamma_{r,R}(G)$ is also NP-hard. In this section, we present a two-phased algorithm for finding [r,R]-dominating set.

Our algorithm is based on Algorithm 2 of Guha and Khuller [9].First we find a distance r-dominating set, then we add chains of vertices to let it satisfy the connectedness requirement.

At the initial state of the first phase all nodes are colored white. Then we pick a node at each step and color it black, coloring gray all the white nodes which are at distance at most r of the very chosen node. A *piece* is defined as a white node or a black connected component(a connected component whose vertices are all colored black). At each step we pick a node to color black that gives the maximum(nonzero)reduction in the number of pieces.

By using the similar method described in [9] we will prove that at the end of this phase if no vertex gives a nonzero reduction to the number of pieces, then there are no white nodes left.

In the second phase we have a collection of black connected components that we need to connect. Recursively connect pairs of black components by choosing a chain of vertices, until there is one black connected component. Our final solution is the set of black vertices that form the connected component.

Lemma 1 *At the end of the first phase there are no white vertices left.*

Proof. Suppose there is a white vertex v at the end of the first phase. If v has a white vertex as its neighbor, then coloring v black will reduce the number of white vertices by two, and increase the number of black components by one, thus picking v will reduce the number of pieces. Otherwise, v has a grey neighbor, then there is a black vertex at distance exactly r from v, and one neighbor, for example u, of this black vertex will be at distance exactly $r-1$ from v, color u black, then the number of white vertices reduces by one without increasing the number of black components. The result follows. □

Lemma 2 *At the end of the first phase if there is more than one black component, then there is always a pair of black components that can be connected by choosing a chain of at most $\lceil \frac{2r+1}{R} \rceil - 1$ vertices.*

Proof. Consider the shortest path connecting two black components. Then the length of this path must be within $2r+1$, and it may be greater than R, in order to satisfy the requirement of "$[r, R]$-domination", at most $\lceil \frac{2r+1}{R} \rceil - 1$ vertices should be added to the ultimate vertex set. □

For the convenience of analysis, we use $N_r(x)$ denote $\{y \in V(G) : d_G(x,y) \leq r, x \neq y\}$, where $d_G(x,y)$ means the distance of x and y in graph G. The maximum r-degree of G, $\Delta_r(G)$, is defined as $\max\{|N_r(x)| : x \in V(G)\}$. We have the following:

Theorem 1 *If $R < 2r+1$, then the performance ratio of the above algorithm is at most* $\ln \Delta_r + \lceil \frac{2r+1}{R} \rceil - 1$.

Proof. Define OPT as the optimal $[r, R]$-dominating set of the given graph and let v be an arbitrary vertex, besides itself, v dominates at at most $\Delta_r + 1$ vertices. Each time we include a vertex which are at distance at most R from a chosen vertex into OPT, at most Δ_r new distinct vertices are dominated. Then $n \leq (\Delta_r + 1) + \Delta_r \cdot (|OPT| - 1)$, so we get $|OPT| \geq (n-1)/\Delta_r$.

Let a_i be the number of pieces left after the ith iteration, and $a_0 = n$. Consider the ith iteration. The optimal solution can connect a_i pieces and decrease the number of pieces by $a_i - 1$. Hence in the following iteration, the greedy procedure is guaranteed to pick a node which can reduce the number of pieces by at least $\lceil a_i - 1/|OPT| \rceil$. This gives us the recurrence relation

$$a_{i+1} \leq a_i - \lceil (a_i - 1)/|OPT| \rceil \leq a_i(1 - \frac{1}{|OPT|}) + \frac{1}{|OPT|}.$$

Solving it, we get

$$\begin{aligned} a_i &\leq a_0(1 - \tfrac{1}{|OPT|})^i + \tfrac{1}{|OPT|} \textstyle\sum_{j=0}^{i-1} (1 - \tfrac{1}{|OPT|})^j \\ &= (a_0 - 1)(1 - \tfrac{1}{|OPT|})^i + 1. \end{aligned}$$

Let $i = |OPT| \cdot \ln \frac{a_0 - 1}{|OPT|}$, we have

$$\begin{aligned} a_i &\leq (a_0 - 1)(1 - \tfrac{1}{|OPT|})^i + 1 \\ &= (a_0 - 1)(1 - \tfrac{1}{|OPT|})^{|OPT| \ln \frac{a_0 - 1}{|OPT|}} + 1 \\ &\leq (a_0 - 1)(\tfrac{1}{e})^{|OPT| \ln \frac{a_0 - 1}{|OPT|}} + 1 \\ &= (a_0 - 1)\tfrac{a_0 - 1}{|OPT|} + 1 \\ &= |OPT| + 1. \end{aligned}$$

After $|OPT| \cdot \ln \frac{a_0-1}{|OPT|}$ iterations, the number of pieces left is less than $|OPT|+1$. Suppose we stop after choosing a_f more nodes, then the number of pieces left to connect is at most $|OPT|+1-a_f$. We connect the remaining pieces by adding $|OPT|+1-a_f-1$ chains of at most $\lceil\frac{2r+1}{R}\rceil - 1$ vertices in the second phase(Lemma 2).

The total number of nodes chosen is at most

$$\begin{aligned} & |OPT| \cdot \ln \tfrac{a_0-1}{|OPT|} + a_f + (\lceil\tfrac{2r+1}{R}\rceil - 1)(|OPT|+1-a_f-1) \\ \leq \; & (\ln \Delta_r + \lceil\tfrac{2r+1}{R}\rceil - 1) \cdot |OPT| - (\lceil\tfrac{2r+1}{R}\rceil - 2) \cdot a_f \end{aligned}$$

Since $R < 2r+1$, then $\lceil\frac{2r+1}{R}\rceil - 2 \geq 0$, and the conclusion follows. □

Remark. When $R = 2r+1$, the problem is equal to find distance r-dominating set, from basics of *Set Cover* we know that there is an algorithm with performance ratio within $H(\Delta_r + 1)$, where $H(\cdot)$ is harmonic function.

3 Bounds

For the reason that an $[r_1, R]$-dominating set is also an $[r_2, R]$-dominating set if $r_1 \leq r_2$, then it is easy to have the following theorem :

Theorem 2 *If $r_1 \leq r_2$, then $\gamma_{r_2,R} \leq \gamma_{r_1,R}$.*

In [4] and [6], it's showed that:

$$\gamma \leq \gamma_c \leq 3\gamma - 2,$$

$$\gamma \leq \gamma_w \leq 2\gamma - 1,$$

$$\gamma_w \leq \gamma_c \leq 2\gamma_w - 1.$$

If we rewrite them in words of $[r, R]$-domination number, that is

$$\gamma_{1,3} \leq \gamma_{1,1} \leq 3\gamma_{1,3} - 2,$$

$$\gamma_{1,3} \leq \gamma_{1,2} \leq 2\gamma_{1,3} - 1,$$

$$\gamma_{1,2} \leq \gamma_{1,1} \leq 2\gamma_{1,2} - 1.$$

More generally, we have the following :

Theorem 3 *If $R_1 \leq R_2$, then*

$$\gamma_{r,R_2} \leq \gamma_{r,R_1} \leq \left\lceil \frac{R_2}{R_1} \right\rceil (\gamma_{r,R_2} - 1) + 1.$$

Proof. The first inequality follows trivially. To establish the second, let D_1 be an $[r, R_2]$-dominating set of cardinality γ_{r,R_2}. If we want to connect the vertices in D_1 by adding paths with endpoints in D_1, the required length of the paths is no more than R_2. More precisely, if the length is less than R_1, the result is obtained. Otherwise, if the length is greater than R_1, similar to lemma 2 we know that we need to include $\left\lceil \frac{R_2}{R_1} \right\rceil - 1$ more vertices to each of these paths to shorten the path length to R_1. Since the number of these paths is within $\gamma_{r,R_2} - 1$, then $\gamma_{r,R_1} \leq (\lceil \frac{R_2}{R_1} \rceil - 1)(\gamma_{r,R_2} - 1) + \gamma_{r,R_2} = \left\lceil \frac{R_2}{R_1} \right\rceil (\gamma_{r,R_2} - 1) + 1$. □

Theorem 4 *Let r, R_1, R_2, c_2 be positive integers with $R_1 \leq R_2$, x nonnegative integer with $x \leq \left\lceil \frac{R_2}{R_1} \right\rceil (c_2 - 1) + 1 - c_2$. Then there exists a tree T with $\gamma_{r,R_2}(T) = c_2$ and $\gamma_{r,R_1}(T) = c_2 + x$.*

Proof. For convenience, define a structure $T(r, m)$ with positive integers r, m : let C_i be a chain with vertices $v_{i,1}, v_{i,2}, ..., v_{i,r-1}$ for $i = 1, ..., m$, connect these chains by joining $v_{1,1}, v_{2,1}, ..., v_{m,1}$ to a new vertex v_0. Thus, the resulting tree is of type $T(r, m)$ and its central vertex is v_0.

We write x in the form of $x = a \cdot (\left\lceil \frac{R_2}{R_1} \right\rceil - 1) + b$ with $0 \leq b < \left\lceil \frac{R_2}{R_1} \right\rceil - 1$, and the tree T is constructed as follows.

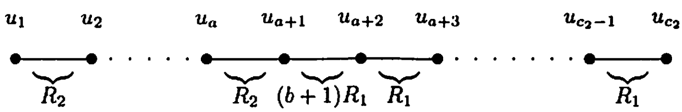

Figure 1: $\gamma_{r,R_1}(T) = c_2 + x$

Fix $m \geq 1$, and let $T_i \cong T(r, m)$, for $i = 1, ..., c_2$, with the central vertex of T_i denoted by u_i. For $1 \leq j \leq a$, connect u_j, u_{j+1} by adding a path with length R_2 and the endpoints of which are u_j, u_{j+1}. For $j = a + 1$, connect u_j, u_{j+1} by adding a path with length $(b+1)R_1$ and the endpoints of which are u_j, u_{j+1}. For $a + 1 < j < c_2$, connect u_j, u_{j+1} by adding a path with length R_1 and the endpoints of which are u_j, u_{j+1}.

Obviously $\{u_1, u_2, ..., u_{c_2}\}$ is a minimum $[r, R_2]$-dominating set of T, the $[r, R_1]$-domination number of T is $c_2 + a \cdot (\left\lceil \frac{R_2}{R_1} \right\rceil - 1) + (\left\lceil \frac{(b+1)R_1}{R_1} \right\rceil - 1) = c_2 + a \cdot (\left\lceil \frac{R_2}{R_1} \right\rceil - 1) + b = c_2 + x$. □

Let $x = \left\lceil \frac{R_2}{R_1} \right\rceil (c_2 - 1) + 1 - c_2$, we can see that the upper bound of Theorem 3 is tight.

We now turn our attention to upper bound of $\gamma_{r,R}$ and examine its relation with the order of graphs. For ordinary distance domination, it's proved in [11] that:

Lemma 3 *For $r \geq 1$, if G is a graph of order $n \geq r+1$, then $\gamma_r(G) \leq \frac{n}{r+1}$.*

We generalize it to $[r, R]$-domination:

Theorem 5 *For $r, R \geq 1$, if G is a graph of order $n \geq r + 1, R$, then $\gamma_{r,R}(G) \leq \dfrac{n}{\lceil R/2 \rceil}$. Furthermore, $\gamma_{r,R}(G) \leq n/2$ when $R = 2$.*

Proof. If $R \geq 3$, since a $\lfloor (R-1)/2 \rfloor$-dominating set is also an $[r, R]$-dominating set, therefore $\gamma_{r,R} \leq \gamma_{\lfloor (R-1)/2 \rfloor} \leq \dfrac{n}{\lfloor (R-1)/2 \rfloor + 1} = \dfrac{n}{\lceil R/2 \rceil}$. If $R < 3$, $\gamma_{r,R}(G) \leq n$ is obvious. It's proved in [6] that $\gamma_w \leq n/2$, that is $\gamma_{1,2} = \gamma_w \leq n/2$, from Theorem 2, we have $\gamma_{r,2}(G) \leq \gamma_{1,2} \leq n/2$. Thus, the result is obtained. □

For specified graph, this upper bound may be improved.

4 Conclusion

In this paper, we generalize the concept of ordinary weakly connected dominating set to $[r, R]$-dominating set. We present a two-phased algorithm for finding $[r, R]$-dominating set in general graph which achieves a performance ratio of $\ln \Delta_r + \lceil \frac{2r+1}{R} \rceil - 1$. The bound for size of minimum $[r, R]$-dominating set, $\gamma_{r,R}$, is also obtained, which is less than $\dfrac{n}{\lceil R/2 \rceil}$. One line of future work is to examine parameter of $\gamma_{r,R}$ in random graph as in [5].

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References

[1] G.J.Chang and G.L.Nemhauser, The k-domination and k-stability problems on graphs, *Technical Report TR-540,School of Operations Research and Industrial Engineering*, Cornell University, Ithaca, NY, 1982.

[2] Y.P.Chen and A.L.Liestman, Approximating minimum size weakly-connected dominating sets for clustering mobile ad hoc networks, *In: Proceedings of 3rd ACM International Symposium on Mobile Ad-Hoc Networking and Computing*, 165-172, 2002.

[3] B.Das and V.Bharghavan, Routing in ad-hoc net- works using minimum connected dominating sets, *IEEE International Conference on Communications*, 376-380, 1997.

[4] P.Duchet and H.Meyniel, On Hadwiger's number and stability number, *Ann. Discrete Math.*, 13:71-74,1982.

[5] W.Duckworth and B.Mans, Randomised algorithms for finding small weakly-connected dominating sets of regular graphs, *Lecture Notes in Computer Science*, 2653: 83-95, 2003.

[6] J.E.Dunbar, J.W.Grossman, J.H.Hattingh, S.T. Hedetniemi, A.A. McRae, On weakly connected domination in graphs, *Discrete Math.*, 167/168: 261-269, 1997.

[7] M.R.Garey and D.S.Johnson, *Computers and Intractability: A Guide to the Theory of NP- Completeness*, Freeman, SanFrancisco, 1978.

[8] J.W.Grossman, Dominating sets whose closed stars form spanning trees, *Discrete Math.*, 169: 83-94, 1997.

[9] S.Guha, and S.Khuller, Approximation Algorithms for Connected Dominating Sets, *Algorithmica*, 20: 374-387,1998.

[10] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs: Advanced topics*, Marcel Dekker, New York, 1998.

[11] M.A.Henning, O.R.Ollermann and H.C.Swart, Bounds on distance domination numbers, *J. Combin. Inform. System Sci.*, 16: 11-18,1991.