A Geometric Approach to the Reflections of Regular Maps

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Abstract

A reflection of a regular map on a Riemann surface fixes some simple closed curves, which are called *mirrors*. Each mirror passes through some of the geometric points (vertices, face-centers and edge-centers) of the map such that these points form a periodic sequence which we call the *pattern* of the mirror. For every mirror there exist two particular conformal automorphisms of the map that fix the mirror setwise and rotate it in opposite directions. We call these automorphisms the *rotary automorphisms* of the mirror. In this paper we first introduce the notion of pattern and then describe the patterns of mirrors on surfaces. We also determine the rotary automorphisms of mirrors. Finally, we give some necessary conditions under which all reflections of a regular map are conjugate.

Keywords: Riemann surface, regular map, reflection, pattern, rotary automorphism.

Mathematics Subject Classifications: 05C10, 30F10.

1 Introduction

A compact Riemann surface X of genus $g > 1$ is called *symmetric* if it admits an anti-conformal involution $T\colon X \to X$, which is called a *symmetry* of X. The fixed point set of T consists of k disjoint simple closed curves on X, and these curves are called the *mirrors* of T. Here k is an integer and by Harnack's theorem $0 \leq k \leq g+1$. Let $\mathcal{M}$ be a regular map on X. A *reflection* of $\mathcal{M}$ is a symmetry of X that leaves $\mathcal{M}$ invariant and fixes some mirrors. Every mirror of a reflection of $\mathcal{M}$ passes through some of the geometric points of $\mathcal{M}$ such that these points form a periodic sequence which we call the *pattern* of the mirror. By geometric points we mean the vertices, the face-centers and the edge-centers of $\mathcal{M}$. If M is a mirror of a reflection of $\mathcal{M}$, then there exist two particular conformal automorphisms of $\mathcal{M}$ that fix M setwise and rotate it in opposite directions. These automorphisms are inverses of each other and we call them the *rotary automorphisms* of M.

In this paper we first introduce the notion of pattern and then describe the patterns of mirrors on surfaces. Then we define the rotary automorphisms of a mirror and we determine such automorphisms according to the patterns of the corresponding mirrors. Finally, we focus on conjugacy classes of reflections and we give some necessary conditions under which all reflections of a regular map are conjugate.

The idea of patterns of mirrors first appeared in Klein [4]. In the context of Riemann surfaces we know of no other work where patterns of mirrors have been discussed, although Coxeter has used this idea in geometry, (see [2], Chapter 4).

2 Preliminaries

Non-Euclidean Crystallographic Groups. A *non-Euclidean crystallographic (NEC) group* is a discrete subgroup Λ of $PGL(2,\mathbb{R})$, the group of isometries of the hyperbolic plane $\mathcal{U}$, such that $\mathcal{U}/\Lambda$ is compact. If Λ is contained in $PSL(2,\mathbb{R})$, the group of conformal isometries of $\mathcal{U}$, then it is called a *Fuchsian group*.

In this paper we deal with NEC groups generated by the reflections in the sides of hyperbolic triangles. If T is a hyperbolic triangle with angles π/l, π/m, π/n, then the NEC group Γ^* generated by the reflections in the sides of T is called the *NEC triangle group* $\Gamma^*(l,m,n)$. Here each of l, m and n is a positive integer greater than one, and $1/l+1/m+1/n<1$. The group Γ^* has a presentation

$$\langle p,q,r \mid p^2=q^2=r^2=(pq)^l=(qr)^m=(rp)^n=1\rangle.$$

The subgroup Γ of Γ^* consisting of conformal isometries is called the *Fuchsian triangle group* $\Gamma[l,m,n]$, and it has a presentation

$$\langle x,y \mid x^l=y^m=(xy)^n=1\rangle.$$

See [5] and [7] for details.

Automorphisms of Riemann Surfaces. A compact Riemann surface X of genus $g > 1$ is conformally equivalent to $\mathcal{U}/K$, where $\mathcal{U}$ is the hyperbolic plane and K is a torsion free Fuchsian group. An *automorphism* of X is a conformal or anti-conformal homeomorphism of X onto itself. A finite group G acts as a group of automorphisms of X if and only if G is isomorphic to Δ/K, where Δ is an NEC group that contains K as a normal subgroup. Thus, there is an epimorphism from Δ to G with kernel K. Such an epimorphism is called *smooth*. All automorphisms of X form a group under composition of maps and we will denote it by $Aut^{\pm}X$ and the subgroup consisting of conformal automorphisms by $Aut^{+}X$. So, $Aut^{+}X$ and $Aut^{\pm}X$ are isomorphic to $N^{+}(K)/K$ and $N^{\pm}(K)/K$, respectively. Here $N^{+}(K)$ and $N^{\pm}(K)$ denote the normalizers of K in $PSL(2,\mathbb{R})$ and $PGL(2,\mathbb{R})$, respectively.

Maps and Regular Maps. A *map* $\mathcal{M}$ is an embedding of a finite graph $\mathcal{G}$ in a Riemann surface X such that the components of $X - \mathcal{G}$, which are called the *faces* of $\mathcal{M}$, are each homeomorphic to an open disc. In our maps we require $\mathcal{G}$ to be connected and every edge of $\mathcal{G}$ to have two vertices. We also require X to be orientable, compact, connected and without boundary. The *genus* of $\mathcal{M}$ is defined to be the genus of the underlying surface X. We define a *dart* to be a pair, consisting of a vertex v and an edge directed towards v. In our case, every edge will give two darts. Since X has no boundary, each dart has two sides, which are called *blades*. In Figure 1, a is a dart and b is a blade. $\mathcal{M}$ is said to be of *type* (m,n) if every face of $\mathcal{M}$ has n sides and m edges meet at every vertex. An *automorphism* of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence. All automorphisms of $\mathcal{M}$ form a group under composition of maps and we will denote it by $Aut^{\pm}\mathcal{M}$ and the subgroup consisting of conformal automorphisms by $Aut^{+}\mathcal{M}$. $\mathcal{M}$ is said to be *regular* if $Aut^{+}\mathcal{M}$ is transitive on the darts. If $\mathcal{M}$ admits an involution R that fixes the mid-point of an edge and interchanges the two darts without interchanging the two neighboring faces, then $\mathcal{M}$ is called *reflexible* and R is called a *reflection* of $\mathcal{M}$. $\mathcal{M}$ is reflexible if and only if $Aut^{\pm}\mathcal{M}$ is transitive on the blades.

Figure 1

From the map $\mathcal{M}$ we may derive a second map $\mathcal{M}^*$, called the *dual map* of $\mathcal{M}$, on X. If $\mathcal{M}$ has F faces, E edges and V vertices, then $\mathcal{M}^*$ has F vertices, one in the interior of each face of $\mathcal{M}$; E edges, one crossing each edge of $\mathcal{M}$; and V faces, one surrounding each vertex of $\mathcal{M}$. In our maps we require $\mathcal{M}^*$ to be regular. So the vertices and the face-centers of $\mathcal{M}^*$ are the face-centers and the vertices of $\mathcal{M}$, respectively. Clearly, their edge-centers are the same. The map $\mathcal{M}^*$ is of type (n, m) by construction.

In [3], it was shown that if $\mathcal{M}$ is a regular map of type (m, n) on a Riemann surface X, then X is uniformized by a normal subgroup of the Fuchsian triangle group $\Gamma[2, m, n]$. Most triangle groups are maximal and in this paper we will be concerned with maps $\mathcal{M}$ of type (m, n) such that the triangle group $\Gamma[2, m, n]$ is maximal. So, by [6], $\mathcal{M}$ will be reflexible and $Aut^{\pm}\mathcal{M}$ will be isomorphic to $Aut^{\pm}X$. The group $Aut^{\pm}\mathcal{M}$ lifts to the NEC triangle group $\Gamma^*(2, m, n)$. Γ^* has a presentation

$$\langle p, q, r \mid p^2 = q^2 = r^2 = (pq)^2 = (qr)^m = (rp)^n = 1\rangle, \qquad (2.1)$$

and $Aut^{\pm}\mathcal{M}$ is the image of Γ^* by a smooth homomorphism Φ. The group of conformal automorphisms of $\mathcal{M}$, $Aut^{+}\mathcal{M}$, can be generated by $\Phi(qr) = x$ and $\Phi(rp) = y$, which satisfy

$$x^m = y^n = (xy)^2 = 1,$$

where x is an anticlockwise rotation about a vertex and y is an anticlockwise rotation about a face-center.

3 Patterns of Mirrors

Let X be a Riemann surface of genus $g > 1$ and let $\mathcal{M}$ be a regular map of type (m, n) on X. Let F be a face of $\mathcal{M}$. If we join the center of F to the centers of the edges and the vertices surrounding F by geodesic arcs, we obtain a subdivision of F into $2n$ triangles. Each triangle has angles $\pi/2$, π/m and π/n, and will be called a $(2, m, n)$-*triangle*. In this way, we obtain a triangulation of X. Note that there are as many $(2, m, n)$-triangles as the order of $Aut^{\pm}\mathcal{M}$, and the reflexibility of $\mathcal{M}$ implies that $Aut^{\pm}\mathcal{M}$ is transitive on these triangles.

Let T be a $(2, m, n)$-triangle on X and let P, Q and R denote the reflections in the sides of T. Suppose that these reflections satisfy the relations

$$P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^m = (RP)^n = 1. \qquad (3.1)$$

The reflections P, Q and R generate $Aut^{\pm}\mathcal{M}$. However, the relations in (3.1) do not give a presentation if $1/m + 1/n < 1/2$ and hence to get a presentation for $Aut^{\pm}\mathcal{M}$ we need at least one more relation.

The *geometric points* of $\mathcal{M}$ are the vertices, the edge-centers and the face-centers. Following Coxeter [2], we will label all vertices with **0**, edge-centers with **1**, and face-centers with **2**. Then the pair of any two successive geometric points on a mirror is either **01**, **02** or **12** (or in reverse order). Let T^* be a $(2, m, n)$-triangle. Then we see that each corner of T^* either is a vertex, a face-center or an edge-center of $\mathcal{M}$. So we can label the corners of T^* with **0**, **1** and **2**. Then each of the pairs **01**, **02** and **12** corresponds to one of the sides of T^*. We will call the corresponding sides of T^* the **01**-*side*, the **02**-*side* and the **12**-*side*. See Figure 2.

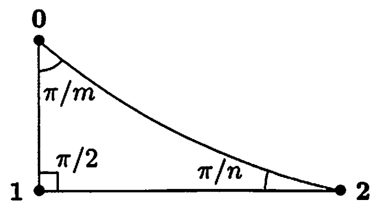

Figure 2

Let M be a mirror on X. Then M passes through some geometric points of $\mathcal{M}$. From the picture of $\mathcal{M}$, we can easily observe that these geometric points form a periodic sequence of the form

$$\underbrace{a_1a_2\ldots a_{k-1}a_k}_{1}\ \underbrace{a_1a_2\ldots a_{k-1}a_k}_{2}\ \ldots \underbrace{a_1a_2\ldots a_{k-1}a_k}_{N} \tag{3.2}$$

which we call the *pattern* of M, where $a_i \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ and $1 \le i \le k$. We call each repeated part $a_1a_2\ldots a_{k-1}a_k$ of (3.2) a *link* of the pattern. So, a link contains no repeats and the pattern of a mirror is a chain consisting of finite links. We define the *order* of a pattern to be the number of its links. We abbreviate the pattern (3.2) to $(a_1a_2\ldots a_{k-1}a_k)^N$. So, the abbreviated form of **01010101** is $(\mathbf{01})^4$.

As an example, we give the patterns corresponding to the spherical maps in Table 1, which also appear in [2].

Since the maps $(3, 4)$ and $(4, 3)$ are duals of each other, the patterns for the map $(4, 3)$ can be deduced from the patterns corresponding to the map $(3, 4)$ by interchanging **0**s and **2**s. The same discussion applies to the maps $(5, 3)$ and $(3, 5)$.

It is known that the regular maps of genus one are of types $(4, 4)$, $(3, 6)$ and $(6, 3)$. In this case, on the same torus there may be more than one maps with the same type. This is because in the Euclidean plane there exist similar polygons of different sizes. However, this situation does not

Table 1: Spherical Maps and Patterns

Map Type	Number of Mirrors	Pattern
$(3,3)$	6	**010212**
$(3,4)$	3	$(\mathbf{12})^4$
	6	$(\mathbf{0102})^2$
$(3,5)$	15	$(\mathbf{010212})^2$

occur in the sphere and the hyperbolic plane. Let $\mathcal{M}$ be a regular map of type $(4,4)$ and let R be a reflection of $\mathcal{M}$. It is not difficult see that the pattern of each mirror of R is either $(\mathbf{01})^k$, $(\mathbf{02})^k$, or $(\mathbf{12})^k$, where k is a positive integer.

If $\mathcal{M}$ is a regular map of type $(3,6)$, then by examining the picture of $\mathcal{M}$ we can see that the pattern of any mirror on the corresponding torus is either $(\mathbf{0102})^k$ or $(\mathbf{12})^{3k}$, where k is a positive integer. Since the maps $(6,3)$ and $(3,6)$ are duals of each other, the pattern of any mirror on the torus that underlies a regular map of type $(6,3)$ is either $(\mathbf{2120})^k$ or $(\mathbf{10})^{3k}$, where k is a positive integer.

Now let $\mathcal{M}$ be a regular map of type (m,n) on a Riemann surface of genus $g > 1$. We know that $Aut^{\pm}\mathcal{M}$ can be generated by three reflections P, Q, R obeying (3.1). The patterns of the mirrors of these reflections can easily be determined by examining the picture of $\mathcal{M}$ according to the parities of m and n. As an example, let m and n be odd. In this case every mirror has pattern $(\mathbf{010212})^k$, where k is a positive integer.

Similarly, the patterns for the other cases can easily be determined, and we give the results in Table 2.

Table 2: Patterns of Mirrors

Case	Reflections	Pattern Link
m and n odd	P, Q, R	**010212**
m odd n even	P	**12**
	Q, R	**0102**
m and n even	P	**12**
	Q	**01**
	R	**02**
m even n odd	P, R	**0212**
	Q	**01**

Notes on Table 2:

(*i*) In the table, we give only one link for each pattern.

(*ii*) In the case where m and n are odd, all reflections of $\mathcal{M}$ are conjugate in $Aut^{\pm}\mathcal{M}$ and any mirror on the surface has pattern $(\mathbf{010212})^k$, where k is a positive integer.

(*iii*) In the case where m is odd and n is even, Q and R are conjugate in $Aut^{\pm}\mathcal{M}$. However, P may not be conjugate to them and it has at least one mirror with pattern $(\mathbf{12})^k$. Similarly, each of Q and R has at least one mirror with pattern $(\mathbf{0102})^k$. In each case, k is a positive integer. Similar discussions apply to the other cases. (In the last section we will see that in the cases where m and n are not both odd, all reflections of $\mathcal{M}$ may be conjugate in $Aut^{\pm}\mathcal{M}$.)

As a result, the pattern of a mirror on a surface is obtained from one of the following links: **12**, **02**, **01**, **0102**, **0212**, **010212**.

4 Rotary Automorphisms

Let $\mathcal{M}$ be a regular map of type (m, n) on a Riemann surface X of genus $g > 1$, and let M be a mirror of a reflection of $\mathcal{M}$. Suppose that the order of the pattern of M is greater than one. Then there exist two conformal automorphisms of $\mathcal{M}$, each of which fixes M setwise and cyclically permutes the links of the pattern of M. These automorphisms are inverses of each other and they rotate M in opposite directions. We will call them the *rotary automorphisms* of M. So each rotary automorphism of M generates a maximal cyclic subgroup of $Aut^{+}\mathcal{M}$ whose order is the same as the order of the pattern of M. If the order of the pattern of M is one, i.e. the pattern of M consists of one link, then M has just one rotary automorphism, which is the identity.

We will now determine the possible rotary automorphisms of M in terms of the generators of $Aut^{\pm}\mathcal{M}$ in (3.1). Let M have pattern $(\mathbf{12})^k$, where k is a positive integer. As we saw earlier, this pattern occurs in the cases where n is even. Using geometrical techniques we can find one of the rotary automorphisms of M as follows. From the picture of $\mathcal{M}$ it follows that M passes through k faces of $\mathcal{M}$. Let us denote these faces by $F_1, F_2, \ldots, F_k$. Let us divide each of these faces into $2n$ $(2, m, n)$-triangles as in §3. Let T_1 be one of these triangles contained in F_1 such that its **12**-side lies on M. Let P, Q and R be the reflections in the sides of T_1. These three reflections generate $Aut^{\pm}\mathcal{M}$ and satisfy (3.1). The automorphism $(RP)^{(\frac{n}{2}-1)}R$ of $\mathcal{M}$ reflects T_1 onto another triangle, say T_2, in F_1 such that the **12**-side of T_2 lies on M, and Q reflects T_2 onto another triangle, say T_3, in F_2. Therefore,

$Q(RP)^{(\frac{n}{2}-1)}R$ maps T_1 to T_3 and hence F_1 to F_2. It is not difficult to see that $Q(RP)^{(\frac{n}{2}-1)}R$ has order k and cyclically permutes the faces F_i $(i = 1, \ldots, k)$. So, it fixes M setwise and cyclically permutes the links of the pattern of M. Therefore, $Q(RP)^{(\frac{n}{2}-1)}R$ is a rotary automorphism for M. We illustrate this in Figure 3, where $m = 3$ and $n = 8$.

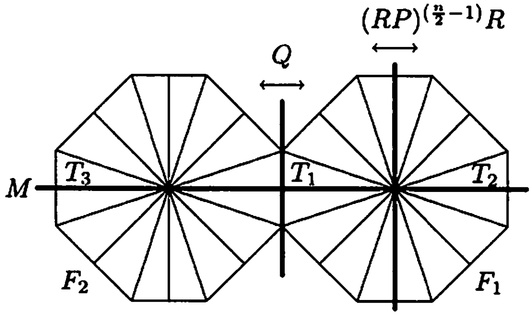

Figure 3

Remark 4.1 Note that in the above discussion each of the reflections $(RP)^{(\frac{n}{2}-1)}R$ and Q has a mirror orthogonal to M, and these reflections have been chosen in order that $Q(RP)^{(\frac{n}{2}-1)}R$ has order k. If we had chosen another mirror M^* that has the same pattern as M, then we would have seen that each rotary automorphism of M^* is conjugate to $Q(RP)^{(\frac{n}{2}-1)}R$. So, each pattern corresponds to a conjugacy class of rotary automorphisms.

In the same way, we can find the rotary automorphisms corresponding to the other patterns, and we give the results in Table 3. In the table, the rotary automorphism in each line represents the conjugacy class of the rotary automorphisms corresponding to the given pattern.

Example 4.1 Let X be Klein's surface, i.e. the Riemann surface of genus 3 with 168 conformal automorphisms. It is known that X underlies a regular map $\mathcal{M}$ of type $(3, 7)$ and $Aut^{\pm}\mathcal{M}$ is isomorphic to $PGL(2, 7)$, which has a presentation

$$\langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^3 = (RP)^7 = (PRQ)^8 = 1 \rangle.$$

Let M be a mirror on X. If we choose P, Q and R as the reflections in the sides of a convenient $(2, 3, 7)$-triangle on X, then M has a rotary automorphism $S = QRPRQ(PR)^3Q(RP)^3$, which has order 3. Since the order of a rotary automorphism is equal to the order of the corresponding

Table 3: Patterns and Rotary Automorphisms

Case	Pattern Link	Rotary Automorphism
1	**12**	$(RP)^{(\frac{n}{2}-1)}RQ$
2	**02**	$(PR)^{(\frac{n}{2}-1)}P(QR)^{(\frac{m}{2}-1)}Q$
3	**01**	$(RQ)^{(\frac{m}{2}-1)}RP$
4	**0102**	$(PR)^{(\frac{n}{2}-1)}P(QR)^{\frac{m-1}{2}}P(RQ)^{\frac{m-1}{2}}$
5	**010212**	$(QR)^{\frac{m-1}{2}}P(RQ)^{\frac{m-1}{2}}(PR)^{\frac{n-1}{2}}Q(RP)^{\frac{n-1}{2}}$
6	**0212**	$(PR)^{\frac{n-1}{2}}Q(RP)^{\frac{n-1}{2}}(QR)^{(\frac{m}{2}-1)}Q$

pattern, we see that M has pattern $(\mathbf{010212})^3$. This result also follows from Klein [4]. When the order of $Aut^{\pm}\mathcal{M}$ is too large, the order of S can be determined by using a computer algebra system such as MAGMA.

Remark 4.2 In the above example, with the aid of MAGMA we can also observe that

$$\langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^3 = (RP)^7 = S^3 = 1 \rangle$$

is a presentation for $PGL(2,7)$, where $S = QRPRQ(PR)^3Q(RP)^3$. Thus, $S^3 = 1$ is a defining relation and so we have another presentation for $PGL(2,7)$.

5 Conjugacy Classes of Reflections

Let $\mathcal{M}$ be a regular map of type (m, n) on a Riemann surface of genus $g > 1$, and let P, Q and R be the generators of $Aut^{\pm}\mathcal{M}$ in (3.1). Using dihedral groups we can see that if m and n are odd, then P, Q and R are conjugate in $Aut^{\pm}\mathcal{M}$. It follows from [1] that if m and n are not both odd, then every regular map of genus two has two or three conjugacy classes of reflections. A natural question is whether this situation occurs for any regular map of genus $g > 1$. In this section, we will see that it does not occur always. Also, in the cases where m and n are not both odd, we will give some necessary conditions under which all reflections of a regular map are conjugate.

Lemma 5.1 *Let $\mathcal{M}$ be a regular map of type (m, n) on a Riemann surface of genus $g > 1$ and let α and β be two reflections of $\mathcal{M}$. Suppose that A and B are two mirrors fixed by α and β, respectively. If A and B both contain a pair* $\mathbf{ij}$ $(\mathbf{i}, \mathbf{j} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\})$ *in common, then A and B have the same length and pattern. Furthermore, α and β are conjugate in $Aut^{\pm}\mathcal{M}$.*

Proof. The pair **ij** is either **01**, **02** or **12** (or in reverse order). Let A and B contain a pair **01** in common. This means that each of A and B contains the **01**-side of at least one $(2, m, n)$-triangle. Let T_A and T_B be two $(2, m, n)$-triangles whose **01**-sides lie on A and B respectively. Since $\mathcal{M}$ is regular and reflexible, $Aut^{\pm}\mathcal{M}$ is transitive on the $(2, m, n)$-triangles and so there exists an automorphism $f \in Aut^{\pm}\mathcal{M}$ such that $f(T_A) = T_B$. It follows that $f(A) = B$ and as f is an isometry, A and B have the same length. Since f maps the vertices (resp. edge-centers, face-centers) to vertices (resp. edge-centers, face-centers), A and B have the same pattern. Similarly, we can show that if **ij** is **02** or **12**, then A and B have the same length and pattern.

Let f be the automorphism above and x be a point on A. Then we see that $f(x) \in B$, $\alpha(x) = x$ and $\beta(f(x)) = f(x)$. It follows that

$$(f^{-1}\beta f)(x) = f^{-1}(\beta(f(x)) = f^{-1}(f(x)) = x,$$

and so every point on A is fixed by α and $f^{-1}\beta f$. Thus, $\alpha f^{-1}\beta f$ pointwise fixes A. As $\alpha f^{-1}\beta f$ is conformal, it must be the identity and hence $\alpha = f^{-1}\beta f$. So, α and β are conjugate in $Aut^{\pm}\mathcal{M}$. □

Proposition 5.1 *Let $\mathcal{M}$ be a regular map on a Riemann surface X of genus $g > 1$, and let M_1 and M_2 be two mirrors on X with different patterns. Suppose that R_1 and R_2 are the reflections fixing M_1 and M_2, respectively. Then R_1 and R_2 are conjugate in $Aut^{\pm}\mathcal{M}$ if and only if each of them has at least two mirrors that have the same patterns as M_1 and M_2.*

Proof. Let R_1 and R_2 be conjugate in $Aut^{\pm}\mathcal{M}$. Then there exists $\alpha \in Aut^{\pm}\mathcal{M}$ such that $R_1 = \alpha^{-1}R_2\alpha$. Since

$$M_1 = R_1(M_1) = (\alpha^{-1}R_2\alpha)(M_1) = \alpha^{-1}(R_2[\alpha(M_1)]),$$

it follows that $\alpha(M_1) = R_2(\alpha(M_1))$, and hence $\alpha(M_1)$ is fixed by R_2 and it has the same pattern as M_1. So, R_2 has at least two mirrors that have the same patterns as M_1 and M_2. Similarly, the mirror $\alpha^{-1}(M_2)$ is fixed by R_1 and has the same pattern as M_2. So, R_1 has at least two mirrors that have the same patterns as M_1 and M_2.

The converse follows from Lemma 5.1. □

We now give the following corollaries. Their proofs follow easily from Lemma 5.1, Proposition 5.1 and the discussions in §3.

Corollary 5.1 *Let $\mathcal{M}$ be a regular map on a Riemann surface X of genus $g > 1$. Then,* (*i*) *If M_1 and M_2 are two mirrors on X with the same pattern, then the reflections fixing them are conjugate in $Aut^{\pm}\mathcal{M}$.*

(*ii*) *There are at most three classes of mirrors on X such that the mirrors in each class have the same length and pattern.*

(*iii*) *$Aut^{\pm}\mathcal{M}$ is transitive on the mirrors with the same pattern.* □

Corollary 5.2 *Let $\mathcal{M}$ be a regular map of type (m,n) of genus $g > 1$. Let all reflections of $\mathcal{M}$ be conjugate in $Aut^{\pm}\mathcal{M}$. Then,*

(*i*) *If m and n are both even, then every reflection of $\mathcal{M}$ has at least three mirrors.*

(*ii*) *If m and n have different parities, then every reflection of $\mathcal{M}$ has at least two mirrors.* □

Theorem 5.1 *Let $\mathcal{M}$ be a regular map of type (m,n) on a Riemann surface X of genus $g > 1$, where m is odd and n is even. Let P, Q and R be the generators of $Aut^{\pm}\mathcal{M}$ in (3.1). If $n \equiv 0 \pmod 4$ and the order of $(RP)^{(\frac{n}{2}-1)}RQ$ is odd, then all reflections of $\mathcal{M}$ are conjugate in $Aut^{\pm}\mathcal{M}$.*

Proof. Since m is odd, Q and R are conjugate in $Aut^{\pm}\mathcal{M}$. We will now show that P and Q are conjugate. Let T be a $(2,m,n)$-triangle on X and let M be the mirror that contains the **12**-side of T. We can consider P, Q and R as the reflections in the sides of T. Let M_Q be the mirror that contains the **01**-side of T. Clearly, it is fixed by Q. Since n is even there is a mirror M_{P^*} that intersects M orthogonally at the corner **2** of T and is fixed by the reflection $P^* = (RP)^{(\frac{n}{2}-1)}R$. (See Figure 4, where $m = 3$ and $n = 8$). P^* is conjugate to P if $(n/2) - 1$ is odd and to R if $(n/2) - 1$ is even. Since $n \equiv 0 \pmod 4$, $(n/2) - 1$ is odd and so P^* is conjugate to P.

The reflections P^* and Q generate a dihedral group D_N, where N is the order of $P^*Q = (RP)^{(\frac{n}{2}-1)}RQ$. By hypothesis, N is odd and hence P^* and Q are conjugate in $Aut^{\pm}\mathcal{M}$. Since P^* and P are conjugate we see that P and Q are conjugate and so all reflections of $\mathcal{M}$ are conjugate in $Aut^{\pm}\mathcal{M}$. □

In the case where m is even and n is odd we give the following theorem, which can be proved in the same way.

Theorem 5.2 *Let $\mathcal{M}$ be a regular map of type (m,n) on a Riemann surface X of genus $g > 1$, where m is even and n is odd. Let P, Q and R be the generators of $Aut^{\pm}\mathcal{M}$ in (3.1). If $m \equiv 0 \pmod 4$ and the order of $(RQ)^{(\frac{m}{2}-1)}RP$ is odd, then all reflections of $\mathcal{M}$ are conjugate in $Aut^{\pm}\mathcal{M}$.*
□

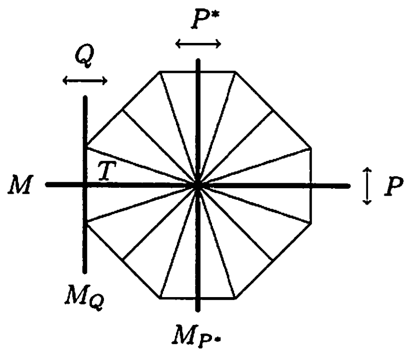

Figure 4

Theorem 5.3 *Let $\mathcal{M}$ be a regular map of type (m,n) on a Riemann surface X of genus $g > 1$, where m and n are even. Let P, Q and R be the generators of $Aut^{\pm}\mathcal{M}$ in (3.1). If the orders of $(RP)^{(\frac{n}{2}-1)}RQ$ and $(RQ)^{(\frac{m}{2}-1)}RP$ are odd, then in each case below, P, Q and R are conjugate in $Aut^{\pm}\mathcal{M}$:*

(*i*) $m \equiv 0 \pmod 4$, $n \equiv 2 \pmod 4$,

(*ii*) $m \equiv 2 \pmod 4$, $n \equiv 0 \pmod 4$,

(*iii*) $m \equiv 2 \pmod 4$, $n \equiv 2 \pmod 4$.

Proof. *Case*(*i*): $m \equiv 0 \pmod 4$, $n \equiv 2 \pmod 4$. Let T be a $(2,m,n)$-triangle on X and let M be the mirror that contains the **01**-side of T. We can consider P, Q and R as the reflections in the sides of T. So, M is fixed by Q. (See Figure 5, where $m = 4$ and $n = 6$).

Since m is even there is a mirror M_{Q^*} that intersects M orthogonally at the corner **0** of T and is fixed by the reflection $Q^* = (RQ)^{(\frac{m}{2}-1)}R$. This reflection is conjugate to Q if $(m/2)-1$ is odd and to R if $(m/2)-1$ is even. Since $m \equiv 0 \pmod 4$, $(m/2)-1$ is odd and so Q^* is conjugate to Q. Similarly, P fixes a mirror M_P, which is perpendicular to M at the corner **1** of T. By hypothesis, the order of PQ^* is odd and hence P and Q^* are conjugate in $Aut^{\pm}\mathcal{M}$. We know that Q^* and Q are conjugate and hence P and Q are conjugate in $Aut^{\pm}\mathcal{M}$.

Now consider the mirror M_P of P. The mirror M of Q is perpendicular to M_P at the corner **1** of T. Since n is even, there is a mirror M_{R^*} that intersects M_P at the corner **2** of T and is fixed by the reflection $R^* = (RP)^{(\frac{n}{2}-1)}R$. R^* is conjugate to R if $(n/2)-1$ is odd and to P if $(n/2)-1$

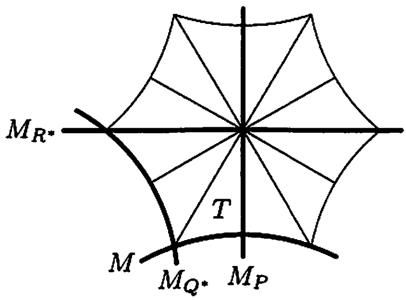

Figure 5

is even. Since $n \equiv 2 \pmod 4$, $(n/2)-1$ is odd and so R^* is conjugate to R. By hypothesis, the order of R^*Q is odd and so R^* and Q are conjugate in $Aut^{\pm}\mathcal{M}$. Since R^* is conjugate to R we see that R is conjugate to Q and therefore all reflections of $\mathcal{M}$ are conjugate in $Aut^{\pm}\mathcal{M}$.

Cases *(ii)* and *(iii)* can be proved similarly. □

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