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Upon the existence of a removable circuit in a simple 3-connected eulerian graph.

We use Tutte's theory of cleavage units or 3-blocks to prove a result on sufficient conditions for a simple 3-connected eulerian graph to contain a circuit from which we can delete the edges and leave a graph which is a nodally 3-connected graph.
Keywords: 3-blocks, cleavage units, removable circuit.

Notation

Let G be a graph and let J be a subgraph of G, we shall write $J \subseteq G$.
$d_G(v)$ is the degree of v in G.
$\delta(G)$ is the minimum degree in G.
$\tau(G)$ denotes the set of vertices of degree 1 in a tree G.
A Θ-graph is a tree T for which $\tau(T) \leq 3$.
$W(J, K)$ denotes the set of vertices of attachment of a subgraph K in a subgraph J of G.
$G[X]$ denotes the subgraph induced by $X \subseteq V(G)$.
H_J^c is the complement of H in J.
$K_{2,3}^+$ is the graph obtained from the complete bipartite graph $K_{2,3}$ by adding an edge between the vertices of the first partition.
$V_3(G)$ denotes the set of vertices with degree at least three.
(H, K) denotes either a separation or a cleavage.
$\mathcal{R}(G) = \{R_1, \ldots, R_s\}$ denotes the set of cleavage units of a 2-connected graph G.
For a 2-connected graph G, G^a denotes the graph that results after adding all possible virtual edges.

Introduction

All graphs will be finite and loopless. For a graph G, $\delta(G)$ or δ will denote the **minimum degree** in G. A vertex $v \in V(G)$ is a **node** vertex if $d_G(v) \geq 3$. Let $V_3(G)$ denote the set of all node vertices of G. If G is a connected graph then a **vertical** n**-separation** in G is a pair (J, K) of edge disjoint subgraphs for which:

(i) $G = J \cup K$;

(ii) $|V(J) \cap V(K)| = n$;

(iii) $V(J) \backslash V(K) \neq \emptyset$ and $V(K) \backslash V(J) \neq \emptyset$.

A graph G on at least $m + 1$ vertices is said to be m**-connected** if it does not have any vertical n-separations for $n < m$. Note that this is not quite the same terminology as in [1]. A **block graph** is a connected graph that does not contain any 1-separations. Let G be a block graph. A **subdivision** of an edge $e = xy \in E(G)$ is the result of deleting e and then adding a new vertex z to $G - e$ with edges xz and yz. A **subdivision of** G is a graph that can be obtained from G by a sequence of subdivisions. A **nodal** n**-separation** is a pair of edge disjoint subgraphs (J, K) which satisfy the following properties:

(i) $G = J \cup K$;

(ii) $|V(J) \cap V(K)| = n$;

(iii) both $V_3(J) \backslash V_3(K) \neq \emptyset$ and $V_3(K) \backslash V_3(J) \neq \emptyset$.

We shall say that a block graph with at least two nodes is **nodally** m**-connected** if there do not exist any nodal n-separations, for $n < m$, in any subdivision of G. In Figure 1 are three nodally 3-connected graphs.

The following question is due to A. Hobbs [4], 'Does every 2-connected eulerian graph with minimum degree at least 4 contain a circuit C for which $G - E(C)$ is a 2-connected graph?' The answer to this question is no. The counter-example of Figure 2 was discovered by N. Robinson and independently by B. Jackson [5]. In [5] Jackson proves Theorem 1.

Theorem 1 *Let G be a simple 2-connected graph with $\delta \geq k \geq 4$ and let $e \in E(G)$. Then there exists a circuit C in $G - e$ such that $|E(C)| \geq k - 1$ and $G - E(C)$ is 2-connected.*

C. Thomassen and B. Toft [8] proved Theorem 2 in a paper about non-separating induced circuits in graphs.

Theorem 2 *Let G be a simple 2-connected graph with $\delta \geq 4$. Then G contains an induced circuit C such that $G - V(C)$ is connected and $G - E(C)$ is 2-connected.*

Theorem 3 is an earlier result due to W. Mader [7].

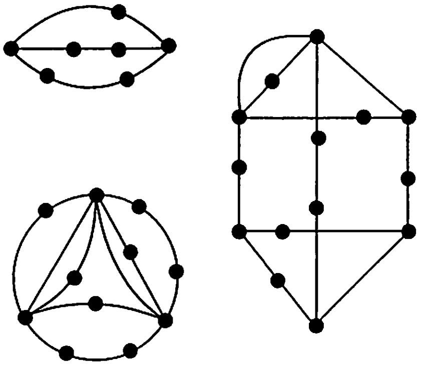

Figure 1: Three nodally 3-connected graphs.

Theorem 3 *Let G be a simple k-connected graph for which $\delta \geq k + 2$. Then G contains a circuit C such that $G - E(C)$ is a k-connected graph.*

Jackson's result has been strengthened by M. Lemos and J. Oxley [6].

Theorem 4 *Let G be a simple 2-connected graph, H be a block that is a subgraph of G, and k an integer exceeding three. Suppose that $d(v) \geq k$ for all $v \in V(G) - V(H)$. Then either*

(i) $G - E(C)$ is 2-connected for every circuit C that is edge disjoint from H; or

(ii) G has a circuit C that is edge disjoint from H such that $G - E(C)$ is 2-connected and when $k \geq 5$ the length of C is at least $k + 1$.

We shall make use of the following Corollary.

Corollary 5 *Let G be a simple 2-connected graph with $\delta(G) \geq 4$. Then there exists a circuit C in G such that $G - E(C)$ is 2-connected and not a circuit.*

Proof 5 *By Theorem 4, there exists a circuit C in G such that $G - E(C)$ is 2-connected. Put $H = G - E(C)$. Suppose that H is a circuit and that $V(H) = V(C) = V(G)$. Since G is simple and $\delta(G) \geq 4$, $|V(G)| \geq 5$ and thus there exist two chords of C, $D_1 = G[\{u, v\}]$ and $D_2 = G[\{x, y\}]$, of G such that $x \notin V(D_1)$. Let $H[u, v]$ be the uv-path in H that does not include x. Then $H' = H[u, v] + uv$ is a 2-connected subgraph of G. Since $\delta(G - E(H')) \geq 2$ there exists a circuit C' that is edge disjoint from H' in G and, by Theorem 4, such that $G - E(C')$ is 2-connected. Since $d_G(x) = 4$ and $H' \subset G - E(C')$, $G - E(C')$ is not a circuit.*

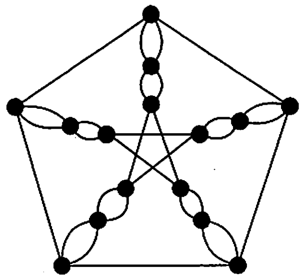

Figure 2:

Let $H \subseteq G$. Let $W(G,H)$ denote the set of **vertices of attachment** of H in G, $W(G,H) = \{v \in V(H) | \exists e = uv \in E(G) - E(H)\}$. For $X \subseteq V(G)$, let $G[X]$ be the subgraph **induced** by X, that is with vertex set X and $E(G[X]) = \{e = uv \in E(G)$ and $u, v \in X\}$. Let H_G^c or H^c be the **complement** of H in G, that is the subgraph with $V(H_G^c) = (V(G) \backslash V(H)) \cup W(G,H)$ and $E(H_G^c) = E(G) \backslash E(H)$. We shall say that J is an H**-attached** subgraph of G, for $H \subseteq G$, if $W(G,J) \subseteq V(H)$. We shall say that J is a H**-bridge** if the following conditions are satisfied:

(i) J is not a subgraph of H.

(ii) J is an H-attached subgraph.

(iii) no proper subgraph of J satisfies (i) and (ii).

Let a $K_{2,3}^+$**-graph** be the graph obtained from the complete bipartite graph $K_{2,3}$ by adding an edge between the vertices of the first partition. Let $J_1, J_2 \cong K_{2,3}^+$. For $i = 1, 2$, let $x_i, y_i, z_i \in V(J_i)$ such that $d_{J_i}(x_i) = d_{J_i}(y_i) = d_{J_i}(z_i) = 2$. Let a $K_{2,3}^+$**-pair graph** be the graph, J, constructed from J_1 and J_2 by identifying z_1 and z_2 to a single vertex z. Suppose that J occurs as a subgraph of G. Then J is called a $K_{2,3}^+$-pair bridge on $\{x_1, y_1, x_2, y_2\}$ if $W(G,J) = \{x_1, y_1, x_2, y_2\}$, x_1, y_1, x_2, y_2 are all distinct and $x_1y_1 \notin E(G)$ and $x_2y_2 \notin E(G)$. We shall say that G contains a $K_{2,3}^+$-pair bridge if there exists a $K_{2,3}^+$-pair bridge on $\{x_1, y_1, x_2, y_2\}$ for some $x_1, x_2, y_1, y_2 \in V(G)$.

In this paper we prove Theorem 6.

Theorem 6 *Let G be a simple 3-connected eulerian graph. Suppose that G does not contain a $K_{2,3}^+$-pair bridge. Then there exists a circuit C in G such that $G - E(C)$ is a nodally 3-connected graph.*

The next result is a Corollary of two results from [10] (Theorem III.9 and Theorem III.10). For a tree T, let $\tau(T) = \{v \in V(T) | d_T(v) = 1\}$.

Corollary 7 *Let H be a block subgraph of G and let X be a path subgraph or a block subgraph such that X is edge disjoint from H in G, $|V(X) \cap V(H)| \geq 2$ and if X is a path subgraph then $\tau(X) \subseteq V(X) \cap V(H)$. Then $H \cup X$ is a block subgraph of G.*

A **branch** B of a graph G is a path subgraph in G for which the node vertices of G in P are precisely the end vertices of P. Let v be a node vertex in a graph G. Let L_1,L_2,...,L_s be the set of branches of G that have v as an end vertex. Let $u_i \in \tau(L_i)\backslash\{v\}$, for $1 \leq i \leq s$. Then $G -_3 v$ is the graph $G - \bigcup_{i=1}^{s}(L_i\backslash\{u_i\})$.

Theorem 8 and Corollary 9 are from [1], Problem 2.1.6.

Theorem 8 *If G is a tree with maximum degree $\Delta \geq k$, then G has at least k-vertices of degree one.*

Corollary 9 *A non-trivial tree has at least two vertices of degree 1.*

Theorem 10 *Let G be a graph and let H be a 2-connected subgraph of G. Let $\{x, y\} \subseteq V(H)$ be such that there do not exist any $\{x, y\}$-branches in H. Let $d_H(x) \geq 2$, $d_H(y) \geq 2$ and let $d_H(u)$ be even for each $u \in V(H)\backslash\{x, y\}$. Let $a \in V(H)$ such that $a \notin \{x, y\}$. If $H' = H \cup B$, where B is an $\{x, y\}$-branch, is nodally 3-connected and $|V_3(H')| \geq 5$ then there exists an xy-path P in H such that $a \in V(P)$ and there exists a circuit in $H - E(P)$.*

Proof 10 *Suppose that the Theorem is false and let $H \subseteq G$ be a counter-example. Let $v \in V_3(H')\backslash\{x, y, a\}$. Because $H' -_3 v$ is 2-connected there exists an xy-path P in $H -_3 v$ such that $a \in V(P)$. Since $H - E(P)$ contains no circuits $H - E(P)$ is a forest. However, by Theorem 8, the component of $H - E(P)$ to which v belongs has at least four vertices of degree 1, a contradiction as each vertex of $V(H - E(P))\backslash\{x, y\}$ has even or zero degree.*

Let $[x, y]$ denote the empty graph with two vertices. Let G be a connected graph. A **2-separation** at $[x, y]$ in G is a pair (J, K) of edge disjoint subgraphs which satisfy the following properties:

(i) $G = J \cup K$;

(ii) $V(J) \cap V(K) = \{x, y\}$;

(iii) both $|E(J)| \geq 2$ and $|E(K)| \geq 2$.

A **split** at $[x, y]$ in G is a pair (J, K) of edge disjoint subgraphs which satisfy properties (i) and (ii) of a 2-separation and

(iii') both $|E(J)| \geq 1$ and $|E(K)| \geq 1$.

A **cleavage** at xy is a 2-separation (J, K) at $[x, y]$ for which not both J and K are split at $[x, y]$ and not both J and K are separable. We call $[x, y]$ a set of **hinge** vertices. Let (J, K) be a cleavage at $[x, y]$. The **cleavage graphs** at xy are the graphs $J + xy$ and $K + xy$. The added edge xy is called a **virtual edge**. The following Theorem is from [10], Theorem IV.21.

Theorem 11 *The cleavage graphs at xy are block graphs.*

The **cleavage units** of G are the minimal cleavage graphs obtained by recursively constructing cleavage graphs from cleavage graphs. If G is 3-connected then G has just one cleavage unit, itself. Figure 3 is an example of a 2-connected graph and its cleavage units.

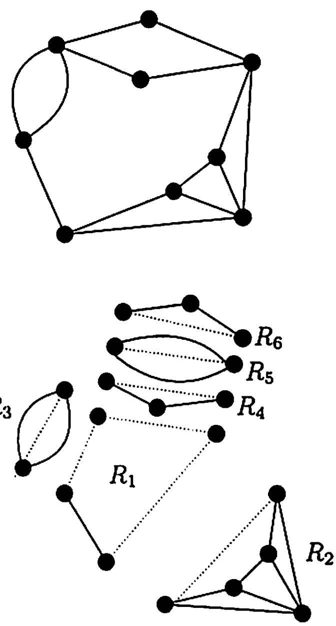

Figure 3: A 2-connected graph G and the cleavage units of G.

The original graph G with all possible virtual edges added is called the **augmented graph** and is denoted G^a. The cleavage graphs and cleavage units of G are subgraphs of G^a.

The following Theorem is proved in [9], Theorem 11.63, see also [10].

Theorem 12 *Let G be a 2-connected graph with at least three edges. Then each cleavage unit of G is either a simple 3-connected graph, a bond graph with at least three edges or a circuit graph with at least three edges. Each edge of G belongs to just one cleavage unit, and each virtual edge of G^a to exactly two.*

Let $\mathcal{R}(G) = \{R_1, \ldots, R_s\}$ be the set of cleavage units of G. By Theorem IV.29 and Theorem IV.44, from [10], there exists a tree T for which $V(T) = \{r_i$ iff $R_i \in \mathcal{R}(G)\}$ and $r_i r_j \in E(T)$ if and only if there exists a virtual edge common to both R_i

and R_j. We call T the **cleavage unit tree** of G. Let (J, K) be a cleavage at $[x, y]$ in G. Let R_i, $R_j \in \mathcal{R}(G)$ be such that R_i and R_j have the virtual edge $e = xy$ in common. Then $e = r_i r_j \in E(T)$. It follows from the recursive definition of the cleavage units of G that the cleavage unit trees of the cleavage graphs $J + e$ and $K + e$ are precisely the components of $T - e$, and moreover, that any subtree of T is the cleavage unit tree of a cleavage graph of G. For a graph G, which is 2-connected but not 3-connected, let the **end cleavage units** of G be the cleavage units of G that have precisely one virtual edge. These cleavage units correspond to the vertices of degree 1 in the cleavage unit tree of G.

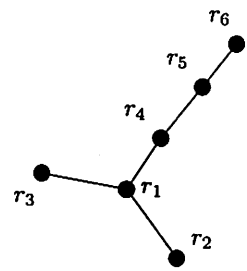

Figure 4: The cleavage unit tree of the graph of Figure 3.

Theorem 13 is proved in [10], Theorem IV.20.

Theorem 13 *Let a 2-connected graph G be the union of two edge disjoint subgraphs J and K with just two vertices, b and c in common. Let J and K have each at least one edge. Then J and K are both connected graphs.*

Theorem 14 *Let G be a 2-connected graph and let H be a 2-connected subgraph of G. If R_i is a cleavage unit of H which is not a circuit graph then $V(R_i) \subseteq V(R_i')$ for a cleavage unit R_i' of G.*

Proof 14 *Suppose that there exists a cleavage (J_1, J_2) at $[x, y]$ in G such that $V(R_i) \cap (V(J_1) \backslash \{x, y\}) \neq \emptyset$ and $V(R_i) \cap (V(J_2) \backslash \{x, y\}) \neq \emptyset$. By Theorem 12, R_i is either a bond graph with at least three edges or a simple 3-connected graph. Thus, $[x, y]$ is not a separating set in H^a and there exist vertices $a \in (V(R_i) \cap V(J_1)) \backslash \{x, y\}$ and $b \in (V(R_i) \cap V(J_2)) \backslash \{x, y\}$ that are joined by a path P in $R_i - \{x, y\}$. Since P is not a path in $G - \{x, y\}$ there exists an $a' \in V(J_1) \cap V(R_i)$ and a $b' \in V(J_2) \cap V(R_i)$ such that $a'b' \in E(P)$ is a virtual edge of H^a. Then there exists a cleavage (L_1, L_2) at $[a', b']$ in H which can be labelled such that $V(R_i) \subseteq V(L_1)$. By Theorem 13, since H is 2-connected, there exists an $a'b'$-path Q in L_2. But now, Q is an $a'b'$-path in $G - \{x, y\}$, a contradiction.*

The following Theorem can be found in [10], Theorem IV.45.

Theorem 15 *Let R_i and R_j be cleavage units of G which are adjacent in the cleavage unit tree of G. Then they are not both bond graphs and not both circuit graphs.*

Let G be a 2-connected graph and let $e = xy \in E(G)$. The cleavage unit to which e belongs is called the **leading 3-block** with respect to e. We can classify graphs that are 2-connected but not 3-connected with respect to e as follows. Let H be the $\{x, y\}$-bridge for which $V(H) = \{x, y\}$ and $E(H) = \{e\}$. If G has two or more $\{x, y\}$-bridges other than H then G is of Type I and the leading 3-block with respect to e is an r-bond with at least three edges. If G has just one $\{x, y\}$-bridge other than H and that bridge is separable then G is of Type II and the leading 3-block with respect to e is a circuit graph with at least three edges. If G has just one $\{x, y\}$-bridge other than H and that bridge is not separable then G is of Type III and the leading 3-block with respect to e is a simple 3-connected graph.

A connected graph is called a **string of blocks** if, for some integer $k \geq 2$, we can enumerate its blocks as $B_1, B_2, \ldots, B_k$ and its cut vertices as $v_1, v_2 \ldots v_{k-1}$ so that the following condition holds. The cut vertex v_j belongs to B_j and B_{j+1} but to no other block, for each $1 \leq j \leq k-1$. If an edge e is added with one end in $V(B_1)\backslash\{v_1\}$ and one end in $V(B_k)\backslash\{v_k\}$ then e is said to close the string of blocks. If G is of Type II with respect to e then $G - e$ is a string of block closed by e.

Theorem 16 is from [10], Theorem IV.30.

Theorem 16 *Let G be a 2-connected graph. Let x be a vertex of G that belongs to two distinct cleavage units R_i and R_j of G. Then x is an end vertex in G of each edge in the unique path in the cleavage unit tree T of G joining r_i to r_j.*

Theorem 17 *Let G be a 2-connected graph and let T be the cleavage unit tree of G. Let $r_i, r_j, r_k \in V(T)$ such that r_i, r_j, r_k is a path in T. If $[x, y]$ is a hinge common to both R_i and R_k then R_j is a bond graph on $\{x, y\}$ with at least three edges.*

Proof 17 *Let $e_1 = r_i r_j \in E(T)$ and $e_2 = r_j r_k \in E(T)$. Let T_i and $T - T_i$ be the two components of $T - e_1$ such that $r_i \in V(T_i)$ and $r_j \in V(T - T_i)$. Let T_j and T_k be the two components of $(T - T_i) - e_2$ labelled such that $r_j \in V(T_j)$ and $r_k \in V(T_k)$. Let G_j be the cleavage graph of G whose cleavage unit tree is T_j. By Theorem 16, $e_1 = xy = e_2$. Hence, G_j is of Type I with respect to e_1 and R_j is a bond graph, as required.*

In Theorem 18, there is a slight abuse of notation, in that a cleavage unit R_s is changed to a new cleavage unit R'_s by replacing an edge e with a new virtual edge. We use R_s to denote both cleavage units and this is the practice throughout the paper.

Theorem 18 *Let G be a graph and let H be a 2-connected subgraph of G. Let $\mathcal{R}(H) = \{R_1, \ldots, R_s\}$ and let T be the cleavage unit tree of H. Let R_s be the leading 3-block with respect to an edge $e = xy$. Let H' be the graph that results from subdividing e in H, one or more times, to create a branch B. Let T' be the cleavage unit tree of H'. If H is of Type I or Type III with respect to e then $\mathcal{R}(H') = \mathcal{R}(H) \cup R'_{s+1}$, where $R'_{s+1} = B + xy$ is a circuit graph, $V(T') = V(T) \cup \{r'_{s+1}\}$ and $E(T') =$*

$E(T) \cup \{r_s r'_{s+1}\}$. If H is of Type II with respect to e, then $\mathcal{R}(H') = (\mathcal{R}(H) \backslash R_s) \cup R'_s$, where R'_s is a circuit graph for which $V(R_s) \subseteq V(R'_s)$, and $T \cong T'$.

Proof 18 *Proof is by induction on $s = |\mathcal{R}(H)|$. Suppose that $s = 1$. If H is of Type I or Type III with respect to e, then $H - e$ is 2-connected, and because B is not split at $\{x, y\}$, $(B, H - e)$ is a cleavage at xy. Therefore, the cleavage units of H' are H and $B + xy$ and the result holds. If H is a circuit graph then $(H - e) \cup B$ is a circuit graph and the result holds. Hence, suppose that the Theorem is true for all 2-connected graphs with less than $s \geq 2$ cleavage units. By Corollary 9, we can choose $r_i \in V(T)$ such that $d_T(r_i) = 1$ and $i \neq s$. Let $e_i \in E(T)$ be incident with r_i and let $e_i = x_i y_i$ be the virtual edge of G^a. Let (J, K) be the cleavage at $[x_i, y_i]$ in G such that $K = R_i - e_i$. By the induction hypothesis, the result holds for the cleavage graph $J + e_i$ and thus for H. The result now holds for all s by induction.*

Theorem 19 *Let G be a 2-connected graph and let T be the cleavage unit tree of G. Let $e_i = r_i r_j \in E(T)$. Let (J, K) be the cleavage of G at $e_i = x_i y_i$. Put $G' = G + e$ and let T' be the cleavage unit tree of G'. Suppose that $e_i \notin E(T')$. Then one end vertex of e belongs to $V(J) \backslash \{x_i, y_i\}$ and one end vertex of e belongs to $V(K) \backslash \{x_i, y_i\}$.*

Proof 19 *Suppose that both end vertices of e belong to either $V(J)$ or $V(K)$, say $V(J)$. If $e \neq x_i y_i$ then $(J + e, K)$ is a cleavage at $[x_i, y_i]$ in G', a contradiction. Suppose that $e = x_i y_i$. If K is not split in G at $[x_i, y_i]$ then $(J + e, K)$ is a cleavage in G', and if J is not split at $[x_i, y_i]$ in G then $(J, K + e)$ is a cleavage in G', in each case a contradiction.*

A **Θ-graph** is a tree with at most three vertices of degree 1.

Theorem 20 *Let G be a 2-connected graph and let H be a 2-connected subgraph of G. Let T be the cleavage unit tree of H. Let $R_1 \in \mathcal{R}(H)$ such that R_1 is not a circuit graph. For $R_a, R_b \in \mathcal{R}(H)$, $R_a \neq R_b$, let $v_a \in V(R_a)$ and $v_b \in V(R_b)$. Let P be the $r_a r_b$-path in T. Let v_a and v_b be chosen such that $r_1 \in V(P)$. Let e_a be the first edge of P and let e_b be the last edge of P (e_a and e_b not necessarily distinct). Suppose that v_a and v_b have been chosen such that v_a is not an end vertex of the virtual edge e_a in H^a, and v_b is not an end vertex of the virtual edge e_b in H^a. Let Q be a path in G for which $V(Q) \cap V(H) = \tau(Q) = \{v_a, v_b\}$. Let (J_i, K_i) be a cleavage at $[x, y]$ in H chosen such that $V(R_1) \subseteq V(K_i)$. If $\{v_a, v_b\} \cap (V(J_i) \backslash \{x, y\}) \neq \emptyset$ then let v_a and v_b be labelled such that $v_b \in V(J_i) \backslash \{x, y\}$. Let $L \subseteq J_i$ be a Θ-graph chosen such that $x, y \in \tau(L)$ and either, if $v_b \in V(J_i) \backslash \{x, y\}$ then $v_b \in V(L)$ and, subject to $x, y \in \tau(L)$, $\tau(L) \subseteq \{v_b, x, y\}$, or otherwise if $v_b \notin V(J_i) \backslash \{x, y\}$, $\tau(L) = \{x, y\}$. Put $H' = K_i \cup L \cup Q$. Then H' is 2-connected and there exists an $R'_1 \in \mathcal{R}(H')$ such that $V(R_1) \subset V(R'_1)$ and R'_1 is not a circuit graph.*

Proof 20 *Let e be the virtual edge at $[x, y]$ for the cleavage (J_i, K_i). Since $r_1 \in V(P)$, r_a and r_1 belong to the same component of $T - e$. Let $L = P_1 \cup P_2$, where P_1 is an xy-path and P_2 is either null or the third branch of L with $\tau(P_2) = \{v, v_b\}$. By Corollary*

7, $H' = K_i \cup L \cup Q = K_i \cup P_1 \cup P_2 \cup Q$ is a 2-connected graph. Let T_P be the cleavage unit tree of $K_P = K_i \cup P_1$. Let P' be the path in T_P such that P' is an $r_a r_b$-path if $r_b \in V(T_P)$ and P' is an $r_a r_i'$-path otherwise ($r_a \in V(T_P)$ by Theorem 18). Then $r_1 \in V(P')$. Put $Q' = P_2 \cup Q$ if $v_b \in V(J_i) \backslash \{x, y\}$ or $Q' = Q$ otherwise. Put $\tau(Q') = \{v_a, v'\}$ ($v' = v_b$ or $v' = v$). Since K_P is a 2-connected graph, and Q' is not split at $\{v_a, v'\}$ in H', either (K_P, Q') is a cleavage at $v_a v'$ in H' or $|E(Q')| = 1$. Put $H^* = (H' - (Q' - \{v_a, v'\})) + v_a v'$ so that H^* is the cleavage graph of H' at $v_a v'$ if $|E(Q')| > 1$ and $H^* = H'$ if $|E(Q')| = 1$. By Theorem 18, R_1 is a cleavage unit of K_P. Therefore, by Theorem 14, $V(R_1) \subseteq V(R_1')$, for some $R_1' \in \mathcal{R}(H')$. Since $v_a v' \in E(H^*)$, v' and v_a belong to the same cleavage unit of H^*. Suppose that some $e' \in E(P')$ is a virtual edge of H^* at $[x', y']$ say. Then v' and v_a would belong to different components of $H^* - \{x', y'\}$, a contradiction. Therefore, $V(R_1) \subset V(R_1')$, as required.

Main result

Theorem 6 *Let G be a simple 3-connected graph, in which the degree of each vertex is even and with $\delta(G) \geq 4$. Suppose that G does not contain a $K_{2,3}^+$-pair bridge. Then there exists a circuit C in G such that $G - E(C)$ is a nodally 3-connected graph.*

Proof 6 *Suppose that the Theorem is false and let G be a counter-example. By Corollary 5, we can choose a circuit C in G for which $G - E(C)$ is 2-connected and is not a circuit graph. Put $H = G - E(C)$. Let T be the cleavage unit tree of H. Some of the vertices of degree 1 in T may correspond to cleavage units that are circuit graphs. Let T' be the cleavage unit tree that is left after all such vertices of degree 1 have been deleted. Suppose that T' is trivial with $V(T') = \{r\}$. Let R be the cleavage unit of H corresponding to r in T'. Then R is either a simple 3-connected graph, a circuit graph with at least three edges or a bond graph with at least three edges. By choice of C, R is not a circuit graph, but now H is a nodally 3-connected graph, a contradiction. Let C be chosen such that the following conditions hold:*

(a) the graph H is 2-connected.

(b) subject to (a), there exists a cleavage unit R_1 that is not a circuit graph.

(c) subject to (b), $|V_3(R_1)|$ is maximum.

(d) subject to (c), $|E(C)|$ is minimum.

Let $f = r_1 r' \in E(T')$, such that if $d_{T'}(r') = 1$ then R' is not a 3-bond, such an r' exists for otherwise, R_1 is a simple 3-connected graph, by Theorem 15, and thus H is nodally 3-connected. Let $f = xy$ be the virtual edge of H^a and let (J_1, K_1) be the cleavage of H at f. Because G is simple and 3-connected there exists an edge $e = v_a v_b \in E(C)$ such that $v_a \in V(J_1) \backslash \{x, y\}$ and $v_b \in V(K_1) \backslash \{x, y\}$. Put $Q = v_a, e, v_b$. Let $v_a \in V(R_a)$ and $v_b \in V(R_b)$ for R_a, R_b cleavage units of H. Let P_T be the $r_a r_b$-path in T. Let $r_i \in V(T')$ such that $d_{T'}(r_i) = 1$ and $i \neq 1$.

Let $e_i \in E(T')$ such that e_i is incident to r_i. Let T_i be the component of $T - e_i$ that contains r_i. Let (J_i, K_i) be the cleavage at $e_i = x_i y_i$ in H. Let $J_i^!$ be the cleavage graph $J_i + e_i$ and let $K_i^!$ be the cleavage graph $K_i + e_i$. Let (J_i, K_i) be labelled such that the cleavage unit tree of $J_i^!$ is T_i.

Claim 1 *If R_i is an m-bond then $m = 3$.*

Proof 1 *Suppose that the Claim is false. By Theorem 12, $m \geq 3$, so suppose that $m \geq 4$. Since G is simple, each edge of R_i, except perhaps one, is a virtual edge and, as $d_{T'}(r_i) = 1$, each vertex of $T_i - r_i$ corresponds to a cleavage unit of H which is a circuit graph. Thus J_i is a subdivision of an $(m-1)$-bond graph with node vertices x_i and y_i. Let B be an $x_i y_i$-branch in J_i chosen such that if $\{v_a, v_b\} \cap V(J_i) \neq \emptyset$ then $\{v_a, v_b\} \cap V(J_i) \subseteq V(B)$. Put $K_Q = K_i \cup B \cup Q$. Because (J_i, K_i), B and Q satisfy Theorem 20, there exists a cleavage unit R_1' of K_Q such that $V(R_1) \subset V(R_1')$. By taking two of the branches of J_i which are not B we obtain a circuit C' in K_Q^c. Therefore, G and K_Q satisfy Theorem 4 and we can choose C' in K_Q^c such that $G - E(C')$ is 2-connected. By Theorem 14, C' contradicts the choice of C by Condition (c).*

By Theorem 15, no two adjacent vertices of T' can both represent circuit graphs. Hence, each vertex of degree 1 in T' that is not r_1 represents a cleavage unit of H that is isomorphic to either a 3-bond or a simple 3-connected graph.

Claim 2 *R_i is not a simple 3-connected graph.*

Proof 2 *Suppose that the Claim is false. Let P be an $x_i y_i$-path in J_i. By Theorem 10, if $\{v_a, v_b\} \cap V(J_i) \neq \emptyset$ then P can be chosen such that $\{v_a, v_b\} \cap V(J_i) \subseteq V(P)$ and there exists a circuit C' in $J_i - E(P)$. The graph $K_P = K_i \cup P$ is a subdivision of $K_i^!$ and thus K_P is 2-connected. Put $K_Q = K_P \cup Q$. Then (J_i, K_i), Q and P satisfy Theorem 20, and there exists a cleavage unit R_1' of K_Q such that R_1' is not a circuit graph and $V(R_1) \subset V(R_1')$. By Corollary 7, because $v_a, v_b \in V(K_P)$, $K_P \cup C$ is 2-connected. Therefore, by Theorem 4, C' can be chosen such that $G - E(C')$ is 2-connected. By Theorem 14, C' contradicts the choice of C by Condition (c).*

Hence, by Claim 1 and Claim 2, every vertex of degree 1 in T', other than perhaps r_1, represents a cleavage unit which is isomorphic to a 3-bond. Let $r_{l,i} \in V(T') \backslash \{r_1\}$ such that $d_{T'}(r_{l,i}) = 1$. If $d_{T'}(r_1) = 1$ then let $P' = r_{l,i}, r_l, r_h, \ldots, r_m, r_k$, where $r_k = r_1$, be a path in T', and if $d_{T'}(r_1) \neq 1$ then let $P' = r_{l,i}, r_l, r_h, \ldots, r_m, r_k, r_{k,i}$ be a path in T', r_l, r_h, r_m, r_k and $r_{k,i}$ not all necessarily distinct, such that $d_{T'}(r_{k,i}) = 1$ and $r_1 \in V(P')$. In both cases let P' be chosen such that every $r \in N_{T'}(r_l) \backslash \{r_h\}$ is such that $d_{T'}(r) = 1$ and if $r_k \neq r_1$ every $r \in N_{T'}(r_k) \backslash \{r_m\}$ is such that $d_{T'}(r) = 1$. Suppose that $r_l \neq r_h$ and let $e_l = r_l r_h$. Let (L, M) be the cleavage at e_l in H with hinges $[x_l, y_l]$, labelled such that R_l is a cleavage unit of the cleavage graph $L' = L + e_l$. Let T_l be the component of $T - e_l$ that includes r_l. The cleavage units of L' are R_l, the cleavage units that are isomorphic to 3-bonds that correspond to vertices *of degree 1 in $V(T') \cap N_{T_l}(r_l)$, and the cleavage units that are isomorphic to circuit graphs that map to vertices of $V(T_l)$ with degree 1 in T. Let $\{r_{l,1}, \ldots, r_{l,p-1}\} = (N_{T_l}(r_l) \cap V(T'))\backslash\{r_h\}$ so that $d_{T'}(r_{l,i}) = 1$, for $1 \le i \le p-1$. Let $V(R_{l,i}) = \{x_{l,i}, y_{l,i}\}$ and let $e_{l,i} = r_l r_{l,i}$, for $1 \le i \le p-1$. Let $T_{l,i}$ be the component of $T - e_{l,i}$ which includes $r_{l,i}$. Then $T_{l,i}$ has at most three vertices, $r_{l,i}$ and one or two vertices which have degree 1 in T. For $1 \le i \le p-1$, let $(X_{l,i}, Y_{l,i})$ be the cleavage at $e_{l,i}$ in H, such that $T_{l,i}$ is the cleavage unit tree of the cleavage graph $X_{l,i} + e_{l,i}$. Then $X_{l,i}$ is the union of two $x_{l,i}y_{l,i}$-branches, that is $X_{l,i}$ is a circuit graph.*

Suppose that each $r_i \in V(T')\backslash\{r_1\}$ is adjacent to r_1, so that T' is a star graph with centre vertex $r_1 = r_l$. By Theorem 12, Theorem 15 and Condition (b) in the choice of circuit, R_1 is a simple 3-connected graph. Now, H can be obtained from R_1 by replacing each virtual edge $e_{l,i}$ of R_1 by a subgraph $X_{l,i}$, for $1 \le i \le p-1$, and the virtual edge e_l by two $x_l y_l$-branches. Therefore, H is nodally 3-connected, a contradiction. Hence, $r_{l,i}$ can be chosen non-adjacent to r_1, $r_1 \ne r_l$.

***Claim 3** If L' is nodally 3-connected with $|V_3(L')| \ge 4$ then $|V_3(L')| = 4$ and $\{v_a, v_b\} \cap (V(L)\backslash\{x_l, y_l\}) \ne \emptyset$.*

***Proof 3** Suppose that the Claim is false. Let P be an $x_l y_l$-path in L. If $\{v_a, v_b\} \cap (V(L)\backslash\{x_l, y_l\}) \ne \emptyset$ then $|V_3(L')| \ge 5$, and thus, by Theorem 10, P can be chosen such that $\{v_a, v_b\} \cap V(L) \subseteq V(P)$ and so that there exists a circuit C' in $L - E(P)$. Then $M_P = M \cup P$ is a subdivision of the cleavage graph $M + e_l$ and is thus 2-connected. Put $M_Q = M_P \cup Q$. Then (L, M), Q and P satisfy Theorem 20 and there exists a cleavage unit R_1' of M_Q such that R_1' is not a circuit graph and $V(R_1) \subset V(R_1')$. By Corollary 7, because $v_a, v_b \in V(M_P)$, $M_P \cup C$ is 2-connected. Therefore, by Theorem 4, C' can be chosen such that $G - E(C')$ is 2-connected. By Theorem 14, C' contradicts the choice of C by Condition (c).*

***Claim 4** If R_l is a circuit graph then L' is a string of blocks closed by e_l and each block is a circuit graph.*

***Proof 4** Suppose first that some edge f of R_l is not a virtual edge. For $t_1, t_2 \in V(R_l)$, let W be a longest t_1t_2-path in R_l such that $f \in E(W)$ and no edge of W is a virtual edge. Since $e_l, e_{l,i} \in E(R_l)$, W can be labelled such that $t_1 \notin \{x_l, y_l\}$. Let $e_{l,j} = t_0 t_1$ be the virtual edge of R_l incident to t_1 and let $r_l r_{l,j} = e_{l,j} \in E(T')$. In H, there are exactly two t_0t_1-branches in $X_{l,j}$. Therefore, since $d_{R_l}(t_1) = 2$, $d_H(t_1) = 3$, a contradiction, as $d_H(v)$ is even for each $v \in V(H)$. Hence, each edge of R_l is a virtual edge. Let $V(R_l)$ be labelled $x_l = v_{l,1}, v_{l,2}, \ldots, v_{l,p} = y_l$ as $R_l - e_l$ is traversed from x_l to y_l. For $1 \le i \le p-1$, the virtual edge $e_{l,i} = v_{l,i}v_{l,i+1} \in E(R_l)$ joins r_l to $r_{l,i}$ in T'. Now $d_{R_l}(v_{l,i+1}) = 2$ and so the vertex $v_{l,i+1}$ belongs to precisely $X_{l,i}$ and $X_{l,i+1}$, for $1 \le i \le p-2$, but to no other $X_{l,j}$, $j \ne i, i+1$. Thus, L' is a string of blocks $X_{l,1}, X_{l,2}, \ldots, X_{l,p-1}$ closed by e_l and each block is a circuit graph.*

Let T'' be the tree that results by deleting each vertex of T' of degree 1 that is not r_1. Recall that if $d_{T'}(r_1) = 1$ then $P' = r_{l,i}, r_l, r_h, \ldots, r_m, r_k$, where $r_k = r_1 = r_a$, *and if $d_{T'}(r_1) \neq 1$ then $P' = r_{l,i}, r_l, r_h, \ldots, r_m, r_k, r_{k,i}$, such that $d_{T'}(r_{k,i}) = 1$ and $r_1 \in V(P')$. Put $e_k = r_k r_m$. Let (K, N) be the cleavage at e_k with hinges $[x_k, y_k]$ in H such that $V(R_k) \subseteq V(K)$. Put $K' = K + e_k$.*

Claim 5 *T'' is a path graph and we may assume that $v_b \in V(L')\backslash\{x_l, y_l\}$ and $v_a \in V(K')\backslash\{x_k, y_k\}$. If R_l is a circuit graph then R_l is a 3-circuit.*

Proof 5 *Suppose that the Claim is false. If $\{v_a, v_b\} \cap V(L) \neq \emptyset$ then let e be labelled such that $v_b \in V(L)$. If R_l is a simple 3-connected graph then L' is a nodally 3-connected graph, and by Claim 3, $v_b \in V(L)\backslash\{x_l, y_l\}$. Therefore, suppose that R_l is a circuit graph either with at least four edges ($p \geq 3$) or with three edges ($p = 2$) and such that $v_b \notin V(L)$. By Claim 4, L is a string of blocks and each block $X_{l,i}$ of L is a circuit graph that can be thought of as two $v_{l,i}v_{l,i+1}$-branches, for $1 \leq i \leq p-1$. Let $v_1, v_2 \in V(L) \cap V(C)$, let S be a v_1v_2-path in L and let P be an x_ly_l-path in L. If $v_b \in V(L)$ let v_b belong to a branch B_b of L. Since G is simple, $V(C) \cap (V(X_{l,i})\backslash\{v_{l,i}, v_{l,i+1}\}) \neq \emptyset$ for $1 \leq i \leq p-1$, and we can choose v_1, v_2, S and P such that the following four statements hold:*
(i) *the edges of S belong to at most one branch of any $X_{l,i}$.*
(ii) *$V(C) \cap V(S) = \{v_1, v_2\}$.*
(iii) *$E(S) \cap E(B_b) = \emptyset$.*
(iv) *$E(S) \cap E(P) = \emptyset$ and $v_b \in V(P)$.*
Put $M_P = M \cup P$. Let S_1 and S_2 be two $\{v_1, v_2\}$-bridges in C labelled such that $e \notin E(S_1)$. Put $C' = S_1 \cup S$ and $M_Q = M_P \cup Q$. Then (L, M), P and Q satisfy Theorem 20, and therefore there exists a cleavage unit R'_1 of M_Q such that $V(R_1) \subset V(R'_1)$. By Theorem 4, because $C' \subseteq M_Q^c$ there exists a circuit $C'' \subseteq M_Q^c$ such that $G - E(C'')$ is 2-connected. Therefore, by Corollary 14, C'' contradicts the choice of C by Condition (c).

Hence, $v_b \in V(L)$ and if R_l is a circuit graph, $p = 2$ and R_l is a 3-circuit. If $r_1 \neq r_k$ then by applying the above arguments to K', it follows that $v_a \in V(K')\backslash\{x_k, y_k\}$. Hence, $v_a \in V(K')\backslash\{x_k, y_k\}$ and T'' is a path graph.

By Claim 5, T' is a tree in which a path T'' can be identified such that all vertices of T' either lie on the path or are are adjacent to a vertex on the path.

Claim 6 *Neither R_l nor R_k is a circuit graph.*

Proof 6 *Suppose that the Claim is false and that R_l is a circuit graph. By Claim 5, $|E(R_l)| = 3$. If S and P can be chosen in L such that statements (i),(ii),(iii) and (iv) of Claim 5 hold then the result follows as for in Claim 5. Hence, v_b belongs to the same branch as either all the vertices of $V(C) \cap V(X_{l,1})$ or all the vertices of $V(C) \cap V(X_{l,2})$. Another consequence is that $v_{l,2} \notin V(C)$. By symmetry, we can assume that $v_1, v_b \in V(X_{l,1})$, $v_1, v_b \in V(B_1)$ for a branch B_1 of $X_{l,1}$. Let B_2 be a $v_{l,2}v_{l,3}$-branch of $X_{l,2}$ for which $V(B_2) \cap V(C) \neq \emptyset$. Let $v_c \in V(B_2)$ such that v_c is the first vertex of C that occurs on B_2, in traversing B_2 from $v_{l,2}$ to $v_{l,3}$. Let S and P be chosen such that statements (i) and (ii) of Claim 5 hold and* *such that* $E(S) \cap E(P) = \emptyset$ *and* $E(P) \cap E(B_2) = \emptyset$. *Thus* $E(S) \cap E(B_2) \neq \emptyset$ *if* $E(S) \cap E(X_{l,2}) \neq \emptyset$. *Put* $Q_1 = B_1[v_{l,1}, v_b] \cup Q$ *and* $Q_2 = S_2[v_a, v_c] \cup B_2[v_c, v_{l,3}]$. *By Corollary 7,* $M_P = M \cup P$, *and thus* $M'_P = M_P \cup Q_1$, *and thus* $M^* = M_P \cup Q_1 \cup Q_2$ *are 2-connected. Put* $K^* = (M^*)^c_G$. *Then,* $d_{K^*}(v)$ *is even, for every* $v \in V(K^*)$ *and therefore, there exists a circuit* $C^* \subseteq K^*$ *such that* $H^* = G - E(C^*)$ *is 2-connected. By Theorem 18, Theorem 14 and Condition (c),* $V(R_1) = V(R_1^*)$ *for a cleavage unit* R_1^* *of* H^*. *Let* S_1 *and* S_2 *be as for in Claim 5, then*

$$K^* = B_1[v_b, v_{l,2}] \cup B_2[v_{l,2}, v_c] \cup C \backslash (V(S_2[v_a, v_c]) \backslash \{v_c\}), \tag{1}$$

and

$$V(K^*) \subseteq V(C) \cup \{v_{l,2}\}. \tag{2}$$

Hence, by Condition (d) in the choice of C, *as* $v_a \notin V(K^*)$, $V(K^*) = V(C^*)$.

We now show that $H^* = M^*$ *(or* $C^* = K^*$*). Suppose that* $H^* \neq M^*$ *and that there exists a* $v \in V(K^*)$ *such that* $d_{K^*}(v) \geq 4$. *From Equation (1),* $d_{H^*}(v) = d_{K^* - E(C^*)}(v) = 2$. *Let* $v' \in W(G, K^*)$, *be chosen such that there exists a* $v'v$*-path in* $K^* - E(C^*)$, *such a* v' *must exist since* H^* *is 2-connected. However, by Equation (2),* $W(G, K^*) \subseteq (V(C) \cup \{v_{l,2}\}) \cap V(M^*)$ *and thus* $d_{K^*}(v') = 2$, *which is a contradiction, as* $V(C^*) = V(K^*)$ *and thus* $d_{K^* - E(C^*)}(v') = 0$. *Hence,* $H^* = M^*$.

Suppose that $v_{l,1}$ *and* v_a *do not belong to the same cleavage unit of* H. *Then substituting* $v_{l,1}$ *as* v_b, (L, M), P *and* Q_1 *satisfy Theorem 20 and therefore, there exists an* $R'_1 \in \mathcal{R}(M'_P)$ *such that* $V(R_1) \subset V(R'_1)$ *and* R'_1 *is not a circuit graph. By Corollary 14,* R'_1 *is contained in a cleavage unit of* H^*, *a contradiction to the choice of* C, *by Condition (c). Hence, we may assume that* $v_{l,1}$ *and* v_a *belong to the same cleavage unit. If* $v_{l,3}$ *and* v_a *do not belong to the same cleavage unit then we can swap* Q_1 *and* $v_{l,1}$ *with* Q_2 *and* $v_{l,3}$ *in the above argument to obtain a contradiction. Thus,* $v_{l,3}$ *also belongs to* R_a.

We now prove that H^* *is 3-connected to obtain a contradiction. Suppose that* $R_a \neq R_1$. *By Theorem 17, because* $[x_l, y_l] = [v_{l,1}, v_{l,3}]$ *is a hinge set common to* R_a, R_h *and* R_l, *and* $r_1 \in V(P_T[r_h, r_a])$, $R_1 = R_h$ *is a bond graph and* r_a, r_1, r_l *is a subpath of* T. *By Theorem 15,* R_a *is not a bond graph and by Condition (c) in the choice of circuit, since* R_1 *is a bond graph,* R_a *is not a simple 3-connected graph. Therefore, by Theorem 12,* R_a *is a circuit graph. But now, a contradiction results if the rôles of* v_a *and* v_b *are reversed (since* $V(C) \cap V(R_l) = \emptyset$*).*

Hence, $R_a = R_1$. *If* R_1 *is a bond graph then* H *is isomorphic to a subdivision of a graph obtained from* C_3 *by replacing each edge with a multiple edge of size at least two. In this case* H *is 3-connected, a contradiction. Thus, by Theorem 12, we may assume that* R_1 *is a simple 3-connected graph. Suppose that* (L_0, M_0) *is a nodal 2-separation at* $[x_0, y_0]$ *in* H^*. *Since* M_P *is nodally 3-connected we can label* (L_0, M_0) *such that* $V_3(M_P) \subseteq V_3(M_0)$. *Thus, as* $H^* = M^*$ *and* $v_{l,1}, v_{l,3} \in V_3(M_P)$, *the only node of* $V(L_0) \backslash \{x_0, y_0\}$ *is* v_a. *Since* v_a *belongs to a component of* $H^* - \{x_0, y_0\}$, v_a *belongs to an* x_0y_0*-branch* B *in* M_P. *Since,* $V(Q_1) \cap V(M_p) = \{x_l, v_a\}$ *and*

$V(M_p) \cap V(Q_2) = \{y_l, v_a\}$, $\{x_0, y_0\} = \{x_l, y_l\}$ and B is an $x_l y_l$-branch. But this is a contradiction, as $R_1 = R_a$ is not a circuit graph.

If $r_k \neq r_1$ then the above arguments apply with the rôles of r_k and r_l reversed.

By choice of C, R_1 is a simple 3-connected graph. By Claim 3 and Claim 6, $R_l \cong K_4$. Let $V(R_l) = \{x_l, y_l, u_1, u_2\}$. Put $M' = M + e_l$.

***Claim 7** We can assume that $|V(C) \cap (V(L)\backslash\{x_l, y_l\})| \geq 2$.*

***Proof 7** Suppose that the Claim is false and that $|V(C) \cap (V(L)\backslash\{x_l, y_l\})| = 1$. Since G is simple and 3-connected, $V(C) \cap (V(X_{l,i})\backslash\{x_{l,i}, y_{l,i}\}) \neq \emptyset$, where $[x_{l,i}, y_{l,i}]$ is the hinge of $(X_{l,i}, Y_{l,i})$, for $1 \leq i \leq q$. Therefore, $V(C) \cap (V(L)\backslash\{x_l, y_l\}) = V(C) \cap (V(X_{l,1})\backslash\{x_{l,1}, y_{l,1}\}) = \{v_b\}$. Since $R_l \cong K_4$, u_1u_2 is a virtual edge of R_l, and we can label $x_{l,1} = u_1$ and $y_{l,1} = u_2$ so that $V(X_{l,1}) = \{v_b, u_1, u_2\}$. Thus $L \cong K^+_{2,3}$. Put $C' = X_{l,1}$ and $H' = G - E(C')$. Put $B_1 = x_l, u_1, y_l$, $B_2 = x_l, u_2, y_l$ and $H_C = M \cup B_1 \cup B_2 = (H - E(C')) - \{v_b\}$. By Corollary 7, $H' = H_C \cup C$ is 2-connected and Condition (a) in the choice of circuit is satisfied by C'. Both R_1 and R_h are (not necessarily distinct) cleavage units of M' and thus, by Theorem 14, there exists a cleavage unit R'_1 of H' such that $V(R_1) \subseteq V(R'_1)$ and R'_1 is not a circuit graph. Thus, Condition (b) is satisfied. By Condition (c), we may assume that $V(R'_1) = V(R_1)$. By Condition (d), since $|E(C')| = 3$ and G is simple, $|E(C)| = 3$. Let $V(C) = \{v_a, v_b, v_c\}$.*

By Claim 5, both v_a and v_c belong to $V(K)\backslash\{x_k, y_k\}$, for otherwise replacing e with either v_bv_c or v_av_c results in a contradiction. Thus, if $R_k \neq R_1$ then the result holds with R_k and R_l swapped. Hence $R_k = R_1$.

We now apply Claims 1 through to 6 to H'. Suppose that $R_h \neq R_1$. If R_h is a bond graph then $x_ly_l \in E(G)$ and since B_1 and B_2 are both x_ly_l-branches R_h is replaced by a cleavage unit R'_h in H' that is a bond graph with at least 4-edges, a contradiction to Claim 1. If R_h is not a bond graph then R_h is either a circuit graph or a simple 3-connected graph, $V(R_h) \not\subseteq V(R'_1)$ and hence, Claim 7 holds for H'. Therefore, $R_k = R_1 = R_h$. Let (J_1, J_2) be a nodal 2-separation at $[x', y']$ in H' chosen such that $V(R_1) \subseteq V(J_1)$. Then $x_l, y_l \in V(J_1)$ and there is precisely one end cleavage unit, R'_i say of J_2 such that $v_b \in V(R'_i)$. Then $J_2 \cong K^+_{2,3}$ and $x'y' \notin E(G)$.

We now claim that x_l,y_l,x' and y' are all distinct. If $[x_l, y_l] = [x', y']$ then either there is a vertical 2-separation in G at $[x_l, y_l]$ or $G = K \cup L \cup C$, in both cases a contradiction. Suppose that $|\{x_l, y_l\} \cap \{x', y'\}| = 1$ and, without loss of generality, let $\{x_l, y_l\} \cap \{x', y'\} = \{y_l\} = \{x'\}$. Put $C^* = v_b, v_a, v_c, y_l, u_2, u_1, v_b$. Put $G' = (G - \{v_a, v_b, v_c, u_1, u_2\}) \cup \{a_1, a_2, a_3\}$, where $a_1 = x_ly_l$, $a_2 = y_ly'$ and $a_3 = x_ly'$. Then $H^* = G - E(C^*)$ is a subdivision of G' which is vertically 3-connected, and thus H^* is nodally 3-connected, a contradiction. Hence, $\{x_l, y_l\} \cap \{x', y'\} = \emptyset$. But now $L \cup K$ is a $K^+_{2,3}$-pair bridge in G, a contradiction.

***Claim 8** $V(C) \cap (V(L)\backslash\{x_l, y_l\}) \subseteq V(X_{l,1})\backslash\{x_{l,1}, y_{l,1}\}$.*

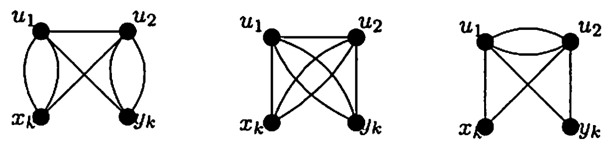

Figure 5:

Proof 8 *Suppose that the Claim is false and let* $v_L \in V(C) \cap ((V(L)\backslash\{x_l, y_l\})\backslash(V(X_{l,1})\backslash\{x_{l,1}, y_{l,1}\}))$. *Put* $C' = X_{l,1}$ *and let* $H' = G-E(C')$. *Because* L' *is nodally 3-connected,* $L_{C'} = (L'-E(C'))\backslash(V(X_{l,1})\backslash\{x_{l,1}, y_{l,1}\})$ *is 2-connected. Therefore, by Corollary 7,* $H'' = L_{C'} \cup M$ *is 2-connected. Because* G *is simple and 3-connected,* $V(C) \cap (V(R_i)\backslash W(H^a, R_i)) \neq \emptyset$, *for each* R_i *that is an end cleavage unit of* H. *Therefore, because* $v_L \notin V(C')$, $|V(C) \cap V(H'')| \geq 2$. *By Corollary 7,* $H' = H'' \cup C$ *is 2-connected. Hence, Condition (a) is satisfied by* C'. *Because* G *is simple and* L *has a spanning subgraph which is a subdivision of one of the graphs of Figure 5 we can choose* v_L *and an* $x_l y_l$*-path* P_1 *in* $L_{C'}$ *such that* $v_L \in V(P_1)$. *Put* $M_P = M \cup P_1$. *By Theorem 18,* $R_1 \in \mathcal{R}(M_P)$. *Therefore, by Theorem 14, there exists an* $R_1' \in \mathcal{R}(H')$ *such that* $V(R_1) \subseteq V(R_1')$ *and* R_1' *is not a circuit graph. Thus, Conditions (b) and (c) in the choice of circuit are satisfied by* C'. *By Condition (d),* $|E(C)| \leq |E(C')|$. *Therefore,* $V(C) - (V(X_{l,1})\backslash\{x_{l,1}, y_{l,1}\}) = \{v_a, v_L\}$, $|E(C')| = |E(C)|$ *and Condition (d) is satisfied by* C'. *Because* P_1 *is a* Θ*-graph,* P_1, v_a, v_L *and* (L, M) *satisfy Theorem 20. Therefore* $V(R_1) \subset V(R_1')$, *a contradiction.*

By Claim 7 and Claim 8, there exist w_1, $w_2 \in V(X_{l,1}) \cap V(C)$. *Let* Q_w *be a* $w_1 w_2$*-path in* $X_{l,1}$ *and let* w_1, w_2 *and* Q_w *be chosen such that* $|E(Q_w)|$ *is minimum. Let* S_1 *and* S_2 *be two* $\{w_1, w_2\}$*-bridges in* C, *labelled such that* $e \in E(S_2)$. *Put* $C_Q = S_1 \cup Q_w$. *Because* $Q_w \subseteq X_{l,1}$ *has been chosen such that* $|E(Q_w)|$ *is minimum,* $L - E(Q_w)$ *is connected. By Claim 8,* $v_b, u_1, u_2 \in V(X_{l,1})$. *Because* $|E(Q_w)|$ *is minimum either* $u_1 \notin V(Q_w)$ *or* $u_2 \notin V(Q_w)$, *by symmetry we may assume that* $u_1 \notin V(Q_w)$. *Then,* $B_1 = x_l, u_1, y_l$ *is an* $x_l y_l$*-path in* L. *Let* P_1 *be a* $v_b u_1$*-path in* $X_{l,1} - E(Q_w)$. *By choice of* P_1, $V(B_1) \cap V(P_1) = \{u_1\}$. *Hence,* $B_1 \cup P_1 \cup Q$ *is a* Θ*-graph. Because* M' *is 2-connected and* $M_P = M \cup P_1$ *is a subdivision to* M', M_P *is 2-connected. Therefore, by Theorem 7,* $U = M_P \cup B_1 \cup Q$, *is 2-connected. By Theorem 4, because* $C_Q \subseteq U^c$ *there exists a circuit* C' *in* U^c *such that* $G - E(C')$ *is 2-connected. Hence, because* (L, M) *and* $B_1 \cup P_1 \cup Q$ *satisfy Theorem 20,* C_Q *contradicts the choice of* C *by Condition (c).*

It seems quite likely that Theorem 6 is true for all simple 3-connected eulerian graphs.

Conjecture 1 *Let G be a simple 3-connected eulerian graph. Then there exists a circuit C such that $G - E(C)$ is nodally 3-connected.*

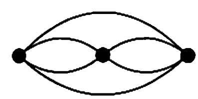

Figure 6:

Conjecture 1 may also be true for multigraphs and indeed for all graphs of order 4 or more, however it is not true for $|V(G)| = 3$; a circuit cannot be removed from the graph of Figure 6 without leaving either a graph without any node vertices, or a graph with a 1-separation.

Conjecture 2 *Let G be a 3-connected eulerian graph for which $|V(G)| \geq 4$. Then there exists a circuit C such that $G - E(C)$ is nodally 3-connected.*

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