On Multi-Color Partitions with Distinct Parts

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Abstract: Given integers $m \geq 2$, $r \geq 2$, let $q_m(n)$, $q_0^{(m)}(n)$, $b_r^{(m)}(n)$ denote respectively the number of m-colored partitions of n into distinct parts, distinct odd parts, and parts not divisible by r. We obtain recurrences for each of the above-mentioned types of partition functions.

Key Words: multi-color partitions

1. Introduction

If n and r are natural numbers with $r \geq 2$, let $q(n)$, $q_0(n)$, $b_r(n)$ denote respectively the number of partitions of n into distinct parts, distinct odd parts, and parts not divisible by r. (It is well-known that $q(n)$, $q_0(n)$, $b_r(n)$ also count respectively the number of partitions of n into odd parts, the number of self-conjugate partitions of n, the number of partitions of n such that no part occurs r or more times.) The function $b_r(n)$ is called the number of r-regular partitions of n. Note that $b_2(n) = q(n)$.

Let the integer $m \geq 2$. In this note, we obtain numerous recurrences concerning the m-color analogues of the above-mentioned partition functions. We denote the functions to be studied $q_m(n)$, $q_0^{(m)}(n)$, $b_r^{(m)}(n)$ respectively.

For example, let us list the partitions of 3 into distinct parts in two colors. These are as follows:

$$3\ ,\ \bar{3}\ ,\ 2+1\ ,\ 2+\bar{1}\ ,\ \bar{2}+1\ ,\ \bar{2}+\bar{1}\ .$$

Thus we have $q_2(3) = 6$. Furthermore, since only the first two of these six partitions consist entirely of odd parts, we have $q_0^{(2)}(3) = 2$. Also, since the last four of these partitions consist of parts not divisible by 3, we have $b_3^{(2)}(3) = 4$.

The symbol $p(n)$, which occurs in one of our theorems, denotes the ordinary partition function. Let the integer $t \geq 2$. The symbol $r_t(n)$, which occurs in several theorems, denotes the number of representations of n as the sum of t squares of integers. (Representations that differ only in the order of summands are considered distinct.)

2. Preliminaries

Let $x \in C$, $|x| < 1$. If $k \in Z$, let $\omega(k) = k(3k-1)/2$.
We will make use of the following ientities:

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{\omega(k)} \tag{1}$$

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{k(k+1)/2} \tag{2}$$

$$\prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}} = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{1-x^n} = \sum_{k=0}^{\infty}x^{k(k+1)/2} \tag{3}$$

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1})^2 = \sum_{k=-\infty}^{\infty}x^{k^2} = 1+2\sum_{k=1}^{\infty}x^{k^2} \tag{4}$$

$$\prod_{n=1}^{\infty}(1-x^{2n})^t(1+x^{2n-1})^{2t} = \sum_{k=0}^{\infty}r_t(k)x^k \tag{5}$$

$$\prod_{n=1}^{\infty}(1-x^n)^3(1-x^{2n-1})^2 = \sum_{k=-\infty}^{\infty}(1-6k)x^{\omega(k)} \tag{6}$$

$$\prod_{n=1}^{\infty}(1-x^{2n})^3(1-x^{2n-1})^5 = \sum_{k=-\infty}^{\infty}(1-6k)x^{\omega(k)} \tag{7}$$

$$\prod_{n=1}^{\infty}(1-x^n)^{-1} = \sum_{n=0}^{\infty}p(n)x^n \tag{8}$$

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}(1-x^{2n-1})^{-1} = \sum_{n=0}^{\infty}q(n)x^n \tag{9}$$

$$\prod_{n=1}^{\infty}(1+x^{2n-1}) = \sum_{n=0}^{\infty} q_0(n)x^n \tag{10}$$

$$\prod_{n=1}^{\infty} \frac{1-x^{rn}}{1-x^n} = \sum_{n=0}^{\infty} b_r(n)x^n \tag{11}$$

$$q(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } \; n=\omega(\pm m) \\ 0 \pmod 2 & \text{otherwise} \end{cases} \tag{12}$$

Remarks: Identities (1), (2), (3) are due to Euler, Jacobi, Gauss respectively. (See [1].) (4) follows from the Jacobi Triple Product Identity, taking $z=1$. (5) follows from (4). (6) and (7), which are equivalent, are consequences of the Gordon-Watson quintuple product identity. (See [3].) (8) through (11) are well-known generating function identities. (12) follows from (1) and (9).

Before presenting our main results, we begin with a convolution-type theorem that will be used to prove several identities.

Theorem A If the integer $m \geq 2$, let $f_m(n)$ and $g(x)$ be functions such that

$$\sum_{n=0}^{\infty} f_m(n)x^n = \prod_{n=1}^{\infty} g(x^n)^m$$

where both members of the above identity converge absolutely for $|x| < 1$. Then for $1 \leq j \leq m-1$, we have

$$f_m(n) = \sum_{k=0}^{n} f_{m-j}(n-k)f_j(k) \ .$$

Proof: By hypothesis, we have

$$\sum_{n=0}^{\infty} f_m(n)x^n = \prod_{n=1}^{\infty} g(x^n)^{m-j} \prod_{n=1}^{\infty} g(x^n)^j = \sum_{n=0}^{\infty} f_{m-j}(n)x^n \sum_{n=0}^{\infty} f_j(n)x^n \, .$$

The conclusion now follows by matching coefficients of like powers of x. ■

3. Partitions into Distinct Parts in m Colors

Definition 1: If $m \geq 1$, let $q_m(n)$ denote the number of partitions of n into distinct parts in m colors.

Generating Function:

$$\sum_{n=0}^{\infty} q_m(n)x^n = \prod_{n=1}^{\infty}(1+x^n)^m \tag{13}$$

Remarks: Identity (13) follows from (9) and from Definition 1.

Our first theorem is a recurrence for $q_2(n)$.

Theorem 1

$$\sum_{k=-\infty}^{\infty} (-1)^k q_2(n-\omega(k)) = \begin{cases} 1 & \text{if } n = m(m+1)/2 \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Using (13) with $m = 2$, we have

$$\sum_{n=0}^{\infty} q_2(n)x^n = \prod_{n=1}^{\infty}(1+x^n)^2 = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1-x^n)^2} .$$

This implies

$$\sum_{n=0}^{\infty} q_2(n)x^n \prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{1-x^n} .$$

The conclusion now follows from (1) and (3), matching coefficients of like powers of x.

The next theorem generalizes Theorem 1.

Theorem 2 If $m \geq 3$, then

$$\sum_{k=-\infty}^{\infty} (-1)^k q_m(n-\omega(k)) = \sum_{j\geq 0} q_{m-2}(n-j(j+1)/2) .$$

Proof:

$$\prod_{n=1}^{\infty}(1+x^n)^m = \prod_{n=1}^{\infty}(1+x^n)^{m-2}\prod_{n=1}^{\infty}(1+x^n)^2 = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1-x^n)^2}\prod_{n=1}^{\infty}(1+x^n)^{m-2} .$$

so that

$$\prod_{n=1}^{\infty}(1+x^n)^m\prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{(1-x^n)}\prod_{n=1}^{\infty}(1+x^n)^{m-2} .$$

The conclusion now follows if we invoke (13), (1), and (3) and match coefficients of like powers of x. ■

Our next theorem is a recurrence for $q_3(n)$.

Theorem 3

$$q_3(n) + 2\sum_{k=1}^{\infty} q_3(n-k^2) = \begin{cases} 1 & \text{if } n = m(m+1)/2 \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Replacing x by $-x$ in (4), we have

$$\prod_{n=1}^{\infty}(1-x^{2n})(1-x^{2n-1})^2 = \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2} .$$

Therefore, invoking (13) with $m = 3$, we have

$$\sum_{n=0}^{\infty} q_3(n)x^n \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2} = \prod_{n=1}^{\infty}(1+x^n)^3\prod_{n=1}^{\infty}(1-x^{2n})(1-x^{2n-1})^2 =$$

$$\prod_{n=1}^{\infty}(1-x^{2n-1})^{-3}(1-x^{2n})(1-x^{2n-1})^2 = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}}$$

using (9) and (3). the conclusion now follows, matching coefficients of like powers of x. ■

The following theorem generalizes Theorem 3.

Theorem 4 If $m \geq 4$, then

$$\sum_{k=-\infty}^{\infty} q_m(n-k^2) = \sum_{j=0}^{\infty} q_{m-3}\left(n - \frac{j(j+1)}{2}\right) .$$

Proof: We saw in the proof of Theorem 3 that

$$\prod_{n=1}^{\infty}(1+x^n)^3 \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2} = \sum_{k=0}^{\infty} x^{\frac{k(k+1)}{2}} .$$

If we multiply this identity by $\prod_{n=1}^{\infty}(1+x^n)^{m-3}$, invoke (13) and match coefficients of like powers of x, the conclusion follows. ■

The next theorem is another recurrence for $q_3(n)$.

Theorem 5

$$\sum_{k\geq 0}(-1)^k(2k+1)q_3(n-\frac{k(k+1)}{2}) = \begin{cases} (-1)^m(2m+1) & \text{if } n = m(m+1) \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Setting $m = 3$ in (13), we have

$$\sum_{n=0}^{\infty} q_3(n)x^n = \prod_{n=1}^{\infty}(1+x^n)^3 = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^3}{(1-x^n)^3} .$$

This implies

$$\prod_{n=1}^{\infty}(1-x^n)^3 \sum_{n=0}^{\infty} q_3(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n})^3 .$$

The conclusion now follows if we invoke (2) and match coefficients of like powers of x. ■

The following theorem generalizes Theorem 5.

Theorem 6 If $m \geq 4$, then

$$\sum_{k\geq 0}(-1)^k(2k+1)q_m(n-\frac{k(k+1)}{2}) = \sum_{j\geq 0}(-1)^j(2j+1)q_{m-3}(n-j(j+1)) .$$

Proof:

$$\prod_{n=1}^{\infty}(1+x^n)^m \prod_{n=1}^{\infty}(1-x^n)^3 = \prod_{n=1}^{\infty}(1+x^n)^{m-3} \prod_{n=1}^{\infty}(1-x^{2n})^3 .$$

The conclusion now follows from (13) and (2), matching coefficients of like powers of x. ■

The next theorem states a congruential property of $q_p(n)$ when p is prime.

Theorem 7 If p is prime, then

$$q_p(n) \equiv \begin{cases} q(n/p) \pmod{p} & \text{if } p|n \\ 0 \pmod{p} & \text{otherwise} \end{cases} .$$

Proof: Identity (13) implies

$$\sum_{n=0}^{\infty} q_p(n)x^n = \prod_{n=1}^{\infty}(1+x^n)^p \equiv \prod_{n=1}^{\infty}(1+x^{pn}) \pmod{p} .$$

Now (9) implies

$$\sum_{n=0}^{\infty} q_p(n)x^n \equiv \sum_{n=0}^{\infty} q(n/p)x^n \pmod{p} .$$

Matching coefficients of like powers of x, we have $q_p(n) \equiv q(n/p) \pmod{p}$. This last statement is equivalent to the conclusion, since by definition, $q(\alpha) = 0$ if α is not a non-negative integer. ■

Corollary 1

$$q_2(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n = 2\omega(\pm k) \\ 0 \pmod{2} & \text{otherwise} \end{cases} .$$

Proof: Theorem 7 implies $q_2(n) \equiv q(n/2) \pmod{2}$ if $n = 2m$. The conclusion now follows from (12). ■

The next several theorems state reduction formulas that express $q_m(n)$ in terms of $q_k(n)$, where $k < m$.

Theorem 8 If $1 \le j \le m-1$, then

$$q_m(n) = \sum_{k=0}^{n} q_{m-j}(n-k)q_j(k) .$$

Proof: This follows from Theorem A, with $f_m(n) = q_m(n)$, and $g(x^n) = 1 + x^n$. ■

Theorem 9 If $m \geq 2$, then

$$\sum_{k=-\infty}^{\infty} (-1)^k q_m(n-\omega(k)) = \sum_{j=-\infty}^{\infty} q_{m-1}(n-2\omega(j)) .$$

Proof:

$$\prod_{n=1}^{\infty}(1+x^n)^m \prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{\infty}(1+x^n)^{m-1} \prod_{n=1}^{\infty}(1-x^{2n}) .$$

Invoking (13), we have

$$\sum_{n=0}^{\infty} q_m(n)x^n \prod_{n=1}^{\infty}(1-x^n) = \sum_{n=0}^{\infty} q_{m-1}(n)x^n \prod_{n=1}^{\infty}(1-x^{2n}) .$$

The conclusion now follows from (1), matching coefficients of like powers of x. ■

The following theorem is yet another recurrence for $q_2(n)$.

Theorem 10

$$\sum_{j=-\infty}^{\infty} (1-6j)q_2(n-\omega(j)) = \begin{cases} (-1)^k(2k+1) & \text{if } n = k(k+1)/2 \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Identities (6) and (9) imply

$$\prod(1+x^n)^2 \sum_{j=-\infty}^{\infty} (1-6j)q_n(n-\omega(j)) = \prod_{n=1}^{\infty}(1-x^n)^3 . \qquad (14)$$

The conclusion now follows if we invoke (13) and (2) and match coefficients of like powers of x. ■

The next theorem generalizes Theorem 10.

Theorem 11 If $m \geq 3$, then

$$\sum_{j=-\infty}^{\infty} (1-6j)q_n(n-\omega(j)) = \sum_{k\geq 0}(-1)^k(2k+1)q_{m-2}(n-\frac{k(k+1)}{2}) .$$

Proof: The conclusion follows if we multiply (14) by $\prod_{n=1}^{\infty}(1+x^n)^{m-2}$, invoke (13) and (2) and match coefficients of like powers of x. ■

The last theorem in this section links $q_m(n)$ and $p(n)$.

Theorem 12

$$\sum_{k=0}^{[n/2]} q_m(n-2k)p(k) = \sum_{j=0}^{n} q_{m-1}(n-j)p(j) \ .$$

Proof:

$$\prod_{n=1}^{\infty}(1+x^n)^m \prod_{n=1}^{\infty}(1-x^{2n})^{-1} = \prod_{n=1}^{\infty}(1+x^n)^{m-1} \prod_{n=1}^{\infty}(1-x^n)^{-1} \ .$$

Invoking (13) and (8), we have

$$\sum_{n=0}^{\infty} q_m(n)x^n \sum_{n=0}^{\infty} p(\frac{n}{2})x^n = \sum_{n=0}^{\infty} q_{m-1}(n)x^n \sum_{n=0}^{\infty} p(n)x^n \ .$$

The conclusion now follows by matching coefficients of like powers of x.
■

Table 1 below lists $q_m(n)$ where $1 \leq m \leq 5$ and $0 \leq n \leq 20$.

n	$q_1(n)$	$q_2(n)$	$q_3(n)$	$q_4(n)$	$q_5(n)$
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	2	6	13	24	40
4	2	9	24	51	95
5	3	14	42	100	206
6	4	22	73	190	425
7	5	32	120	344	835
8	6	46	192	601	1575
9	8	66	302	1024	2880
10	10	93	465	1702	5121
11	12	128	702	2768	8885
12	15	176	1046	4422	15095
13	18	238	1536	6948	25165
14	22	319	2226	10752	41240
15	27	426	3195	16424	66562
16	32	562	4536	24782	105945
17	38	736	6378	36972	166480
18	46	960	8896	54602	258560
19	54	1242	12306	79872	397235
20	64	1598	16896	115805	604162

Table 1: $q_m(n)$

4. Partitions into Distinct Odd Parts in m Colors

Definition 2 If $m \geq 1$, let $q_0^{(m)}(n)$ denote the number of partitions of n into distinct odd parts in m colors.

Generating Function:

$$\sum_{n=0}^{\infty} q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1+x^{2n-1})^m . \tag{15}$$

Remarks: Identity (15) follows from (10) and from Definition 2. We will find it convenient to employ an alternate form of (15), obtained by replacing x by $-x$, namely:

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^m . \qquad (16)$$

Our first theorem in this section is a recurrence for $q_0^{(2)}(n)$.

Theorem 13

$$\sum_{j\geq 0}(-1)^{j(j+1)/2} q_0^{(2)}(n-\frac{j(j+1)}{2}) = \begin{cases} (-1)^j & \text{if } n=\omega(\pm j) \\ 0 & \text{otherwise} \end{cases} .$$

Proof: (16) implies

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^{m-2}\prod_{n=1}^{\infty}(1-x^{2n-1})^2 \qquad (17)$$

$$= \prod_{n=1}^{\infty}(1-x^{2n-1})^{m-2}\prod_{n=1}^{\infty}(\frac{1-x^n}{1-x^{2n}})^2$$

so that

$$\prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{1-x^n}\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^{m-2}\prod_{n=1}^{\infty}(1-x^n) .$$

Invoking (1) and (3), we have

$$\sum_{n\geq 0} x^{\frac{n(n+1)}{2}}\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^{m-2}\sum_{n=-\infty}^{\infty} x^{\omega(n)} ,. \qquad (18)$$

The conclusion now follows if we set $m=2$, match coefficients of like powers of x, and simplify. ■

The next theorem generalizes Theorem 13.

<u>Theorem 14</u> If $m \geq 3$, then

$$\sum_{j\geq 0}(-1)^{j(j+1)/2}q_0^{(m)}(n-\frac{j(j+1)}{2}) = \sum_{k=-\infty}^{\infty}(-1)^k q_0^{(m-2)}(n-\omega(k)) .$$

<u>Proof:</u> If $m \geq 3$, then (18) and (16) imply

$$\sum_{n\geq 0} x^{\frac{n(n+1)}{2}} \sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \sum_{n=0}^{\infty}(-1)^n q_0^{(m-2)}(n)x^n \sum_{n=-\infty}^{\infty} x^{\omega(n)} .$$

The conclusion now follows if we match coefficients of like powers of x and simplify. ■

The next theorem is a second recurrence for $q_0^{(2)}(n)$.

<u>Theorem 15</u>

$$\sum_{k\geq 0}(-1)^{n-k(k+1)/2}(2k+1)q_0^{(2)}(n-\frac{k(k+1)}{2}) = \begin{cases} 1 \mp 6j & \text{if } n = \omega(\pm j) \\ 0 & \text{otherwise} \end{cases} .$$

<u>Proof:</u> If we multiply both members of the first equality in (17) by $\prod_{n=1}^{\infty}(1-x^n)^3$, set $m=2$ and invoke (6), we obtain

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(2)}(n)x^n \prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=-\infty}^{\infty}(1-6k)x^{\omega(k)}$$

The conclusion now follows from (2), matching coefficients of like powers of x. ■

The following theorem generalizes Theorem 15.

<u>Theorem 16</u>

$$\sum_{k\geq 0}(-1)^{\frac{k(k-1)}{2}}(2k+1)q_0^{(m)}(n-\frac{k(k+1)}{2}) =$$

$$\sum_{j=-\infty}^{\infty}(-1)^{\omega(j)}(1-6j)q_0^{(m-2)}(n-\omega(j)) .$$

Proof: (16) and (17) imply

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n) = \sum_{n=0}^{\infty}(-1)^n q_0^{(m-2)}(n) \prod_{n=1}^{\infty}(1-x^{2n-1})^2 .$$

If we multiply this last identity by $\prod_{n=1}^{\infty}(1-x^n)^3$ and invoke (2) and (6), we obtain

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n) \sum_{n=0}^{\infty}(-1)^n x^{n(n+1)/2} = \sum_{n=0}^{\infty}(-1)^n q_0^{(m-2)}(n) \sum_{n=-\infty}^{\infty}(1-6n)x^{\omega(n)} .$$

The conclusion now follows if we match coefficients of like powers of x and simplify. ■

The next theorem is a recurrence for $q_0^{(3)}(n)$.

Theorem 17

$$\sum_{j\geq 0}(-1)^{n-j}(2j+1)q_0^{(3)}(n-j(j+1)) = \begin{cases} (-1)^k(2k+1) & \text{if } n=\frac{k(k+1)}{2} \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Setting $m=3$ in (16), we have

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(3)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^3 = \prod_{n=1}^{\infty}\frac{(1-x^n)^2}{(1-x^{2n})^3} .$$

Thus

$$\prod_{n=1}^{\infty}(1-x^{2n})^3 \sum_{n=0}^{\infty}(-1)^n q_0^{(3)}(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^3 .$$

The conclusion now follows from (2), matching coefficients of like powers of x. ■

The next theorem generalizes Theorem 17.

Theorem 18

If $m \geq 4$, then

$$\sum_{j\geq 0}(-1)^{j(j+1)}(2j+1)q_0^{(m)}(n-j(j+1)) =$$

$$\sum_{k\geq 0}(-1)^{k(k+1)/2}(2k+1)q_0^{(m-3)}(n-k(k+1)/2) .$$

Proof: Identity (16) implies

$$\sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{2n-1})^3 \sum_{n=0}^{\infty}(-1)^n q_0^{(m-3)}(n)x^n .$$

If we multiply by $\prod_{n=1}^{\infty}(1-x^{2n})^3$, we obtain

$$\prod_{n=1}^{\infty}(1-x^{2n})^3 \sum_{n=0}^{\infty}(-1)^n q_0^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^3 \sum_{n=0}^{\infty}(-1)^n q_0^{(m-3)}(n)x^n .$$

The conclusion now follows if we invoke (2) and match coefficients of like powers of x. ■

The next theorem is a recurrence for $q_0^{(5)}(n)$.

Theorem 19

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)q_0^{(5)}(n-k(k+1)) = \begin{cases} (-1)^{\omega(\pm k)}(1 \mp 6k) & \text{if } n=\omega(\pm k) \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Replacing x by $-x$ in (7), we have

$$\prod_{n=1}^{\infty}(1+x^{2n-1})^5(1-x^{2n})^3 = \sum_{k=-\infty}^{\infty}(-1)^{\omega(k)}(1-6k)x^{\omega(k)} . \tag{19}$$

The conclusion now follows from (15) and (2), matching coefficients of like powers of x. ■

The next theorem generalizes Theorem 19.

Theorem 20 If $m \geq 6$, then

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)q_0^{(m)}(n-k(k+1)) = \sum_{j=-\infty}^{\infty}(-1)^{\omega(j)}(1-6j)q_0^{(m-5)}(n-\omega(j)) .$$

Proof: If we multiply identity (19) by $\prod_{n=1}^{\infty}(1+x^{2n-1})^{m-5}$, we get

$$\prod_{n=1}^{\infty}(1+x^{2n-1})^m(1-x^{2n})^3 = \sum_{k=-\infty}^{\infty}(-1)^{\omega(k)}(1-6k)x^{\omega(k)}\prod_{n=1}^{\infty}(1+x^{2n-1})^{m-5} .$$

The conclusion now follows from (15) and (2), matching coefficients of like powers of x. ■

The next theorem is an analogue of Theorem 8.

Theorem 21 Let $1 \le j \le m-1$. Then

$$q_0^{(m)}(n) = \sum_{k=0}^{n} q_0^{(m-j)}(n-k) q_0^j(k) .$$

Proof: This follows from Theorem A, with $f_m(n) = q_0^{(m)}(n)$ and $g(x) = 1 + x^{2n-1}$. ■

We conclude this section with several theorems that link $r_t(n)$ with $q_0^{(m)}(n)$.

Theorem 22

$$\sum_{k=-\infty}^{\infty} (-1)^k r_2(n - 2\omega(k)) = \sum_{j=0}^{\infty} (-1)^j (2j+1) q_0^{(4)}(n - j(j+1)) .$$

Proof: If we invoke (5) with $t = 2$ and multiply by $\prod_{n=1}^{\infty}(1 - x^{2n})$, we get

$$\sum_{n=0}^{\infty} r_2(n) x^n \prod_{n=1}^{\infty}(1 - x^{2n}) = \prod_{n=1}^{\infty}(1 - x^{2n})^3 \prod_{n=1}^{\infty}(1 + x^{2n-1})^4 .$$

The conclusion now follows from (1), (2), and (15), matching coefficients of like powers of x. ■

Theorem 23

$$r_3(n) = \sum_{k=0}^{\infty} (-1)^k (2k+1) q_0^{(6)}(n - k(k+1)) .$$

Proof: If we invoke (5) with $t = 3$, we obtain

$$\sum_{n=0}^{\infty} r_r(n) x^n = \prod_{n=1}^{\infty}(1 - x^{2n})^3 \prod_{n=1}^{\infty}(1 + x^{2n-1})^6 .$$

The conclusion now follows from (2) and (15), matching coefficients of like powers of x. ■

Theorem 24

$$r_3(n) = \sum_{k=0}^{\infty}(-1)^{\frac{k(k-1)}{2}}(2k+1)q_0^{(3)}(n - \frac{k(k+1)}{2}) \,.$$

Proof: If we invoke (5) with $t = 3$, replace x by $-x$ and simplify, we have

$$\sum_{n=0}^{\infty}(-1)^n r_3(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^3 \prod_{n=1}^{\infty}(1-x^{2n-1})^3 \,.$$

So that (16) implies

$$\sum_{n=0}^{\infty}(-1)^n r_3(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^3 \sum_{n=0}^{\infty}(-1)^n q_0^{(3)}(n)x^n \,.$$

The conclusion now follows from (2), simplifying and matching coefficients of like powers of x. ■

Remarks: In [2], Ewell obtained the identity:

$$r_3(n) = \sum_{k=-\infty}^{\infty}(-1)^{\omega(k)}(1-6k)q_0(n-\omega(k)) \,.$$

Table 2 below lists $q_0^{(m)}(n)$ for $1 \le m \le 5$ and $0 \le n \le 20$.

n	$q_0^{(1)}(n)$	$q_0^{(2)}(n)$	$q_0^{(3)}(n)$	$q_0^{(4)}(n)$	$q_0^{(5)}(n)$
0	1	1	1	1	1
1	1	2	3	4	5
2	0	1	3	6	10
3	1	2	4	8	15
4	1	4	9	17	30
5	1	4	12	28	56
6	1	5	15	38	85
7	1	6	21	56	130
8	2	9	30	84	205
9	2	12	43	124	315
10	2	13	54	172	465
11	2	16	69	232	665
12	3	21	94	325	960
13	3	26	123	448	1380
14	3	29	153	594	1925
15	4	36	193	784	2651
16	5	46	252	1049	3660
17	5	54	318	1388	5020
18	5	62	391	1796	6775
19	6	74	486	2320	9070
20	7	90	609	3005	12126

Table 2: $q_0^{(m)}(n)$

5. Partititions into parts not divisible by r, in m colors

Definition 3 If $m \ge 2$ and $r \ge 2$, let $b_r^{(m)}(n)$ denote the number of partitions of n in m colors into parts not divisible by r.

Generating Function

$$\sum_{n=0}^{\infty} b_r^{(m)}(n) x^n = \prod_{n=1}^{\infty} \left(\frac{1-x^{rn}}{1-x^n}\right)^m . \tag{20}$$

Remarks: Identity (20) follows from (11) and from Definition 3.

Our first theorem in this section is an analogue of Theorem 5.

Theorem 25 If $1 \le j \le m-1$, then

$$b_r^{(m)}(n) = \sum_{k=0}^{n} b_r^{(m-j)}(n-k) b_r^{(j)}(k) .$$

Proof: This follows from Theorem A, with $f_n(m) = b_r^{(m)}(n)$ and $g(x) = (1-x^r)/(1-x)$. ■

The next theorem is a recurrence concerning $b_r^{(2)}(n)$.

Theorem 26

$$\sum_{k=-\infty}^{\infty} (-1)^k b_r^{(2)}(n-\omega(k)) = \sum_{j=-\infty}^{\infty} (-1)^j b_r(n-r\omega(j)) .$$

Proof: Invoking (20) with $m=2$, we have

$$\sum_{n=0}^{\infty} b_r^{(2)}(n) x^n = \prod_{n=1}^{\infty} \left(\frac{1-x^{rn}}{1-x^n}\right)^2 .$$

Multiplying by $\prod_{n=1}^{\infty}(1-x^n)$, we have

$$\sum_{n=0}^{\infty} b_r^{(2)}(n) x^n \prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{\infty} \frac{1-x^{rn}}{1-x^n} \prod_{n=1}^{\infty}(1-x^{rn}) .$$

Now (11) implies

$$\sum_{n=0}^{\infty} b_r^{(2)}(n) x^n \prod_{n=1}^{\infty}(1-x^n) = \sum_{n=0}^{\infty} b_r(n) x^n \prod_{n=1}^{\infty}(1-x^{rn}) .$$

The conclusion now follows from (1), matching coefficients of like powers of x. ■

The next theorem is a recurrence for $b_r^{(3)}(n)$.

Theorem 27

$$\sum_{j\ge 0}(-1)^j(2j+1) b_r^{(3)}\left(n-\frac{j(j+1)}{2}\right) = \begin{cases} (-1)^k(2k+1) & \text{if } n = \frac{rk(k+1)}{2} \\ 0 & \text{otherwise} \end{cases}$$

Proof: If we invoke (20) with $m = 3$ and multiply by $\prod_{n=1}^{\infty}(1-x^n)^3$, we obtain

$$\prod_{n=1}^{\infty}(1-x^n)^3 \sum_{n=0}^{\infty} b_r^{(3)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{rn})^3 .$$

The conclusion now follows from (2), matching coefficients of like powers of x. ■

Our last theorem is a reduction formula that generalizes Theorem 27.

Theorem 28 If $m \geq 4$, then

$$\sum_{j\geq 0}(-1)^j(2j+1)b_r^{(m)}(n-\frac{j(j+1)}{2}) = \sum_{k\geq 0}(-1)^k(2k+1)b_r^{(m-3)}(n-\frac{rk(k+1)}{2}) .$$

Proof: Identity (20) implies

$$\sum_{n=0}^{\infty} b_r^{(m)}(n)x^n = \prod_{n=1}^{\infty}(\frac{1-x^{rn}}{1-x^n})^3 \prod_{n=1}^{\infty}(\frac{1-x^{rn}}{1-x^n})^{m-3} .$$

so we have

$$\prod_{n=1}^{\infty}(1-x^n)^3 \sum_{n=0}^{\infty} b_r^{(m)}(n)x^n = \prod_{n=1}^{\infty}(1-x^{rn})^3 \sum_{n=0}^{\infty} b_r^{(m-3)}(n)x^n .$$

The conclusion now follows if we invoke (2) and match coefficients of like powers of x. ■

6. References

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