On the spectral radius of graphs with k-vertex cut *

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Abstract. We study the spectral radius of graphs with n vertices and a k-vertex cut and describe the graph which has the maximal spectral radius in this class. We also discuss the limit point of the maximal spectral radius.

Key words: vertex cut, spectral radius, limit point.

AMS classification: 05C15

1. Introduction

The graphs in this paper are simple. The spectral radius, $\rho(G)$, of a graph G is the largest eigenvalue of its adjacency matrix $A(G)$. For results on the spectral radii of graphs, the reader is referred to [4], [5] and [7] and the references therein. When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem, e.g., [1], the spectral radius is simple and has a unique (up to a multiplication by a scalar) positive eigenvector. We shall refer to such an eigenvector as the Perron vector of G. If we add an edge to G, the spectral radius increases.

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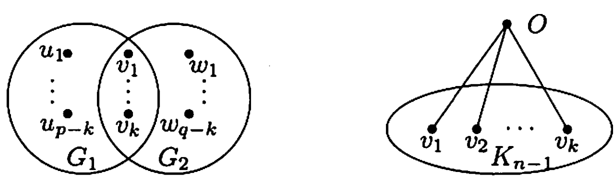
A *vertex cut* of a connected graph G is a subset V' of $V(G)$ such that $G - V'$ is disconnected. A *k-vertex cut* is a vertex cut with k vertices.

Brualdi and Solheid [3] proposed the following problem concerning spectral radii: Given a set $\mathscr{S}$ of graphs, find an upper bound for the spectral radii of graphs in $\mathscr{S}$ and characterize the graphs in which the maximal spectral radius is attained. Berman and Zhang [2], and H. Liu et al [8] studied this question for the graphs with n vertices and k cut vertices, and k cut edges, respectively, and described the graph that has the maximal spectral radius in these classes. In this paper, we investigate the same question for $\mathscr{S} = \mathscr{G}_n^k$, the set of connected graphs with n vertices and a k-vertex cut, where $n \geq k + 2 \geq 3$. Let o be a vertex disjoint with the complete graph K_{n-1}. Then we denote by K_{n-1}^k the graph obtained by joining o with k vertices of K_{n-1}, as in Fig. 1. We show that of all the connected graphs $\mathscr{G}_n^k$, the maximal spectral radius is obtained uniquely at K_{n-1}^k. Finally, we study the limit points of the spectral radii.

2. The main results

Suppose that G_1 and G_2 are two connected graphs with k common vertices $v_1, v_2, \ldots, v_k$.



$G_1 v_1 \cdots v_k G_2$ $\qquad$ K_{n-1}^k

Figure 1.

Then $H = G_1v_1v_2\cdots v_kG_2$ is defined by $V(H) = V(G_1)\cup V(G_2)$, $V(G_1)\cap V(G_2) = \{v_1, v_2, \ldots, v_k\}$, and $E(H) = E(G_1)\cup E(G_2)$, as in Fig. 1.

Let p, $q > k \geq 1$ and $H = K_pv_1v_2\cdots v_kK_q$, where K_p and K_q are complete graphs with $V(K_p) = \{u_1, \ldots, u_{p-k}, v_1, \ldots, v_k\}$ and $V(K_q) = \{w_1, \ldots, w_{q-k}, v_1, \ldots, v_k\}$. Then $V(K_p)\cap V(K_q) = \{v_1, \ldots, v_k\}$. Let $\rho(H)$ be the spectral radius of H. Then $\rho(H) > max\{p, q\} - 1$. Let $\xi = (y_1, \ldots, y_{p-k}, x_1, \ldots, x_k, z_1, \ldots, z_{q-k})^T$ be a Perron vector of H, where, for $1 \leq i \leq k$, $1 \leq j \leq p-k$ and $1 \leq l \leq q-k$, x_i, y_j and z_l correspond to v_i, u_j and w_l, respectively. Then by the symmetry of H we have $y_1 = \cdots = y_{p-k}$, $x_1 = \cdots = x_k$ and $z_1 = \cdots = z_{q-k}$.

Lemma 1. *Let $H = K_pv_1v_2\cdots v_kK_q$, $\rho(H)$, ξ, x_i, y_j and z_l be as above. Write $x = x_i$, $y = y_j$ and $z = z_l$. Then $(p-k)y > z$ and $(q-k)z > y$ if $min\{p, q\} > k+1$ and $\rho(H) \geq max\{p, q\} + k - 1$.*

Proof If $p = q$, then $y = z$, and so $(p-k)y > z$ and $(q-k)z > y$ since $p-k = q-k > 1$. Suppose that $p > q$. Then $\rho(H) \geq p+k-1$. From $A(H)\xi = \rho(H)\xi$ we get $\rho(H)y = (p-k-1)y + kx > (q-k-1)y + kx$ and $\rho(H)z = (q-k-1)z + kx$. Hence $\rho(H)(y-z) > (q-k-1)(y-z)$, which implies that $y > z$. Thus $(p-k)y > z$.

Next we will show that $(q-k)z > y$. From $A(H)\xi = \rho(H)\xi$ we also get $\rho(H)x = (p-k)y + (q-k)z + (k-1)x$ and $\rho(H)y = (p-k-1)y + kx$. Thus $(q-k)z = (\frac{1}{k}(\rho(H) - k+1)(\rho(H) - p+k+1) - p+k)y$.

Let $f(\lambda) = (\lambda-k+1)(\lambda-p+k+1) - k(p-k)$. Then $f'(\lambda) = 2\lambda - (p-2)$, and so $f'(\lambda) > 0$ if $\lambda > \frac{1}{2}(p-2)$. It is easy to verify that $f(\lambda) > 1$ if $\lambda \geq p+k-1$, which implies that $(q-k)z > y$.

Similarly, we can prove that $(p-k)y > z$ and $(q-k)z > y$ if $q > p$. □

Theorem 2. *Of all the connected graphs with n vertices and a k-vertex cut ($n \geq k+2 \geq 3$), the maximal spectral radius is obtained uniquely at K^k_{n-1}.*

Proof We have to prove that if $G \in \mathscr{G}^k_n$, then $\rho(G) \leq \rho(K^k_{n-1})$ with equality only when $G = K^k_{n-1}$. If $\rho(G) \leq n-2$, then $\rho(G) < \rho(K^k_{n-1})$. So suppose $\rho(G) > n-2$. The adjacency matrix of a connected graph is irreducible, so if we add an edge e to a connected graph G, $\rho(G+e) > \rho(G)$. Thus we can assume that the k-vertex cut of G is contained in exactly two blocks and that these two blocks are cliques. Denote these two blocks by K_p and K_q, respectively. Then $p+q = n$ and $p, q > k$. If $p = k+1$ or $q = k+1$, then $G = K^k_{n-1}$. So suppose $p, q > k+1$. Let $V(K_p) =$

$\{u_1, \ldots, u_{p-k}, v_1, \ldots, v_k\}$ and $V(K_q) = \{w_1, \ldots, w_{q-k}, v_1, \ldots, v_k\}$. Then $G = K_p v_1 v_2 \cdots v_k K_q$ and $V(K_p) \cap V(K_q) = \{v_1, \ldots, v_k\}$.

Select some w_l or u_j, say u_{p-k}, of G as the vertex o and delete all edges $u_j u_{p-k}$ $(1 \le j \le p-k-1)$ and join all vertices u_j with w_l $(1 \le j \le p-k-1, 1 \le l \le q-k)$. Then we obtain the graph $\overline{G}$. Obviously $\overline{G} \cong K_{n-1}^k$.

Let ξ be a Perron vector of G, and x, y and z are the coordinates of ξ corresponding to v_i, u_j and w_l, respectively. Then, by a simple calculation, we can obtain

$$\xi^T(A(\overline{G}) - A(G))\xi = 2(p-k-1)((q-k)z - y)y$$

By Lemma 1, we know that $\xi^T(A(\overline{G}) - A(G))\xi > 0$.

Thus $\rho(\overline{G}) = max_{\eta \neq 0} \frac{\eta^T A(\overline{G})\eta}{\eta^T \eta} \ge \frac{\xi^T A(\overline{G})\xi}{\xi^T \xi} > \frac{\xi^T A(G)\xi}{\xi^T \xi} = \rho(G)$. □

The study of the limit points of the eigenvalues of a graph was initiated by Hoffman in [6], where he posed the problem of finding the limits of eigenvalues of graphs. Now we consider the limits of the spectral radius of K_{n-1}^k.

Theorem 3. *Let ρ be the spectral radius of the graph K_{n-1}^k $(n \ge k+2 \ge 3)$. Then*

(i) $n - 2 < \rho < n - 2 + \frac{k^2}{n^2 - 3n - k + 2}$;
(ii) $limit_{n \to \infty}(\rho - (n-2)) = 0$.

Proof Suppose $V(K_{n-1}^k) = \{o, v_1, \ldots, v_k, w_1, \ldots, w_{n-k-1}\}$, where the vertex o only joins with the vertices $v_1, \ldots, v_k$. Let ζ be a Perron vector of K_{n-1}^k in which x, y and z are the coordinates corresponding to the vertices v_i, o and w_l, respectively $(1 \le i \le k, 1 \le l \le n-k-1)$.

From $A(K_{n-1}^k)\zeta = \rho\zeta$ we obtain $\rho y = kx$, $\rho x = y + (k-1)x + (n-k-1)z$ and $\rho z = kx + (n-k-2)z$. Hence we have

$$\rho^3 - (n-3)\rho^2 - (n+k-2)\rho + k(n-k-2) = 0 \qquad (1)$$

Since K_{n-1}^k contains K_{n-1} as a subgraph, $\rho > n-2$, and so we can assume $\rho = n - 2 + \delta$, where $\delta > 0$. Thus, by (1), we have

$$\delta^3 + (2n-3)\delta^2 + (n^2 - 3n - k + 2)\delta - k^2 = 0 \qquad (2)$$

Hence $\delta < \frac{k^2}{n^2 - 3n - k + 2}$. Since $n \ge k + 2 \ge 3$, $n^2 - 3n - k + 2 > 0$. Part (ii) follows from part (i). □

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