ON FAMILIES OF BIPARTITE GRAPHS ASSOCIATED WITH SUMS OF FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we consider the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

1. Introduction

The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n > 2$

$$F_n = F_{n-1} + F_{n-2}$$

where $F_1 = F_2 = 1$.

The Lucas Sequence, $\{L_n\}$, is defined by the recurrence relation, for $n > 2$

$$L_n = L_{n-1} + L_{n-2}$$

where $L_1 = 1$, $L_2 = 3$.

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a $(0,1)-matrix$ if each of its entries is either 0 or 1.

In [4], Minc constructed the $n \times n$ $(0,1)-matrix$ $F(n,k)$ where, $k \leq n+1$, with 1 in the (i,j) position for $i-1 \leq j \leq i+k-1$ and 0 otherwise. Then he showed that $perF(n,k) = g_{n+1}^k$ where g_n^k is the nth generalized order$-k$ Fibonacci number. When $k = 2$, $perF(n,2) = F_{n+2}$.

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Also Lee defined the matrix $\mathcal{L}_n$ as follows [3]:

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & & 0 \\ 1 & 1 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & & \\ 0 & 0 & 1 & 1 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & 0 \\ \vdots & & & & & & 1 \\ 0 & \cdots & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per\mathcal{L}_n = L_{n-1}$$

where L_n is the nth Lucas number.

In this paper, we find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrix is a sum of consecutive Fibonacci or Lucas numbers.

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A $1-$ factor (or *perfect matching*) of a graph with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications. Let $A(G)$ be the adjacency matrix of the bipartite graph G, and let $\mu(G)$ denote the number of 1-factors of G. Then, one can find the following fact in [5]: $\mu(G) \leq \sqrt{perA(G)}$. Also, one can find more applications of permanents in [5].

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the *bipartite adjacent matix* $B(G) = [b_{ij}]$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and 0 otherwise. Then, in [2] and [5], the number of 1-factors of bipartite graph G equals the permanent of its bipartite adjacency matrix.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is *contractible on column* (resp. *row*.) k *if column* (resp. *row*.) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called *the contraction of A on row* *k relative to columns i and j.* Every contraction used in this paper will be on the first column using the first and second rows. We say that A can be contracted to a matrix B if either $B = A$ or if there exist matrices $A_0, A_1, \ldots A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, 2, \ldots, t$. One can find the following fact in [1]: let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then

$$per A = per B. \tag{1.1}$$

2. The Sums of the Fibonacci numbers

In this section, we determine a class of bipartite graphs whose number of 1-factors is the summation of the Fibonacci numbers, $\sum_{i=0}^{n} F_i$.

Let n be positive integer and $n \geq 3$. Let $P_n = [p_{ij}]$ be the $n \times n$ $(0,1)$ −tridiagonal matrix with $p_{ij} = 1$ if and only if $|i - j| \leq 1$. Let $R_n = [r_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $r_{1j} = 1$ if and only if $3 \leq j \leq n$. Now we consider the sum of these matrices and denote by $V_n = [s_{ij}] = P_n + R_n$. Clearly

$$V_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 1 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & 0 & \ldots & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 1. *Let $G(V_n)$ be the bipartite graph with bipartite adjacency matrix $V_n = P_n + R_n$, $n \geq 3$. Then the number of 1-factors of $G(V_n)$ is $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.*

Proof. If $n = 3$, then we have

$$per V_3 = per \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = F_0 + F_1 + F_2 + F_3 = 4.$$

Let V_n^k be the kth contraction of V_n, $1 \leq k \leq n-2$. Since the definition of the matrix V_n, the matrix V_n can be contracted on column 1 so that

$$V_n^1 = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix V_n^1 can be contracted on column 1 and $\sum_{i=0}^{3} F_i = 4$ and $F_4 = 3$,

$$\begin{aligned} V_n^2 &= \begin{bmatrix} 4 & 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \cdots & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^{3} F_i & F_4 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \cdots & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore, the matrix V_n^2 can be contracted on column 1 so that

$$V_n^3 = \begin{bmatrix} \sum_{i=0}^{4} F_i & F_5 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we have

$$V_n^k = \begin{bmatrix} \sum_{i=0}^{k+1} F_i & F_{k+2} & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for $3 \le k \le n-4$. Hence,

$$V_n^{(n-3)} = \begin{bmatrix} \sum_{i=0}^{n-2} F_i & F_{n-1} & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of $V_n^{(n-4)}$ on column 1, gives

$$V_n^{(n-2)} = \begin{bmatrix} \sum_{i=0}^{n-1} F_i & F_n \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we obtain $perV_n = perV_n^{(n-2)} = \sum_{i=0}^{n} F_i$. □

3. On the Lucas Numbers and Their Sums

In this section, we determine two classes of bipartite graphs whose number of 1-factors are the the Lucas numbers and their sums, $\sum_{i=0}^{n-2} L_i$. Now we note that our result on the Lucas numbers is different from the result of Lee.

Firstly, let be positive integer such that $n \ge 4$ and let $C_n = [c_{ij}]$ be the $n \times n$ $(0,1)$ −matrix with $c_{13} = c_{14} = 1$ and $c_{ij} = 1$ for $|i-j| \le 1$ and 0 otherwise. Clearly

$$C_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ \vdots & & & & & & 1 \\ 0 & \dots & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 2. *Let $G(C_n)$ be the bipartite graph with bipartite adjacency matrix C_n, $n \ge 4$. Then the number of 1-factors of $G(C_n)$ is L_n where L_i is the ith Lucas number.*

Proof. If $n = 4$, then we have

$$perC_4 = per\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 7 = L_4.$$

Let C_n^k be the kth contraction of C_n, $1 \le k \le n-2$. Since the definition of the matrix C_n, the matrix C_n can be contracted on column 1 so that

$$C_n^1 = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 1 & 1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix S_n^1 can be contracted on column 1 and $L_2 = 3$, $L_3 = 4$,

$$\begin{aligned} C_n^2 &= \begin{bmatrix} 4 & 3 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} L_3 & L_2 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore, the matrix C_n^2 can be contracted on column 1 so that

$$C_n^3 = \begin{bmatrix} L_4 & L_3 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we reach

$$C_n^k = \begin{bmatrix} L_{k+1} & L_k & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for $3 \leq k \leq n-4$. Hence,

$$C_n^{(n-3)} = \begin{bmatrix} L_{n-2} & L_{n-3} & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of $C_n^{(n-4)}$ on column 1, gives

$$C_n^{(n-2)} = \begin{bmatrix} L_{n-3} + L_{n-2} & L_{n-2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} L_{n-1} & L_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we obtain $perC_n = perC_n^{(n-2)} = L_n$.
So the proof is complete. □

Secondly, let n be positive integer such that $n \geq 4$ and let $K_n = [k_{ij}]$ be the $n \times n$ $(0,1)$ −tridiagonal matrix with entries $k_{ij} = 1$ for $|i-j| \leq 1$ and $2 \leq i,j \leq n$, $k_{11} = 1$ and 0 otherwise. Let D_n be the $n \times n$ $(0,1)-$ matrix with $d_{1j} = 1$ for $3 \leq j \leq n$, $d_{24} = 1$ and 0 otherwise. Now we consider the sum of these matrices, K_n and D_n, and denote by $W_n = [w_{ij}] = K_n + D_n$.

Clearly

$$W_n = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & \cdots & & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & & & 0 \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & & \vdots \\ & & & & & & & & 0 \\ \vdots & & & & & & & & 1 \\ 0 & \cdots & & & \cdots & & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 3. *Let $G(W_n)$ be the bipartite graph with bipartite adjacency matrix $W_n = [w_{ij}] = K_n + D_n$, $n \geq 4$. Then the number of 1-factors of $G(W_n)$ is $\sum_{i=0}^{n-2} L_i = L_n - 1$.*

Proof. If $n = 4$, then we have

$$per W_4 = per \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 6 = L_0 + L_1 + L_2.$$

Let W_n^k be the kth contraction of W_n, $1 \leq k \leq n-2$. Since the definition of the matrix W_n, the matrix W_n can be contracted on column 1 so that

$$W_n^1 = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & \cdots & & 1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & & 1 \\ 0 & \cdots & & & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix W_n^1 can be contracted on column 1 and $\sum_{i=0}^{1} L_i = 3$ and $L_2 = 3$,

$$W_n^2 = \begin{bmatrix} 3 & 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{1} L_i & L_2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Furthermore, the matrix W_n^2 can be contracted on column 1 so that

$$W_n^3 = \begin{bmatrix} \sum_{i=0}^{2} L_i & L_3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we reach

$$W_n^k = \begin{bmatrix} \sum_{i=0}^{k-1} L_i & L_k & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for $4 \le k \le n-4$. Hence,

$$W_n^{(n-3)} = \begin{bmatrix} \sum_{i=0}^{n-4} L_i & L_{n-3} & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of $W_n^{(n-4)}$ on column 1, gives

$$W_n^{(n-2)} = \begin{bmatrix} \sum_{i=0}^{n-3} L_i & L_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we have $perW_n = perW_n^{(n-2)} = \sum_{i=0}^{n-2} L_i$. □

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