λ-Designs with Two Block Sizes

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Abstract

A λ-design on v points is a set of v subsets (blocks) of a v-set such that any two distinct blocks meet in exactly λ points and not all of the blocks have the same size. Ryser's and Woodall's λ-design conjecture states that all λ-designs can be obtained from symmetric designs by a complementation procedure. In this paper, we establish feasibility criteria for the existence of λ-designs with two block sizes in the form of integrality conditions, equations, inequalities, and Diophantine equations involving various parameters of the designs. We use these criteria and a computer to prove that the λ-design conjecture is true for all λ-designs with two block sizes with $\lambda \leq 90$ and $\lambda \neq 45$.

1 Introduction

Definition 1.1. Given integers λ and v, $0 < \lambda < v$, a λ-*design* on v points is a pair $(X, \mathcal{B})$, where X is a set of cardinality v whose elements are called *points* and $\mathcal{B}$ is a set of v subsets of X whose elements are called *blocks*, such that

(i) for all blocks $A, B \in \mathcal{B}$, $A \neq B$, $|A \cap B| = \lambda$, and

(ii) there exist blocks $A, B \in \mathcal{B}$ with $|A| \neq |B|$.

λ-designs were first defined by Ryser [17], [18] and Woodall [26]. The only known examples of λ-designs are obtained from symmetric designs by the following complementation procedure. Let $(X, \mathcal{A})$ be a symmetric (v, k, μ)-design with $\mu \neq k/2$ and fix a block $A \in \mathcal{A}$. Put $\mathcal{B} = \{A\} \cup \{A \Delta B : B \in \mathcal{A}, B \neq A\}$, where Δ denotes the symmetric difference of sets (we refer to this procedure as *complementing* with respect to the block A). Then an elementary counting argument shows that $(X, \mathcal{B})$ is a λ-design on v points with one block of size k and $v-1$ blocks of size 2λ, where $\lambda = k - \mu$. Any λ-design obtained in this manner is called a *type*-1 λ-design.

The λ-*design conjecture* of Ryser [17], [18] and Woodall [26] states that all λ-designs are type-1. The conjecture was proven for $\lambda = 1$ by deBruijn and Erdős [5], for $\lambda = 2$ by Ryser [17], for $3 \leq \lambda \leq 9$ by Bridges and Kramer [2], [3], [14], for $\lambda = 10$ by Seress [20], for $\lambda = 14$ by Tsaur [4], [24], and for $\lambda \leq 34$ by Weisz [25]. S. S. Shrikhande and Singhi [22] proved the conjecture for prime λ and Seress [21] proved it when λ is twice a prime.

Investigating the conjecture as a function of v rather than λ, Ionin and M. S. Shrikhande [11], [12] proved the conjecture for $v = p+1, 2p+1, 3p+1$, and $4p+1$, where p is any prime, Hein [10] proved it for $v = 5p+1$, where $p \not\equiv 2$ or $8 \pmod{15}$ is prime, and Fiala [7], [8] proved it for $v = 6p+1$, p any prime, and $v = 8p+1$, $p \equiv 1$ or $7 \pmod 8$ prime.

The reader interested in λ-designs should consult the last chapter of [13].

2 Preliminary results

In this section, we gather together some results on λ-designs that we shall need later. First, we have the following characterization of type-1 λ-designs.

Theorem 2.1. [26] *A λ-design is type-1 if and only if it has exactly two distinct block sizes, one of which occurs only once.*

Next, we make the following definition.

Definition 2.2. Given a λ-design $(X, \mathcal{B})$ and a point $x \in X$, the *replication number* of x, denoted by r_x, is the number of blocks $A \in \mathcal{B}$ containing x.

Ryser [17] and Woodall [26] independently proved the following theorem.

Theorem 2.3. *If $(X, \mathcal{B})$ is a λ-design on v points, then there exist integers $r_1 > r_2 > 1$ such that every point has replication number r_1 or r_2 and*

$$r_1 + r_2 = v + 1. \tag{1}$$

In addition, if x_1 and y_1 have replication number r_1 and x_2 and y_2 have replication number r_2, then

$$\sum_{A : x_1 \in A} \frac{1}{|A| - \lambda} = \frac{v-1}{r_2 - 1}, \tag{2}$$

$$\sum_{A : x_2 \in A} \frac{1}{|A| - \lambda} = \frac{v-1}{r_1 - 1}, \tag{3}$$

$$\sum_{A:x_1,y_1\in A} \frac{1}{|A|-\lambda} = \frac{r_1-1}{r_2-1}, \tag{4}$$

$$\sum_{A:x_1,x_2\in A} \frac{1}{|A|-\lambda} = 1, \tag{5}$$

$$\sum_{A:x_2,y_2\in A} \frac{1}{|A|-\lambda} = \frac{r_2-1}{r_1-1}, \tag{6}$$

and

$$\frac{1}{\lambda} + \sum_{A\in\mathcal{B}} \frac{1}{|A|-\lambda} = \frac{(v-1)^2}{(r_1-1)(r_2-1)}. \tag{7}$$

We also have the following results.

Theorem 2.4. [19] *A λ-design on v points with replication numbers r_1 and r_2 is type-1 if and only if $r_1(r_1-1)/(v-1)$ or $r_2(r_2-1)/(v-1)$ is an integer.*

Theorem 2.5. [15] *Let $(X,\mathcal{B})$ be a λ-design with replication numbers r_1 and r_2, $r_1 > r_2$. Then $(r_1-1)/(r_2-1) \leq \lambda$ and if $(r_1-1)/(r_2-1) \geq \lambda-1$, then $(X,\mathcal{B})$ is type-1.*

Theorem 2.6. [7], [8], [10], [11], [12] *Let $(X,\mathcal{B})$ be a λ-design with replication numbers r_1 and r_2. If $\gcd(r_1-1, r_2-1) = 1, 2, 3, 4, 5, 6$, or 8, then $(X,\mathcal{B})$ is type-1.*

Theorem 2.7. [21], [23] *Let $(X,\mathcal{B})$ be a λ-design with replication numbers r_1 and r_2. Let $g = \gcd(r_1-1, r_2-1)$. If $\gcd(\lambda, (r_1-r_2)/g) = 1$, 2, or $\lambda/2$, then $(X,\mathcal{B})$ is type-1.*

Let $(X,\mathcal{B})$ be a λ-design on v points. Then Theorem 2.3 implies that every point has replication number r_1 or r_2 for some integers $r_1 > r_2$. Therefore, the set X is partitioned into two subsets, E_1 and E_2, of points having replication numbers r_1 and r_2, respectively. Let $|E_1| = e_1$ and $|E_2| = e_2$. Then

$$e_1 + e_2 = v \tag{8}$$

and counting in two different ways the set of triples $(x, A, B) \in X \times \mathcal{B}^2$ such that $A \neq B$ and $x \in A \cap B$ we obtain

$$e_1 r_1(r_1-1) + e_2 r_2(r_2-1) = v(v-1)\lambda. \tag{9}$$

Solving equations (8) and (9) for e_1 and e_2 (and using (1) to simplify) we obtain

$$e_1 = \frac{\lambda(v-1) - r_2(r_2-1)}{r_1 - r_2} \tag{10}$$

and

$$e_2 = \frac{r_1(r_1-1)-\lambda(v-1)}{r_1-r_2}. \tag{11}$$

Therefore, the right-hand sides of (10) and (11) are positive integers. We also have the following results.

Theorem 2.8. [2] *Let $(X,\mathcal{B})$ be a λ-design with replication numbers r_1 and r_2, $r_1 > r_2$. If $e_1 = 1$, then $(X,\mathcal{B})$ is type-1. Also, $e_1 \neq 2$.*

Theorem 2.9. [19] *Let $(X,\mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2. Then $e_1e_2 \leq \lambda(v-1)$ and $(X,\mathcal{B})$ is type-1 if and only if $e_1e_2 = \lambda(v-1)$.*

Moreover, each block A is partitioned into two subsets, $A' = A \cap E_1$ and $A^* = A \cap E_2$, of points having replication number r_1 and r_2, respectively. Let $|A| = k$, $|A'| = k'$, and $|A^*| = k^*$. Then

$$k' + k^* = k \tag{12}$$

and counting in two ways the set of pairs $(x, B) \in X \times \mathcal{B} \setminus \{A\}$ such that $x \in A \cap B$ we obtain

$$k'(r_1-1) + k^*(r_2-1) = (v-1)\lambda. \tag{13}$$

Solving equations (12) and (13) for k' and k^* we obtain

$$k' = \frac{\lambda(v-1)-k(r_2-1)}{r_1-r_2} \tag{14}$$

and

$$k^* = \frac{k(r_1-1)-\lambda(v-1)}{r_1-r_2}. \tag{15}$$

Thus, $|A'|$ and $|A^*|$ depend only on $|A|$.

Given two distinct points x and y, denote by r_{xy} the number of blocks containing both x and y. We have the following results.

Theorem 2.10. [19] *Let $(X,\mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$. Let $x_1 \in E_1$ and $x_2 \in E_2$. Then more than half of the numbers in $\{r_{x_1y_2} : y_2 \in E_2\}$ are equal to $\lceil r_1(r_2-1)/(v-1)\rceil$ and more than half of the numbers in $\{r_{x_2y_2} : y_2 \in E_2, y_2 \neq x_2\}$ are equal to $\lceil r_2(r_2-1)/(v-1)\rceil$.*

Remark 2.11. Even though r_{xy} is not necessarily constant for constant r_x and r_y, for convenience we set $r_{12} = \lceil r_1(r_2-1)/(v-1)\rceil$ and $r_{22} = \lceil r_2(r_2-1)/(v-1)\rceil$.

Theorem 2.12. [19] *A λ-design $(X,\mathcal{B})$ is type-1 if and only if there exists $x \in X$ such that r_{xy} depends only on r_y.*

3 Integrality conditions

Since type-1 λ-designs have only two different block sizes, it seems natural to study λ-designs possessing only two distinct block sizes. In this section, we begin the study of such designs by establishing integrality conditions involving various parameters of the designs.

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2 and block sizes k_1 and k_2. Then (14) and (15) imply that the number of points in a block of size k_i, $i = 1, 2$, of replication number r_j, $j = 1, 2$, depends only on i and j and is given by the expressions

$$k_1' = \frac{\lambda(v-1) - k_1(r_2 - 1)}{r_1 - r_2}, \tag{16}$$

$$k_1^* = \frac{k_1(r_1 - 1) - \lambda(v-1)}{r_1 - r_2}, \tag{17}$$

$$k_2' = \frac{\lambda(v-1) - k_2(r_2 - 1)}{r_1 - r_2}, \tag{18}$$

and

$$k_2^* = \frac{k_2(r_1 - 1) - \lambda(v-1)}{r_1 - r_2}, \tag{19}$$

respectively. Therefore, the right-hand sides of (16), (17), (18), and (19) must all be nonnegative integers. We also have the following result.

Theorem 3.1. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, and block sizes k_1 and k_2, $k_1 > k_2$. Then $r_2 \leq k_2 < k_1 \leq r_1$.*

Proof. Clearly, we must have $k_1', k_2' \leq e_1$ and $k_1^*, k_2^* \leq e_2$. Using (10), (11), (16), (17), (18), and (19), we obtain the result. □

Given a point x, denote by r_x' the number of blocks of size k_1 that contain x and denote by r_x^* the number of blocks of size k_2 that contain x. We have the following result.

Theorem 3.2. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, and block sizes k_1 and k_2. Then r_x' and r_x^* depend only on r_x. In addition, if we denote the number of blocks of size k_i, $i = 1, 2$, that contain a fixed point of replication number r_j, $j = 1, 2$, by r_1', r_1^*, r_2', and r_2^*, respectively, then*

$$r_1' = \frac{(k_1 - \lambda)[r_1(r_2 - 1) - (k_2 - \lambda)(v-1)]}{(k_1 - k_2)(r_2 - 1)}, \tag{20}$$

$$r_1^* = \frac{(k_2-\lambda)[(k_1-\lambda)(v-1)-r_1(r_2-1)]}{(k_1-k_2)(r_2-1)}, \tag{21}$$

$$r_2' = \frac{(k_1-\lambda)[r_2(r_1-1)-(k_2-\lambda)(v-1)]}{(k_1-k_2)(r_1-1)}, \tag{22}$$

and

$$r_2^* = \frac{(k_2-\lambda)[(k_1-\lambda)(v-1)-r_2(r_1-1)]}{(k_1-k_2)(r_1-1)}. \tag{23}$$

Thus, the right-hand sides of (20), (21), (22), *and* (23) *must all be nonnegative integers. Furthermore,* $\max\{r_1', r_1^*\} \le r_1$ *and* $\max\{r_2', r_2^*\} \le r_2$.

Proof. If $x \in E_1$, then

$$r_x' + r_x^* = r_1 \tag{24}$$

and (2) gives us

$$\frac{r_x'}{k_1-\lambda} + \frac{r_x^*}{k_2-\lambda} = \frac{v-1}{r_2-1}. \tag{25}$$

Therefore, r_x' and r_x^* depend only on r_x. Putting $r_x' = r_1'$ and $r_x^* = r_1^*$ and solving equations (24) and (25) for r_1' and r_1^* we obtain (20) and (21).

If $x \in E_2$, then

$$r_x' + r_x^* = r_2 \tag{26}$$

and (3) gives us

$$\frac{r_x'}{k_1-\lambda} + \frac{r_x^*}{k_2-\lambda} = \frac{v-1}{r_1-1}. \tag{27}$$

Thus, r_x' and r_x^* depend only on r_x. Putting $r_x' = r_2'$ and $r_x^* = r_2^*$ and solving equations (26) and (27) for r_2' and r_2^* we obtain (22) and (23). □

Given two distinct points x and y, denote by r_{xy}' the number of blocks of size k_1 that contain both x and y and denote by r_{xy}^* the number of blocks of size k_2 that contain x and y. We have the following results.

Theorem 3.3. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, and block sizes k_1 and k_2. Let $x, y \in X$, $x \neq y$. Then*

$$r_{xy}' = \frac{(k_1-\lambda)[r_{xy}(r_2-1)-(k_2-\lambda)(r_1-1)]}{(k_1-k_2)(r_2-1)} \tag{28}$$

and

$$r_{xy}^* = \frac{(k_2-\lambda)[(k_1-\lambda)(r_1-1)-r_{xy}(r_2-1)]}{(k_1-k_2)(r_2-1)} \tag{29}$$

if $x, y \in E_1$,

$$r_{xy}' = \frac{(k_1-\lambda)(r_{xy}-(k_2-\lambda))}{k_1-k_2} \tag{30}$$

and

$$r^*_{xy} = \frac{(k_2-\lambda)(k_1-\lambda-r_{xy})}{k_1-k_2} \tag{31}$$

if $x \in E_1$ *and* $y \in E_2$, *and*

$$r'_{xy} = \frac{(k_1-\lambda)[r_{xy}(r_1-1)-(k_2-\lambda)(r_2-1)]}{(k_1-k_2)(r_1-1)} \tag{32}$$

and

$$r^*_{xy} = \frac{(k_2-\lambda)[(k_1-\lambda)(r_2-1)-r_{xy}(r_1-1)]}{(k_1-k_2)(r_1-1)} \tag{33}$$

if $x, y \in E_2$.

Proof. First of all,

$$r'_{xy} + r^*_{xy} = r_{xy}. \tag{34}$$

If $x, y \in E_1$, then (4) gives us

$$\frac{r'_{xy}}{k_1-\lambda} + \frac{r^*_{xy}}{k_2-\lambda} = \frac{r_1-1}{r_2-1}. \tag{35}$$

Solving equations (34) and (35) for r'_{xy} and r^*_{xy} gives us (28) and (29).
If $x \in E_1$ and $y \in E_2$, then (5) gives us

$$\frac{r'_{xy}}{k_1-\lambda} + \frac{r^*_{xy}}{k_2-\lambda} = 1. \tag{36}$$

Solving equations (34) and (36) for r'_{xy} and r^*_{xy} gives us (30) and (31).
If $x, y \in E_2$, then (6) gives us

$$\frac{r'_{xy}}{k_1-\lambda} + \frac{r^*_{xy}}{k_2-\lambda} = \frac{r_2-1}{r_1-1}. \tag{37}$$

Solving equations (34) and (37) for r'_{xy} and r^*_{xy} gives us (32) and (33). □

Corollary 3.4. *Let* $(X, \mathcal{B})$ *be a* λ*-design on* v *points with replication numbers* r_1 *and* r_2, $r_1 > r_2$, *and block sizes* k_1 *and* k_2. *Then the expressions*

$$\frac{(k_1-\lambda)(\lceil\frac{r_1(r_2-1)}{v-1}\rceil-(k_2-\lambda))}{k_1-k_2}, \tag{38}$$

$$\frac{(k_2-\lambda)(k_1-\lambda-\lceil\frac{r_1(r_2-1)}{v-1}\rceil)}{k_1-k_2}, \tag{39}$$

$$\frac{(k_1-\lambda)[\lceil\frac{r_2(r_2-1)}{v-1}\rceil(r_1-1)-(k_2-\lambda)(r_2-1)]}{(k_1-k_2)(r_1-1)}, \tag{40}$$

and

$$\frac{(k_2-\lambda)[(k_1-\lambda)(r_2-1)-\lceil\frac{r_2(r_2-1)}{v-1}\rceil(r_1-1)]}{(k_1-k_2)(r_1-1)} \tag{41}$$

are all nonnegative integers.

Proof. There exist $x \in E_1$ and $y \in E_2$ such that $r_{xy} = r_{12}$ by Theorem 2.10. Substituting this expression into (30) and (31) implies that (38) and (39) are nonnegative integers. Similarly, there exist $x, y \in E_2$, $x \neq y$, such that $r_{xy} = r_{22}$ by Theorem 2.10. Substituting this expression into (32) and (33) implies that (40) and (41) are nonnegative integers. □

Remark 3.5. Even though r'_{xy} and r^*_{xy} depend on r_{xy} and not just on r_x and r_y, for convenience we denote expressions (38), (39), (40), and (41) by r'_{12}, r^*_{12}, r'_{22}, and r^*_{22}, respectively. Note that we must have $r'_{12} \leq \min\{r'_1, r'_2, r_{12}\}$, $r^*_{12} \leq \min\{r^*_1, r^*_2, r_{12}\}$, $r'_{22} \leq \min\{r'_2, r_{22}\}$, and $r^*_{22} \leq \min\{r^*_2, r_{22}\}$.

Theorem 3.6. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Then*

$$v_1 = \frac{(k_1-\lambda)[(k_2+\lambda(v-1))(r_1-1)(r_2-1)-\lambda(k_2-\lambda)(v-1)^2]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)} \tag{42}$$

and

$$v_2 = \frac{(k_2-\lambda)[\lambda(k_1-\lambda)(v-1)^2-(k_1+\lambda(v-1))(r_1-1)(r_2-1)]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)}. \tag{43}$$

Therefore, the right-hand sides of (42) *and* (43) *must both be positive integers. Furthermore, we must have $v_1 \geq \max\{r'_1, r'_2\}$ and $v_2 \geq \max\{r^*_1, r^*_2\}$.*

Proof. First of all,

$$v_1 + v_2 = v. \tag{44}$$

Also, (7) gives us

$$\frac{1}{\lambda} + \frac{v_1}{k_1-\lambda} + \frac{v_2}{k_2-\lambda} = \frac{(v-1)^2}{(r_1-1)(r_2-1)}. \tag{45}$$

Solving equations (44) and (45) for v_1 and v_2 we obtain (42) and (43). □

Corollary 3.7. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, and block sizes k_1 and k_2. Then $(X, \mathcal{B})$ is type-1 if and only if*

$$\frac{(k_1-\lambda)[(k_2+\lambda(v-1))(r_1-1)(r_2-1)-\lambda(k_2-\lambda)(v-1)^2]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)} = 1$$

or

$$\frac{(k_2-\lambda)[\lambda(k_1-\lambda)(v-1)^2-(k_1+\lambda(v-1))(r_1-1)(r_2-1)]}{\lambda(k_1-k_2)(r_1-1)(r_2-1)}=1.$$

Proof. Apply Theorems 2.1 and 3.6. □

4 Equations

In this section, we derive several equations involving various parameters of λ-designs with two block sizes.

Theorem 4.1. *Let $(X,\mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Then*

$$\frac{(k_1-\lambda)[r_1(r_2-1)-(k_2-\lambda)(v-1)]}{(k_1-k_2)(r_2-1)}=\frac{v_1[\lambda(v-1)-k_1(r_2-1)]}{\lambda(v-1)-r_2(r_2-1)}, \quad (46)$$

$$\frac{(k_2-\lambda)[(k_1-\lambda)(v-1)-r_1(r_2-1)]}{(k_1-k_2)(r_2-1)}=\frac{v_2[\lambda(v-1)-k_2(r_2-1)]}{\lambda(v-1)-r_2(r_2-1)}, \quad (47)$$

$$\frac{(k_1-\lambda)[r_2(r_1-1)-(k_2-\lambda)(v-1)]}{(k_1-k_2)(r_1-1)}=\frac{v_1[\lambda(v-1)-k_1(r_1-1)]}{\lambda(v-1)-r_1(r_1-1)}, \quad (48)$$

and

$$\frac{(k_2-\lambda)[(k_1-\lambda)(v-1)-r_2(r_1-1)]}{(k_1-k_2)(r_1-1)}=\frac{v_2[\lambda(v-1)-k_2(r_1-1)]}{\lambda(v-1)-r_1(r_1-1)}. \quad (49)$$

Proof. For each $i,j=1,2$, we count in two different ways the set of pairs $(x,A)\in E_i\times\mathcal{B}$ such that $|A|=k_j$ and $x\in A$ and obtain

$$e_1r_1'=v_1k_1', \quad (50)$$

$$e_1r_1^*=v_2k_2', \quad (51)$$

$$e_2r_2'=v_1k_1^*, \quad (52)$$

and

$$e_2r_2^*=v_2k_2^*. \quad (53)$$

Solving equations (50), (51), (52), and (53) for r_1', r_1^*, r_2', and r_2^*, respectively, using (10), (11), (16), (17), (18), and (19), and equating the results with the right-hand sides of (20), (21), (22), and (23), we obtain (46), (47), (48), and (49). □

A type-1 λ-design not only possesses just two distinct block sizes, it is also the case that one of the blocks is E_1 or E_2. This suggests the following result which can be found in [24].

Theorem 4.2. *Let* $(X, \mathcal{B})$ *be a* λ*-design on* v *points with replication numbers* r_1 *and* r_2, $r_1 > r_2$, v_1 *blocks of size* k_1, *and* v_2 *blocks of size* k_2. *Then* $(X, \mathcal{B})$ *is type-1 if and only if* $k_1' = 0$, $k_1' = e_1$, $k_2' = 0$, *or* $k_2' = e_1$. *If* $e_2 > 1$, *then* $(X, \mathcal{B})$ *is type-1 if and only if* $k_1^* = 0$, $k_1^* = e_2$, $k_2^* = 0$, *or* $k_2^* = e_2$.

Proof. By Theorem 2.8, we may assume that $e_1 > 1$. Suppose that $(X, \mathcal{B})$ is type-1. Suppose that $0 < k_1', k_2' < e_1$. Then (50) and (51) imply that $r_1' = v_1(k_1'/e_1)$ and $r_1^* = v_2(k_2'/e_1)$. Therefore, $v_1, v_2 > 1$, a contradiction by Corollary 3.7.

Suppose $k_1' = 0$. Then $r_1' = 0$ by (50). Let $x \in E_1$. Then (4) and (5) imply that

$$\sum_{A:x,y\in A} \frac{1}{|A| - \lambda} = \frac{r_{xy}}{k_2 - \lambda} = \frac{r_1 - 1}{r_2 - 1}$$

if $y \in E_1$ and

$$\sum_{A:x,y\in A} \frac{1}{|A| - \lambda} = \frac{r_{xy}}{k_2 - \lambda} = 1$$

if $y \in E_2$. Thus, $r_{xy} = (k_2 - \lambda)(r_1 - 1)/(r_2 - 1)$ if $y \in E_1$ and $r_{xy} = k_2 - \lambda$ if $y \in E_2$. Therefore, by Theorem 2.12, $(X, \mathcal{B})$ is type-1. The remaining cases are proven similarly. □

Theorem 4.3. *Let* $(X, \mathcal{B})$ *be a* λ*-design on* v *points with replication numbers* r_1 *and* r_2, $r_1 > r_2$, v_1 *blocks of size* k_1, *and* v_2 *blocks of size* k_2. *Then*

$$e_1 r_1'(r_1' - 1) + e_2 r_2'(r_2' - 1) = \lambda v_1(v_1 - 1), \tag{54}$$

$$e_1 r_1' r_1^* + e_2 r_2' r_2^* = \lambda v_1 v_2, \tag{55}$$

and

$$e_1 r_1^*(r_1^* - 1) + e_2 r_2^*(r_2^* - 1) = \lambda v_2(v_2 - 1). \tag{56}$$

Proof. For $i, j = 1, 2$, count in two ways the number of triples $(x, A, B) \in X \times \mathcal{B}^2$ such that $A \neq B$, $|A| = k_i$, $|B| = k_j$, and $x \in A \cap B$. We obtain (54), (55), and (56). □

5 Inequalities

In this section, we establish some inequalities involving various parameters of λ-designs with two block sizes. First, we shall need some results.

Definition 5.1. Given a real symmetric $n \times n$ matrix A, we will denote the eigenvalues of A (which must be real) by $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. If B is a $m \times m$ matrix with $m \leq n$, then we say that the eigenvalues of B *interlace* the eigenvalues of A if B has only real eigenvalues and if $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$ for $i = 1, \ldots, m$. We say that the interlacing is *tight* if there exists an integer l, $0 \leq l \leq m$, such that $\lambda_i(A) = \lambda_i(B)$ for $i = 1, \ldots, l$ and $\lambda_{n-m+i}(A) = \lambda_i(B)$ for $i = l+1, \ldots, m$.

Theorem 5.2. [9] *Let A be a real symmetric $n \times n$ matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{1,m}^T & \cdots & A_{m,m} \end{pmatrix},$$

where $A_{i,i}$ is square for $i = 1, \ldots, m$. Let $b_{i,j}$ be the average row sum of $A_{i,j}$ for $i, j = 1, \ldots, m$. Let $B = (b_{i,j})$ (we refer to B as the quotient matrix of A with respect to the partition). Then the eigenvalues of B interlace the eigenvalues of A. If $A_{i,j}$ has constant row sums for $i, j = 1, \ldots, m$, then every eigenvalue of B is also an eigenvalue of A. If the interlacing is tight, then $A_{i,j}$ has constant row and column sums for $i, j = 1, \ldots, m$.

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Let N be any matrix whose rows are indexed by the elements of X (points in E_1 coming first), whose columns are indexed by the elements of $\mathcal{B}$ (blocks of size k_1 coming first), and whose (x, A) entry is 1 if $x \in A$ and is 0 otherwise. Thus,

$$N^T N = \begin{pmatrix} (k_1 - \lambda) I_{v_1} + \lambda J_{v_1} & \lambda J_{v_1, v_2} \\ \lambda J_{v_2, v_1} & (k_2 - \lambda) I_{v_2} + \lambda J_{v_2} \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix, J_n denotes the $n \times n$ matrix of all 1's, and $J_{m,n}$ denotes the $m \times n$ matrix of all 1's. Let

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}.$$

The rows of N can be partitioned into E_1 and E_2 and the columns of N can be partitioned into the set of blocks of size k_1 and the set of blocks of size k_2. This induces a partition of A with quotient matrix B given by

$$B = \begin{pmatrix} 0 & 0 & r_1' & r_1^* \\ 0 & 0 & r_2' & r_2^* \\ k_1' & k_1^* & 0 & 0 \\ k_2' & k_2^* & 0 & 0 \end{pmatrix}.$$

Theorem 5.3. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Then every eigenvalue of B is also an eigenvalue of A.*

Proof. The block matrices in the partitioning of A have constant row sums by Theorem 3.2. Now apply Theorem 5.2. □

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Suppose that $e_1, e_2 > 1$ and $r'_1, r^*_1, r'_2, r^*_2 > 0$. For $i = 1, 2$, fix $x_i \in E_i$ and define N_i to be any $(v-1) \times r_i$ matrix whose rows are indexed by the elements of $X \setminus \{x_i\}$ (points in E_1 coming first), whose columns are indexed by the set of blocks in $\mathcal{B}$ that contain x_i (blocks of size k_1 coming first), and whose (x, A) entry is 1 if $x \in A$ and is 0 otherwise. Therefore,

$$N_i^T N_i = \begin{pmatrix} (k_1 - \lambda)I_{r'_i} + (\lambda - 1)J_{r'_i} & (\lambda - 1)J_{r'_i, r^*_i} \\ (\lambda - 1)J_{r^*_i, r'_i} & (k_2 - \lambda)I_{r^*_i} + (\lambda - 1)J_{r^*_i} \end{pmatrix}.$$

For $i = 1, 2$, let

$$A_i = \begin{pmatrix} 0 & N_i \\ N_i^T & 0 \end{pmatrix}.$$

The rows of N_1 can be partitioned into $E_1 \setminus \{x_1\}$ and E_2 and the columns of N_1 can be partitioned into the set of blocks of size k_1 that contain x_1 and the set of blocks of size k_2 that contain x_1. Similarly, the rows of N_2 can be partitioned into E_1 and $E_2 \setminus \{x_2\}$ and the columns of N_2 can be partitioned into the set of blocks of size k_1 that contain x_2 and the set of blocks of size k_2 that contain x_2. This induces partitions of A_1 and A_2 with quotient matrices B_1 and B_2, respectively, given by

$$B_1 = \begin{pmatrix} 0 & 0 & \frac{r'_1(k'_1-1)}{e_1-1} & \frac{r^*_1(k'_2-1)}{e_1-1} \\ 0 & 0 & \frac{r'_1 k^*_1}{e_2} & \frac{r^*_1 k^*_2}{e_2} \\ k'_1 - 1 & k^*_1 & 0 & 0 \\ k'_2 - 1 & k^*_2 & 0 & 0 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & \frac{r'_2 k'_1}{e_1} & \frac{r^*_2 k'_2}{e_1} \\ 0 & 0 & \frac{r'_2(k^*_1-1)}{e_2-1} & \frac{r^*_2(k^*_2-1)}{e_2-1} \\ k'_1 & k^*_1 - 1 & 0 & 0 \\ k'_2 & k^*_2 - 1 & 0 & 0 \end{pmatrix}.$$

Theorem 5.4. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. If $e_1 > 1$ and $r'_1, r^*_1 > 0$, then the eigenvalues of B_1 interlace the eigenvalues of A_1 and if the interlacing is tight, then $(X, \mathcal{B})$ is type-1. If $e_2 > 1$ and*

$r'_2, r^*_2 > 0$, *then the eigenvalues of* B_2 *interlace the eigenvalues of* A_2 *and if the interlacing is tight, then* $(X, \mathcal{B})$ *is type*-1.

Proof. Apply Theorems 5.2 and 2.12. □

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. Suppose that $e_1 > 1$, $e_2 > 2$, and $r'_{12}, r^*_{12}, r'_{22}, r^*_{22} > 0$. By Theorem 2.10, there exist $x_1 \in E_1$ and $x_2, y_2, z_2 \in E_2$ such that $r_{x_1 x_2} = r_{12}$ and $r_{y_2 z_2} = r_{22}$. Define N_{12} to be any $(v-2) \times r_{12}$ matrix whose rows are indexed by the elements of $X \setminus \{x_1, x_2\}$ (points in E_1 coming first), whose columns are indexed by the set of blocks in $\mathcal{B}$ that contain x_1 and x_2 (blocks of size k_1 coming first), and whose (x, A) entry is 1 if $x \in A$ and is 0 otherwise. Similarly, define N_{22} to be any $(v-2) \times r_{22}$ matrix whose rows are indexed by the elements of $X \setminus \{y_2, z_2\}$ (points in E_1 coming first), whose columns are indexed by the set of blocks in $\mathcal{B}$ that contain y_2 and z_2 (blocks of size k_1 coming first), and whose (x, A) entry is 1 if $x \in A$ and is 0 otherwise. Thus,

$$N_{ij}^T N_{ij} = \begin{pmatrix} (k_1 - \lambda) I_{r'_{ij}} + (\lambda - 2) J_{r'_{ij}} & (\lambda - 2) J_{r'_{ij}, r^*_{ij}} \\ (\lambda - 2) J_{r^*_{ij}, r'_{ij}} & (k_2 - \lambda) I_{r^*_{ij}} + (\lambda - 2) J_{r^*_{ij}} \end{pmatrix}.$$

For $(i, j) = (1, 2), (2, 2)$, let

$$A_{ij} = \begin{pmatrix} 0 & N_{ij} \\ N_{ij}^T & 0 \end{pmatrix}.$$

The rows of N_{12} can be partitioned into $E_1 \setminus \{x_1\}$ and $E_2 \setminus \{x_2\}$ and the columns of N_{12} can be partitioned into the set of blocks of size k_1 that contain x_1 and x_2 and the set of blocks of size k_2 that contain x_1 and x_2. Similarly, the rows of N_{22} can be partitioned into E_1 and $E_2 \setminus \{y_2, z_2\}$ and the columns of N_{22} can be partitioned into the set of blocks of size k_1 that contain y_2 and z_2 and the set of blocks of size k_2 that contain y_2 and z_2. This induces partitions of A_{12} and A_{22} with quotient matrices B_{12} and B_{22}, respectively, given by

$$B_{12} = \begin{pmatrix} 0 & 0 & \frac{r'_{12}(k'_1 - 1)}{e_1 - 1} & \frac{r^*_{12}(k'_2 - 1)}{e_1 - 1} \\ 0 & 0 & \frac{r'_{12}(k^*_1 - 1)}{e_2 - 1} & \frac{r^*_{12}(k^*_2 - 1)}{e_2 - 1} \\ k'_1 - 1 & k^*_1 - 1 & 0 & 0 \\ k'_2 - 1 & k^*_2 - 1 & 0 & 0 \end{pmatrix}$$

and

$$B_{22} = \begin{pmatrix} 0 & 0 & \frac{r'_{22} k'_1}{e_1} & \frac{r^*_{22} k'_2}{e_1} \\ 0 & 0 & \frac{r'_{22}(k^*_1 - 2)}{e_2 - 2} & \frac{r^*_{22}(k^*_2 - 2)}{e_2 - 2} \\ k'_1 & k^*_1 - 2 & 0 & 0 \\ k'_2 & k^*_2 - 2 & 0 & 0 \end{pmatrix}.$$

Theorem 5.5. *Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$, v_1 blocks of size k_1, and v_2 blocks of size k_2. If $e_1, e_2 > 1$ and $r'_{12}, r^*_{12} > 0$, then the eigenvalues of B_{12} interlace the eigenvalues of A_{12}. If $e_2 > 2$ and $r'_{22}, r^*_{22} > 0$, then the eigenvalues of B_{22} interlace the eigenvalues of A_{22}.*

Proof. Apply Theorem 5.2. □

Remark 5.6. The spectrum of the matrix A is easily calculated since

$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^TN \end{pmatrix},$$

the spectra of NN^T and N^TN are the same, the eigenvalues of any matrix of the form

$$\begin{pmatrix} (a-c)I_m + cJ_m & cJ_{m,n} \\ cJ_{n,m} & (b-c)I_n + cJ_n \end{pmatrix}$$

are $a - c$ with multiplicity $m - 1$, $b - c$ with multiplicity $n - 1$, and

$$\frac{1}{2}(a + b + c(m + n - 2) \pm$$

$$\sqrt{(a + b + c(m + n - 2))^2 - 4((a - c)(b - c) + cm(b - c) + cn(a - c))}),$$

and the spectrum of A is symmetric with respect to zero. These comments also clearly apply to the calculation of the eigenvalues of the matrices A_1, A_2, A_{12}, and A_{22}.

6 Diophantine equations

In this section, we prove a theorem for λ-designs with two block sizes that is similar to the Bruck-Ryser-Chowla Theorem for symmetric designs [1]. First, we must recall some results from the theory of rational quadratic forms [6].

Definition 6.1. Let G and H be $n \times n$ matrices over $\mathbb{Q}$. Then G and H are *equivalent*, denoted by $G \cong H$, if there exists a matrix S over $\mathbb{Q}$ such that $H = S^TGS$ and $det(S) \neq 0$.

Theorem 6.2. *Any symmetric matrix is equivalent to a diagonal matrix $diag(d_1, d_2, \ldots, d_n)$. If $\pi \in S_n$, then $diag(d_1, d_2, \ldots, d_n) \cong diag(d_{\pi(1)}, d_{\pi(2)}, \ldots, d_{\pi(n)})$. If $c \neq 0$, then $diag(d_1, d_2, \ldots, d_n) \cong diag(c^2d_1, d_2, \ldots, d_n)$.*

If $diag(a_1, a_2, \ldots, a_n) \cong diag(b_1, b_2, \ldots, b_n)$, *then* $diag(a_1, a_2, \ldots, a_n, c) \cong diag(b_1, b_2, \ldots, b_n, c)$. *If*

$$G = \left(\begin{array}{ccc|c} & & & a_1 \\ & H & & \vdots \\ & & & a_{n-1} \\ \hline a_1 & \cdots & a_{n-1} & a_n \end{array}\right)$$

and $det(H) \neq 0$, *then* $G \cong diag(H, det(G)/det(H))$.

Theorem 6.3. *If* $diag(a_1, a_2, \ldots, a_n, c) \cong diag(b_1, b_2, \ldots, b_n, c)$, $c \neq 0$, *then* $diag(a_1, a_2, \ldots, a_n) \cong diag(b_1, b_2, \ldots, b_n)$.

Theorem 6.4. $diag(n, n, n, n) \cong diag(1, 1, 1, 1)$ *for any* $n \in \mathbb{N}$.

We now have the following result.

Theorem 6.5. *Let* $(X, \mathcal{B})$ *be a* λ*-design on* v *points with* v_1 *blocks of size* k_1 *and* v_2 *blocks of size* k_2. *Then*

$$(k_1 - \lambda)^{v_1 - 1}(k_2 - \lambda)^{v_2 - 1}[(k_1 - \lambda)(k_2 - \lambda) + \lambda v_1(k_2 - \lambda) + \lambda v_2(k_1 - \lambda)]$$

is a perfect square and

(a) *If* $v_1 \equiv 0 \pmod 4$ *and* $v_2 \equiv 2 \pmod 4$, *then*

$$diag(k_2 - \lambda, k_2 - \lambda) \cong diag(1, 1).$$

If $v_1 \equiv 2 \pmod 4$ *and* $v_2 \equiv 0 \pmod 4$, *then*

$$diag(k_1 - \lambda, k_1 - \lambda) \cong diag(1, 1).$$

(b) *If* $v_1, v_2 \equiv 2 \pmod 4$, *then*

$$diag(k_2 - \lambda, k_2 - \lambda) \cong diag(k_1 - \lambda, k_1 - \lambda).$$

(c) *If* $v_1, v_2 \equiv 1 \pmod 4$, *then*

$$diag(1, 1, \lambda(k_1 - \lambda)(k_2 - \lambda)) \cong diag(k_1 - \lambda, k_2 - \lambda, \lambda).$$

(d) *If* $v_1 \equiv 1 \pmod 4$ *and* $v_2 \equiv 3 \pmod 4$, *then*

$$diag(k_2 - \lambda, \lambda(k_1 - \lambda)(k_2 - \lambda)) \cong diag(k_1 - \lambda, \lambda).$$

If $v_1 \equiv 3 \pmod 4$ *and* $v_2 \equiv 1 \pmod 4$, *then*

$$diag(k_1 - \lambda, \lambda(k_1 - \lambda)(k_2 - \lambda)) \cong diag(k_2 - \lambda, \lambda).$$

(e) *If* $v_1, v_2 \equiv 3 \pmod 4$, *then*

$$diag(k_1-\lambda, k_2-\lambda, \lambda(k_1-\lambda)(k_2-\lambda)) \cong diag(1,1,\lambda).$$

(f) *If* $v_1 \equiv 0 \pmod 4$ *and* $v_2 \equiv 1 \pmod 4$, *then*

$$diag(k_2-\lambda, \lambda) \cong diag(1, \lambda(k_2-\lambda)).$$

If $v_1 \equiv 1 \pmod 4$ *and* $v_2 \equiv 0 \pmod 4$, *then*

$$diag(k_1-\lambda, \lambda) \cong diag(1, \lambda(k_1-\lambda)).$$

(g) *If* $v_1 \equiv 0 \pmod 4$ *and* $v_2 \equiv 3 \pmod 4$*)*, *then*

$$diag(1, \lambda) \cong diag(k_2-\lambda, \lambda(k_2-\lambda)).$$

If $v_1 \equiv 3 \pmod 4$ *and* $v_2 \equiv 0 \pmod 4$, *then*

$$diag(1, \lambda) \cong diag(k_1-\lambda, \lambda(k_1-\lambda)).$$

(h) *If* $v_1 \equiv 1 \pmod 4$ *and* $v_2 \equiv 2 \pmod 4$, *then*

$$diag(1, k_1-\lambda, \lambda) \cong diag(k_2-\lambda, k_2-\lambda, \lambda(k_1-\lambda)).$$

If $v_1 \equiv 2 \pmod 4$ *and* $v_2 \equiv 1 \pmod 4$, *then*

$$diag(1, k_2-\lambda, \lambda) \cong diag(k_1-\lambda, k_1-\lambda, \lambda(k_2-\lambda)).$$

(i) *If* $v_1 \equiv 2 \pmod 4$ *and* $v_2 \equiv 3 \pmod 4$, *then*

$$diag(k_1-\lambda, k_1-\lambda, \lambda) \cong diag(1, k_2-\lambda, \lambda(k_2-\lambda)).$$

If $v_1 \equiv 3 \pmod 4$ *and* $v_2 \equiv 2 \pmod 4$, *then*

$$diag(k_2-\lambda, k_2-\lambda, \lambda) \cong diag(1, k_1-\lambda, \lambda(k_1-\lambda)).$$

Proof. Let

$$H = N^T N = \begin{pmatrix} (k_1-\lambda)I_{v_1} + \lambda J_{v_1} & \lambda J_{v_1,v_2} \\ \lambda J_{v_2,v_1} & (k_2-\lambda)I_{v_2} + \lambda J_{v_2} \end{pmatrix} \cong I_v.$$

Next, let

$$D = diag(\underbrace{k_1-\lambda, \ldots, k_1-\lambda}_{v_1}, \underbrace{k_2-\lambda, \ldots, k_2-\lambda}_{v_2}, \lambda)$$

and

$$S = \left(\begin{array}{ccc|c} & & & 0 \\ & I_v & & \vdots \\ & & & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right).$$

Then

$$S^T DS = \left(\begin{array}{ccc|c} & & & \lambda \\ & H & & \vdots \\ & & & \lambda \\ \hline \lambda & \cdots & \lambda & \lambda \end{array}\right),$$

$det(D) = \lambda(k_1-\lambda)^{v_1}(k_2-\lambda)^{v_2}$, and $det(H) = (k_1-\lambda)^{v_1-1}(k_2-\lambda)^{v_2-1}[(k_1-\lambda)(k_2-\lambda)+\lambda v_1(k_2-\lambda)+\lambda v_2(k_1-\lambda)]$ is a perfect square.

Case 1. Suppose v_1 and v_2 are both even. Then Theorem 6.2 implies that

$$D \cong diag(H, \frac{det(D)}{det(H)}) \cong diag(H, det(D)) =$$

$$diag(H, \lambda(k_1-\lambda)^{v_1}(k_2-\lambda)^{v_2}) \cong diag(I_v, \lambda).$$

Therefore,

$$diag(\underbrace{k_1-\lambda, \ldots, k_1-\lambda}_{v_1}, \underbrace{k_2-\lambda, \ldots, k_2-\lambda}_{v_2}) \cong diag(\underbrace{1, \ldots, 1}_{v})$$

by Theorem 6.3.

Subcase 1a. If $v_1 \equiv 0 \pmod 4$ and $v_2 \equiv 2 \pmod 4$, then Theorems 6.4 and 6.2 imply that

$$diag(\underbrace{k_1-\lambda, \ldots, k_1-\lambda}_{v_1}, \underbrace{k_2-\lambda, \ldots, k_2-\lambda}_{v_2}) \cong$$

$$diag(\underbrace{1, \ldots, 1}_{v_1}, \underbrace{1, \ldots, 1}_{v_2-2}, k_2-\lambda, k_2-\lambda) \cong$$

$$diag(k_2-\lambda, k_2-\lambda, \underbrace{1, \ldots, 1}_{v-2}) \cong diag(\underbrace{1, \ldots, 1}_{v}).$$

Then Theorem 6.3 implies that

$$diag(k_2-\lambda, k_2-\lambda) \cong diag(1,1).$$

Similarly, if $v_1 \equiv 2 \pmod 4$ and $v_2 \equiv 0 \pmod 4$, then

$$diag(k_1-\lambda, k_1-\lambda) \cong diag(1,1).$$

Remark 6.6. This implies that if $v_1 \equiv 0 \pmod 4$ and $v_2 \equiv 2 \pmod 4$, then $k_2 - \lambda$ is the sum of two perfect squares and if $v_1 \equiv 2 \pmod 4$ and $v_2 \equiv 0 \pmod 4$, then $k_1 - \lambda$ is the sum of two perfect squares.

Subcase 1b. If $v_1, v_2 \equiv 2 \pmod 4$, then Theorems 6.2 and 6.4 imply that

$$diag(\underbrace{k_1-\lambda,\ldots,k_1-\lambda}_{v_1},\underbrace{k_2-\lambda,\ldots,k_2-\lambda}_{v_2}) \cong$$

$$diag(\underbrace{1,\ldots,1}_{v_1-2},k_1-\lambda,k_1-\lambda,\underbrace{1,\ldots,1}_{v_2-2},k_2-\lambda,k_2-\lambda) \cong$$

$$diag(k_1-\lambda,k_1-\lambda,k_2-\lambda,k_2-\lambda,\underbrace{1,\ldots,1}_{v-4}) \cong diag(\underbrace{1,\ldots,1}_{v}).$$

Then Theorems 6.3 and 6.4 imply that

$$diag(k_1-\lambda,k_1-\lambda,k_2-\lambda,k_2-\lambda) \cong diag(1,1,1,1) \cong$$

$$diag(k_1-\lambda,k_1-\lambda,k_1-\lambda,k_1-\lambda).$$

Finally, Theorem 6.3 implies that

$$diag(k_2-\lambda,k_2-\lambda) \cong diag(k_1-\lambda,k_1-\lambda).$$

Remark 6.7. This implies that if $v_1, v_2 \equiv 2 \pmod 4$, then $(k_1-\lambda)(x^2+y^2) = k_2-\lambda$ has a nontrivial rational solution and therefore that $(k_1-\lambda)(x^2+y^2) = (k_2-\lambda)z^2$ has a nontrivial integral solution. This implies that $k_1-\lambda$ is the sum of two perfect squares if and only if $k_2-\lambda$ is the sum of two perfect squares.

Case 2. Suppose v_1 and v_2 are both odd. Then Theorem 6.2 implies that

$$D \cong diag(H,\frac{det(D)}{det(H)}) \cong diag(H,\frac{det(D)}{det(H)}\frac{det(H)}{(k_1-\lambda)^{v_1-1}(k_2-\lambda)^{v_2-1}}) =$$

$$diag(H,\lambda(k_1-\lambda)(k_2-\lambda)) \cong diag(I_v,\lambda(k_1-\lambda)(k_2-\lambda)).$$

Thus,

$$diag(\underbrace{k_1-\lambda,\ldots,k_1-\lambda}_{v_1},\underbrace{k_2-\lambda,\ldots,k_2-\lambda}_{v_2},\lambda) \cong$$

$$diag(\underbrace{1,\ldots,1}_{v},\lambda(k_1-\lambda)(k_2-\lambda)).$$

Theorem 6.5 (c), (d), and (e) are now proven similarly. The remainder of Theorem 6.5 is also proven similarly. □

7 An algorithm

In this section, we develop a simple algorithm that was implemented in Maple [16]. We use the algorithm to prove that all λ-designs with two block sizes with $\lambda \leq 90$ and $\lambda \neq 45$ are type-1.

Let $(X, \mathcal{B})$ be a λ-design on v points with replication numbers r_1 and r_2, $r_1 > r_2$. Let $\rho = (r_1 - 1)/(r_2 - 1) = x/y$, where $\gcd(x, y) = 1$, and let $d = e_1 - r_2$. Then it can be shown [17], [26] that

$$r_1 = \lambda(\rho + 1) - (d+1)(\rho - 1), \tag{57}$$

$$r_2 = \frac{\lambda(\rho+1) - d(\rho - 1)}{\rho}, \tag{58}$$

$$e_1 = \frac{\lambda(\rho+1) + d}{\rho}, \tag{59}$$

and

$$e_2 = \lambda(\rho + 1) - \rho(d+1). \tag{60}$$

Since $e_1, e_2 \geq 1$, using (59) and (60) we have that

$$\rho - \lambda(\rho + 1) \leq d \leq \frac{(\lambda - 1)(\rho + 1)}{\rho}. \tag{61}$$

We also have the following result.

Theorem 7.1. [22] *If $\lambda > 1$, then $y < \lambda$ and $x - y < \lambda$.*

Suppose $(X, \mathcal{B})$ has just two block sizes, k_1 and k_2, $k_1 > k_2$. Then for a fixed value of $\lambda > 1$, Theorems 7.1 and 3.1 and (61) imply that

$$1 \leq y \leq \lambda - 1, \tag{62}$$

$$y + 1 \leq x \leq 2(\lambda - 1), \tag{63}$$

$$\lceil \rho - \lambda(\rho + 1) \rceil \leq d \leq \lfloor \frac{(\lambda - 1)(\rho + 1)}{\rho} \rfloor, \tag{64}$$

$$\max\{\lambda + 1, r_2\} \leq k_2 \leq r_1 - 1, \tag{65}$$

and

$$k_2 + 1 \leq k_1 \leq r_1. \tag{66}$$

Therefore, for a fixed $\lambda \geq 2$, the set of possible λ-designs with two block sizes can be described by a finite set of 6-tuples of the form $(\lambda, y, x, d, k_2, k_1)$ (although a single tuple could correspond to multiple designs) which can be generated using (62), (63), (64), (65), (66), (57), and (58). The algorithm generates this set of tuples and uses the results present in this paper to eliminate tuples that must correspond to nonexistent or type-1 designs. We also use the following result from [20].

For $x \in E_1$, define

$$U_x = \sum_{A:x\in A} (|A| - \lambda - \frac{r_1}{\rho+1})$$

and for $x \in E_2$ define

$$U_x = \sum_{A:x\in A} (|A| - \lambda - \frac{\rho r_2}{\rho+1}).$$

Theorem 7.2.

$$\rho \sum_{x\in E_1} U_x + \sum_{x\in E_2} U_x = \frac{e_1 e_2(\rho-1)^2}{\rho+1}.$$

In this case,

$$U_x = r_1'(k_1 - \lambda - \frac{r_1}{\rho+1}) + r_1^*(k_2 - \lambda - \frac{r_1}{\rho+1}) = U_1$$

for $x \in E_1$ and

$$U_x = r_2'(k_1 - \lambda - \frac{\rho r_2}{\rho+1}) + r_2^*(k_2 - \lambda - \frac{\rho r_2}{\rho+1}) = U_2$$

for $x \in E_2$, so Theorem 7.2 says

$$\rho e_1 U_1 + e_2 U_2 = \frac{e_1 e_2(\rho-1)^2}{\rho+1}.$$

Additionally, let $t_1 = \lceil r_1/(\rho+1)\rceil - r_1/(\rho+1)$, $t_2 = \lceil r_2/(\rho+1)\rceil - r_2/(\rho+1)$, and $C = (x-y)\lfloor [r_1/(\rho+1) - \lambda]/(x-y)\rfloor$. Then we have the following results from [21].

Theorem 7.3.

$$t_1 \le \frac{(\rho-1)(\lambda\rho+\lambda-\rho+d)}{(\rho+1)(\lambda\rho+\lambda-\rho)}$$

and

$$t_2 \le \frac{(\rho-1)(\lambda\rho+\lambda+d)}{(\rho+1)(\lambda\rho+\lambda-1)}.$$

Theorem 7.4.

$$U_1 \ge \frac{t_1(\rho-1)[e_2 - e_1 + 1 - t_1(v-1)]}{2[\rho - t_1(\rho+1) - 1]}.$$

Theorem 7.5.

$$U_2 \geq \frac{t_2(\rho-1)[e_2-e_1-1-t_2(v-1)]}{2[\rho-t_2(\rho+1)-1]}.$$

Theorem 7.6.

$$U_1 \geq (\rho+1)(\frac{r_1}{\rho+1}-\lambda-C)(\lambda+C+x-y-\frac{r_1}{\rho+1}).$$

The algorithm was run for all $12 \leq \lambda \leq 90$ and all tuples except $(45, 1, 4, 3, 81, 90)$ and $(45, 1, 4, 11, 81, 90)$ were eliminated. Thus, we have the following result.

Theorem 7.7. *All λ-designs with two block sizes with $\lambda \leq 90$ and $\lambda \neq 45$ are type-*1.

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