$(K_3 + e, \lambda)$-group divisible designs of type $g^t u^1$ *

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Abstract

Necessary and sufficient conditions are given to the existence for a $(K_3 + e, \lambda)$-group divisible design of type $g^t u^1$.

Keywords: $(K_3 + e, \lambda)$-group divisible design; cyclic partial $(K_3 + e, \lambda)$-GDD; difference leave; Fundamental Construction

1 Introduction

Let G be a simple, connected graph and H a complete multipartite graph. λH denote the graph H with each of its edges replicated λ times. We define H to be of *type* $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ if it has exactly $\sum_{i=1}^{s} u_i$ classes (*groups*) in the multipartition, and there are u_i groups of size g_i for $i = 1, 2, \ldots, s$. Then we define λH to be of the same type as H. A *G-decomposition* of λH is a partition of λH into subgraphs (*blocks*) so that each subgraph is isomorphic to G. We term it as a (G, λ)-*group divisible design* of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$, and it is often called a (G, λ)-GDD for short. The existence problem of $(K_3, 1)$-GDD with group type $g^t u^1$ is completely settled in [1]. The existence spectrum of $(K_4 - e, 1)$-GDD with group type g^t is also determined in [2]. The aim of this paper is to solve the existence problem for $(K_3 + e, \lambda)$-GDD of type $g^t u^1$ for integers g, t, u and λ. In what follows we will denote $(K_3 + e)$ by (a, b, c)-d or (b, a, c)-d, sometimes it is called a *kite* in [4].

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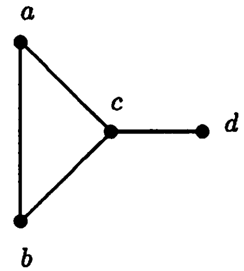

The following lemma is from [4] when $\lambda = 1$. It is easy to check that it still holds when $\lambda > 1$.

Lemma 1.1 *Let g, t, u and λ be nonnegative integers. If there exists a $(K_3{+}e, \lambda)$-GDD of type $g^t u^1$, then the following conditions are all satisfied:*

(1) *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u \geq \lceil g/2 \rceil$, or $t = 1$ and $u = 0$, or $t = 0$;*

(2) *$u \leq \lfloor 3g(t-1)/2 \rfloor$ or $gt = 0$;*

(3) *$\lambda(g^2 t(t-1)/2 + gtu) \equiv 0 \pmod 4$.*

First observe that when $gt = 0$, or $t = 1$ and $u = 0$, the design is trivial, it has no blocks. Hence we assume that g and t are positive and $t \geq 2$ in the following sections.

Theorem 1.2 [4] *The necessary conditions as in Lemma 1.1 for the existence of a $(K_3 + e, 1)$-GDD of type $g^t u^1$ are also sufficient.*

2 Preliminaries

In this section, we will introduce some recursive constructions and some useful lemmas. The following lemmas are from [4] when $\lambda = 1$. It is not difficult to prove that they are still true when $\lambda > 1$.

Theorem 2.1 (Fundamental Construction) *Let $(V, \mathcal{G}, \mathcal{B})$ be a GDD where $\mathcal{G} = \{G_1, \ldots, G_m\}$. Let each $x \in V$ have an associated integer weight $w(x)$. Suppose that for each block $\{x_1, x_2, \ldots, x_k\}$ in $\mathcal{B}$, there is a $(K_3{+}e, \lambda)$-GDD with k groups, having sizes $w(x_1), \ldots, w(x_k)$. Then there is a $(K_3 + e, \lambda)$-GDD whose groups have sizes $\sum_{x \in G_i} w(x)$ for $i = 1, \ldots, m$.*

Lemma 2.2 *If there exist (K_3+e,λ)-GDDs of types g^tu^1 and $(g/s)^sw^1$, then there exists a (K_3+e,λ)-GDD of type $(g/s)^{st}(w+u)^1$.*

Lemma 2.3 *If there exist (K_3+e,λ)-GDDs of types g^tu^1 and g^sx^1 with $u=sg+x$, then there exists a (K_3+e,λ)-GDD of type $g^{t+s}x^1$.*

Lemma 2.4 *Let $(V,\mathcal{G},\mathcal{B})$ be a (K_3+e,λ)-GDD with group type $g_1^{u_1}g_2^{u_2}\cdots g_m^{u_m}$ and $t\geq 3$. If there exists a (K_3+e,λ)-GDD of type $g_i^tu^1$ for each $i=1,2,\ldots,m$, then there exists a (K_3+e,λ)-GDD of type $(|V|)^tu^1$.*

Lemma 2.5 *If $(V,\mathcal{G},\mathcal{B})$ is a (K_3+e,λ)-GDD of type $g_1^{u_1}g_2^{u_2}\cdots g_s^{u_s}$, then there is a (K_3+e,λ)-GDD of type $(ng_1)^{u_1}(ng_2)^{u_2}\cdots(ng_s)^{u_s}$ for any integer $n\geq 1$.*

Next we introduce a simple but very useful lemma.

Lemma 2.6 *Let m and λ be positive integers. If there exists a (K_3+e,λ)-GDD of type g^tu^1, then there exists a $(K_3+e,m\lambda)$-GDD of type g^tu^1.*

Let $D_n=\{d:1\leq d\leq [n/2]\}$. The elements of D_n are called *differences* of Z_n. Let $R=\{\infty_1^{l_1},\ldots,\infty_r^{l_r}\}$ be a multiset where ∞_i appears l_i times for $i=1,2,\ldots,r$ and $R\cap Z_n=\emptyset$. Denote by $\langle Z_n\cup R,\{d_1,d_2,\ldots,d_t\}\rangle$ the graph G with vertex set $V(G)=Z_n\cup R$ and edge set $E(G)=\{\langle d_i\rangle:1\leq i\leq t\}\cup\{\{\infty,j\}:\infty\in R, j\in Z_n\}$ where $\langle d_i\rangle=\{(x,x+d_i):x\in Z_n\}$ if $d_i\neq n/2$ and $\langle n/2\rangle=\{(x,x+n/2):x=0,1,\ldots,n/2-1\}$.

Let S be a set. We define λS to be a multiset in which each element of S appears exactly λ times. Suppose that $B=(a,b,c)$-d and $a,b,c,d\in Z_n$. Define $\Delta B=\{\pm(a-b),\pm(a-c),\pm(b-c),\pm(c-d)\}$ and $\Delta B^+=\{d:d\in\Delta B, 1\leq d\leq[n/2]\}$. Note that ΔB^+ is a multiset.

A cyclic partial (K_3+e,λ)-GDD of type g^t is a triple $(Z_{gt},\mathcal{G},\mathcal{B})$ where $\mathcal{G}=\{\{i,t+i,\ldots,(g-1)t+i\}:0\leq i\leq t-1\}$ and $\mathcal{B}$ is a collection of (K_3+e)-blocks (called *base blocks*) of Z_{gt}, so that:

(1) $\Delta B^+\cap\{0,t,\ldots,(g-1)t\}=\emptyset$ for any $B\in\mathcal{B}$;

(2) $\cup_{B\in\mathcal{B}}\Delta B^+\subseteq\lambda D_{gt}$;

(3) If gt is even, then $gt/2\notin\Delta B^+$ for any $B\in\mathcal{B}$.

Let $\Delta\mathcal{B}=\cup_{B\in\mathcal{B}}\Delta B^+$ and $E=D_{gt}\cap\{0,t,\ldots,(g-1)t\}$. The set $L=\lambda D_{gt}\setminus(\Delta\mathcal{B}\cup\lambda E)$ is called *difference leave* of $(Z_{gt},\mathcal{G},\mathcal{B})$.

Lemma 2.7 *Let $d \in D_n \backslash \{n/2\}$. Then the graph $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{d\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Proof The graph $\langle Z_n, \{d\}\rangle$ is regular of degree 2 and so it can be decomposed into r-cycles. Let $(x_0, x_1, \ldots, x_{r-1})$ be such a cycle. Consider the following $(K_3 + e)$-blocks with the subscript modulo r.

If r is odd: $(\infty_1, x_{2i}, x_{2i+1})$-$\infty_2$, $0 \leq i \leq (r-3)/2$; $(\infty_1, x_{2i+1}, x_{2i+2})$-$\infty_2$, $0 \leq i \leq (r-5)/2$; $(x_{r-2}, x_{r-1}, \infty_1)$-$x_0$; (∞_2, x_0, x_{r-1})-∞_1.

If r is even: $(\infty_1, x_{2i}, x_{2i+1})$-$\infty_2$, $(\infty_1, x_{2i+1}, x_{2i+2})$-$\infty_2$, $0 \leq i \leq (r-2)/2$. ◇

Lemma 2.8 [3] *Let $d_1, d_2, d_3 \in D_n \backslash \{n/2\}$ such that $d_3 = d_2 - d_1$. Then the graph $\langle Z_n \cup \{\infty\}, \{d_1, d_2, d_3\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Lemma 2.9 [3] *Let n be even and $d \in D_n \backslash \{n/2\}$ such that $r = n/gcd(n, d)$ is even. Then the graph $\langle Z_n \cup \{\infty\}, \{d\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Lemma 2.10 [3] *Let n be odd. The graph $\langle Z_n \cup \{\infty_1, \infty_2\}, \{2, 4\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Lemma 2.11 [4] *Let $n \equiv 0 \pmod 4$ and $n > 4$. Then the graph $\langle Z_n \cup \{\infty_1, \infty_2\}, \{2\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Lemma 2.12 *Let n be even and $d \in D_n \backslash \{n/2\}$. Then the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{n/2, n/2\}\rangle$ and $\langle Z_n \cup \{\infty_1^2\}, \{n/2, n/2, d\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Proof For the graph $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{n/2, n/2\}\rangle$, consider the following (K_3+e)-blocks: $(\infty_1, n/2+i, i)$-∞_2, $(\infty_1, i, n/2+i)$-∞_2, $i = 0, 1, \ldots, n/2-1$.

For the graph $\langle Z_n \cup \{\infty_1^2\}, \{n/2, n/2, d\}\rangle$, consider the following (K_3+e)-blocks: $(\infty_1, n/2+i, i)$-$(d+i)$, $(\infty_1, i, n/2+i)$-$(n/2+d+i)$, $i = 0, 1, \ldots, n/2-1$. ◇

Lemma 2.13 *Let $d_1, d_2 \in D_n \setminus \{n/2\}$. Then the graph $\langle Z_n \cup \{\infty_1^2, \infty_2^2, \infty_3^2\}, \{d_1, d_2\}\rangle$ can be decomposed into $(K_3 + e)$-blocks.*

Proof By Lemma 2.7 the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{d_1\}\rangle$ and $\langle Z_n \cup \{\infty_2, \infty_3^2\}, \{d_2\}\rangle$ can be decomposed into $(K_3 + e)$-blocks. ◇

Lemma 2.14 *Let $B = (a,b,0)$-d with $n/2 \notin \Delta B^+$. Then the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2^2\}, \Delta B^+\rangle$ and $\langle Z_n \cup \{\infty_1^2, \dots, \infty_6^2\}, \Delta B^+\rangle$ can be decomposed into (K_3+e)-blocks. When n is even and there are two odd differences in ΔB^+, or n is odd and the differences 2 and 4 are contained in ΔB^+, the graph $\langle Z_n \cup \{\infty_1^2, \dots, \infty_4^2\}, \Delta B^+\rangle$ can also be decomposed into (K_3+e)-blocks.*

Proof Without loss of generality we can assume that $\Delta B^+ = \{a, b, b-a, d\}$. For the graph $\langle Z_n \cup \{\infty_1^2, \infty_2^2\}, \Delta B^+\rangle$, by Lemmas 2.7 and 2.8 the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{d\}\rangle$ and $\langle Z_n \cup \{\infty_2\}, \{a, b, b-a\}\rangle$ can be decomposed into (K_3+e)-blocks.

For the graph $\langle Z_n \cup \{\infty_1^2, \dots, \infty_6^2\}, \Delta B^+\rangle$, by Lemma 2.7 the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{a\}\rangle$, $\langle Z_n \cup \{\infty_2, \infty_3^2\}, \{b\}\rangle$, $\langle Z_n \cup \{\infty_4^2, \infty_5\}, \{b-a\}\rangle$, $\langle Z_n \cup \{\infty_5, \infty_6^2\}, \{d\}\rangle$ can be decomposed into (K_3+e)-blocks.

When n is even, suppose that a, b are odd. By Lemmas 2.7 and 2.9 the graphs $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{b-a\}\rangle$, $\langle Z_n \cup \{\infty_2, \infty_3^2\}, \{d\}\rangle$, $\langle Z_n \cup \{\infty_4\}, \{a\}\rangle$, $\langle Z_n \cup \{\infty_4\}, \{b\}\rangle$ can be decomposed into (K_3+e)-blocks.

When n is odd, we can assume that $a = 2$, $d = 4$. By Lemmas 2.7 and 2.10 the graphs $\langle Z_n \cup \{\infty_1^2, \infty_3\}, \{b-a\}\rangle$, $\langle Z_n \cup \{\infty_2^2, \infty_4\}, \{b\}\rangle$, $\langle Z_n \cup \{\infty_3, \infty_4\}, \{2,4\}\rangle$ can be decomposed into (K_3+e)-blocks. ◇

Lemma 2.15 *Let $(Z_{gt}, \mathcal{G}, \mathcal{B})$ be a cyclic partial $(K_3+e, 2)$-GDD of type g^t with difference leave L where $\mathcal{G} = \{\{i, t+i, \dots, (g-1)t+i\} : 0 \le i \le t-1\}$, in which there exists $B \in \mathcal{B}$ such that ΔB^+ contains two odd differences if $gt \equiv 0 \pmod 2$, or $2, 4 \in \Delta B^+$ if $gt \equiv 1 \pmod 2$. If the graph $\langle Z_{gt} \cup \{\infty_1^2, \dots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks, then there exists a $(K_3+e, 2)$-GDD of type $g^t u^1$ for any integer $u = 2l + w$ where $0 \le l \le 3|\mathcal{B}|$.*

Proof Let $l = 3k + j$ where $j = 0, 1, 2$ and $0 \le k \le |\mathcal{B}| - 1$ when $j = 1, 2$, or $0 \le k \le |\mathcal{B}|$ when $j = 0$.

For the case of $j = 0$, choose k base blocks from $\mathcal{B}$, say $B_1, \dots, B_k$. By Lemma 2.14 arrange the differences of each base block B_l, $1 \le l \le k$, with six different infinite points respectively, saying the resultant collection of (K_3+e)-blocks K_1. Let K_2 denote the collection of (K_3+e)-blocks generated by the graph $\langle Z_{gt} \cup \{\infty_1^2, \dots, \infty_w^2\}, L\rangle$ and the remaining base blocks of $\mathcal{B} \backslash \{B_1, \dots, B_k\}$ (note that the collection of (K_3+e)-blocks generated by the graph $\langle Z_{gt} \cup \{\infty_1^2, \dots, \infty_w^2\}, L\rangle$ is empty-set if $L = \emptyset$). All different infinite points form a group $R_u = \{\infty_1, \dots, \infty_u\}$ where $u = 6k + w = 2l + w$. Then it is easy to verify that $(Z_{gt} \cup R_u, \mathcal{G} \cup \{R_u\}, K_1 \cup K_2)$ is a $(K_3+e, 2)$-GDD of type $g^t u^1$ where $u = 2l + w$.

For the case of $j = 1, 2$, choose k base blocks from $\mathcal{B}\backslash\{B\}$, say $B_1, \ldots, B_k$.

When $j = 1$, $\langle Z_{gt} \cup \{\infty_1^2, \infty_2^2\}, \Delta B^+\rangle$ can be decomposed into $(K_3 + e)$-blocks by Lemma 2.14. When $j = 2$, $\langle Z_{gt} \cup \{\infty_1^2, \infty_2^2, \infty_3^2, \infty_4^2\}, \Delta B^+\rangle$ can be decomposed into $(K_3 + e)$-blocks by Lemma 2.14.

That is to say that we can arrange the four differences of B with $2j$ different infinite points, and denote the obtained $(K_3 + e)$-blocks as $\mathcal{K}_1$. Then by Lemma 2.14 arrange the differences of the base blocks B_l, $1 \leq l \leq k$, with six different infinite points respectively, saying the resultant collection of $(K_3 + e)$-blocks $\mathcal{K}_2$. Let $\mathcal{K}_3$ denote the collection of $(K_3 + e)$-blocks generated by the graph $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ and the other base blocks of $\mathcal{B}\backslash\{B, B_1, \ldots, B_k\}$. All different infinite points form a group $R_u = \{\infty_1, \ldots, \infty_u\}$ where $u = 6k + 2j + w = 2l + w$. Then it is easy to verify that $(Z_{gt} \cup R_u, \mathcal{G} \cup \{R_u\}, \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$ is a $(K_3 + e, 2)$-*GDD* of type $g^t u^1$ where $u = 2l + w$. ⋄

3 The existence of a $(K_3 + e, 2)$-*GDD* of type $g^t u^1$

By Lemma 1.1 we know that the necessary conditions for the existence of a $(K_3 + e, 2)$-*GDD* of type $g^t u^1$ are equivalent to one of the following conditions:

Case 1: $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 4$ and $u \leq \lfloor 3g(t-1)/2 \rfloor$, and when $t = 2$, $u \geq \lceil g/2 \rceil$;

Case 2: $g^2 t(t-1)/2 + gtu \equiv 2 \pmod 4$ and $u \leq \lfloor 3g(t-1)/2 \rfloor$.

Lemma 3.1 *If $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 4$, $u \leq \lfloor 3g(t-1)/2 \rfloor$, and when $t = 2$, $u \geq \lceil g/2 \rceil$, then there is a $(K_3 + e, 2)$-GDD of type $g^t u^1$.*

Proof It follows immediately from Lemma 2.6 and Theorem 1.2. ⋄

Next we mainly treat Case 2. For the sake of convenience, we classify the necessary conditions in Case 2 as follows when g, t and u are all positive and $t \geq 2$:

(I) $g \equiv 1 \pmod 2$, $t \equiv 4 \pmod 8$, and $u \leq \lfloor 3g(t-1)/2 \rfloor$;

(II) $g \equiv 1 \pmod 2$, $t \equiv 1 \pmod 2$, $u \equiv 3g(t-5)/2 \pmod 4$ and $u \leq 3g(t-1)/2$;

$g \equiv 2 \pmod 4$, $t \equiv 1 \pmod 2$, $u \equiv 1 \pmod 2$ and $u \leq 3g(t-1)/2$.

Lemma 3.2 *There exists a $(K_3+e,2)$-GDDof type 1^4u^1 for $0 \leq u \leq 4$.*

Proof Let $X = \{1,2,3,4\} \cup \{\infty_1,\ldots,\infty_u\}$, and $\mathcal{G} = \{\{i\} : 1 \leq i \leq 4\} \cup \{\{\infty_1,\ldots,\infty_u\}\}$. A $(K_3+e,2)$-*GDD* of type 1^4u^1 is constructed by listing its blocks as below:

1^4: (1,2,3)-4 (2,4,1)-3 (2,3,4)-1

1^41^1: (1,2,3)-4 $(1,4,\infty_1)$-2 $(1,2,4)$-∞_1 $(1,\infty_1,3)$-4 $(\infty_1,3,2)$-4

1^42^1: $(1,2,\infty_2)$-3 $(3,4,1)$-∞_1 $(3,\infty_1,2)$-1 $(\infty_2,2,4)$-∞_1 $(4,1,\infty_1)$-2 $(1,\infty_2,3)$-∞_1 $(2,3,4)$-∞_2

1^43^1: $(2,\infty_2,1)$-∞_3 $(3,\infty_2,4)$-∞_1 $(4,\infty_3,2)$-∞_1 $(1,\infty_3,3)$-∞_1 $(4,\infty_2,1)$-∞_1 $(2,\infty_2,3)$-1 $(3,\infty_1,4)$-∞_3 $(3,\infty_3,2)$-∞_1 $(2,4,1)$-∞_1

1^44^1: $(2,\infty_1,1)$-∞_2 $(4,\infty_1,3)$-∞_3 $(2,\infty_2,4)$-∞_3 $(1,\infty_3,3)$-∞_1 $(4,\infty_4,1)$-∞_3 $(3,\infty_4,2)$-∞_3 $(4,\infty_3,2)$-∞_4 $(2,\infty_2,3)$-4 $(1,3,\infty_4)$-4 $(1,2,\infty_1)$-4 $(1,4,\infty_2)$-3

◇

Lemma 3.3 *There exists a $(K_3+e,2)$-GDDof type 3^4u^1 for $0 \leq u \leq 13$.*

Proof Let $X = Z_{12} \cup \{\infty_1,\ldots,\infty_u\}$, $\mathcal{G} = \{\{i,4+i,8+i\} : 0 \leq i \leq 3\} \cup \{\{\infty_1,\ldots,\infty_u\}\}$, $\mathcal{B} = \{(3,1,0)\text{-}5\}$. Then $(X,\mathcal{G},\mathcal{B})$ is a partial cyclic $(K_3+e,2)$-*GDD* of type 3^4 with the difference leave $L = \{1,2,3,5,6,6\}$. We can arrange the differences in L with w different infinite points where $w = 3,4,5,6,7$ by Lemmas 2.7, 2.9, 2.11 and 2.12. That is to say that $\langle Z_{12} \cup \{\infty_1^2,\ldots,\infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks for $w = 3,4,5,6,7$. Then by Lemma 2.15 we get a $(K_3+e,2)$-*GDD* of type 3^4u^1 where $u = 2j+w$, $0 \leq j \leq 3$, $w = 3,4,5,6,7$. This handles the case of $u \geq 3$. For $u = 0$, apply Lemma 2.5 to a $(K_3+e,2)$-*GDD* of type 1^4 from Lemma 3.2. For $u = 1,2$, a $(K_3+e,2)$-*GDD* of type 3^4u^1 is listed in Appendix B. ◇

Lemma 3.4 *Let $g \equiv 1 \pmod 2$ and $u \leq \lfloor 9g/2 \rfloor$. Then there is a $(K_3+e,2)$-GDD of type g^4u^1.*

Proof The conclusion follows by Lemmas 3.2 and 3.3 when $g = 1,3$. When $g > 3$, we treat the case of $0 \leq u \leq 4g$ first. There exists a $TD(5,g)$

when $g > 3$. Give weight 1 to the points of the first four groups of the $TD(5, g)$ and any weight between 0 and 4 to the points of the last group. Now applying Fundamental Construction with a $(K_3 + e, 2)$-GDD of type 1^4w^1 $(0 \le w \le 4)$ from Lemma 3.2, we get a $(K_3 + e, 2)$-GDD of type g^4u^1 where $0 \le u \le 4g$.

Next we consider the case of $4g < u \le \lfloor 9g/2 \rfloor$. We form a $(K_3 + e, 2)$-GDD of type g^4u^1 on the point set $X = Z_{4g} \cup \{\infty_1, \ldots, \infty_u\}$ and group set $\mathcal{G} = \{\{i, 4+i, \ldots, (g-1)4+i\} : 0 \le i \le 3\} \cup \{\{\infty_1, \ldots, \infty_u\}\}$. Let $E = D_{4g} \cap \{0, 4, \ldots, (g-1)4\}$. By Lemmas 2.12 and 2.11 we can decompose $\langle Z_n \cup \{\infty_1^2, \infty_2\}, \{2g, 2g\}\rangle$ and $\langle Z_n \cup \{\infty_2, \infty_3\}, \{2\}\rangle$ into $(K_3 + e)$-blocks, and say K_1. Choose $2g - m$ odd differences from $2(D_{4g} \backslash E)$ where $m \in \{g+1, \ldots, 2g\}$, and by Lemma 2.9 arrange them with one infinite point respectively and denote the resultant $(K_3 + e)$-blocks as K_2. By Lemma 2.7 arrange others differences in $2(D_{4g} \backslash E)$ with three infinite points respectively and denote the resultant $(K_3 + e)$-blocks as K_3. Furthermore, it is not difficult to assure that each infinite point appears two times in all those graphs. Then we can calculate out that the total number of different infinite points is $(5 + 3(g - 2 + m) + 2g - m)/2 = (5g - 1)/2 + m$ where $m \in \{g+1, \ldots, 2g\}$. It is easy to check that $(X, \mathcal{G}, K_1 \cup K_2 \cup K_3)$ is a $(K_3 + e, 2)$-GDD of type g^4u^1 where $4g < u \le \lfloor 9g/2 \rfloor$. ◇

Lemma 3.5 *Let $g \equiv 1 \pmod 2$, $t \equiv 4 \pmod 8$ and $u \le \lfloor 3g(t-1)/2 \rfloor$. Then there exists a $(K_3 + e, 2)$-GDD of type g^tu^1.*

Proof Let $t = 8l + 4$. When $l = 0$, the conclusion follows by Lemma 3.4. Next we consider the case of $l > 0$. By Lemma 3.1 there is a $(K_3 + e, 2)$-GDD of type $(4g)^{2l+1}x^1$ where $0 \le x \le 12gl$. By Lemma 3.4 there is a $(K_3 + e, 2)$-GDD of type g^4w^1 where $0 \le w \le \lfloor 9g/2 \rfloor$. Then apply Lemma 2.2 to get a $(K_3 + e, 2)$-GDD of type g^tu^1 where $u \le \lfloor 3g(t-1)/2 \rfloor$ (since $u = w + x$). ◇

Lemma 3.6 *Let n and s be positive integers such that $n > 8s$. Then there exists a collection $\mathcal{B}$ of $(K_3 + e)$-blocks (base blocks) on Z_n such that $\Delta\mathcal{B} = \cup_{B \in \mathcal{B}} \Delta B^+ = \{1, 2, \ldots, 4s\}$, in which there is a $B \in \mathcal{B}$ such that $\{2, 4\} \in \Delta B^+$.*

Proof Consider the following base blocks $\mathcal{B}$:

When $s \ge 4$: $(4s - i, 2s + i + 1, 0)$-$(2s - 2i)$ where $2 \le i \le s - 3$ (note that the number of $(K_3 + e)$-base blocks in this part is $s - 4$);

$(4s, 2s+1, 0)$-$3s$, $(4s-1, 2s+2, 0)$-$(3s+1)$, $(3s+2, 3s-1, 0)$-1, $(2s, 2s-2, 0)$-4.

When $s = 3$: $(2,6,0)$-10, $(3,11,0)$-9, $(12,5,0)$-1.

When $s = 2$: $(2,6,0)$-7, $(3,8,0)$-1.

When $s = 1$: $(1,3,0)$-4.

It is easy to check that $\Delta\mathcal{B} = \{1, 2, \ldots, 4s\}$ and that there is a $B \in \mathcal{B}$ such that $\{2, 4\} \in \Delta B^+$. ◇

Lemma 3.7 *Let $t \geq 3$ be odd, $u \equiv -(t-5)/2 \pmod 4$ and $u \leq \lfloor 3(t-1)/2 \rfloor$. Then there exists a $(K_3 + e, 2)$-GDD of type $1^t u^1$.*

Proof Let $t = 8s + k$ where $k = 1, 3, 5, 7$. We repeat the base blocks in Lemma 3.6 twice and denote the resultant base blocks as $\mathcal{B}$. It is easy to see that $(Z_t, \{\{i\} : 0 \leq i \leq t-1\}, \mathcal{B})$ is a cyclic partial $(K_3 + e, 2)$-GDD of type 1^t in which there is a base block containing differences 2 and 4. The difference leave L is $2\{4s+1, \ldots, 4s+(k-1)/2\}$ (note that $L = \emptyset$ if $k = 1$).

By Lemma 2.13 arrange each two differences in L with three different infinite points. That is to say that the graph $\langle Z_t \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks where $w = 3(k-1)/2$. Then by Lemma 2.15 we get a $(K_3+e, 2)$-GDD of type $1^t u^1$ for any integer $u = 2j+3(k-1)/2$ where $0 \leq j \leq 3|\mathcal{B}|$. This handles the case of $u \geq 3(k-1)/2$ and $u \equiv -(t-5)/2 \pmod 4$.

For $t = 8s+1$, it handles all case of $u \leq \lfloor 3(t-1)/2 \rfloor$ and $u \equiv 2 \pmod 4$.

For $t = 8s + 3$, it handles the case of $3 \leq u \leq \lfloor 3(t-1)/2 \rfloor$ and $u \equiv 1 \pmod 4$. For $u = 1$, it follows by Lemma 3.5.

For $t = 8s + 5$, it handles the case of $u \geq 6$ and $u \equiv 0 \pmod 4$. For $u = 0, 4$, by Lemma 3.1 there is a $(K_3+e, 2)$-GDD of type $1^{8s}(5+u)^1$. Then by Lemma 2.3 fill in the long group of the GDD with a $(K_3+e, 2)$-GDD of type $1^5 u^1$ from Lemma 3.2 when $u = 0$, or from Appendix A when $u = 4$.

For $t = 8s + 7$, it handles the case of $u \geq 9$ and $u \equiv 3 \pmod 4$. For $u = 3$, by Lemma 3.1 there is a $(K_3 + e, 2)$-GDD of type $1^{8s}10^1$ $(s > 0)$. Then by Lemma 2.3 fill in the long group of the GDD with a $(K_3 + e, 2)$-GDD of type $1^7 3^1$ from Appendix C. For $u = 7$, by Lemma 3.1 there is a $(K_3+e, 2)$-GDD of type $1^{8s}14^1$ $(s > 1)$. Then by Lemma 2.3 fill in the long group of the GDD with a $(K_3+e, 2)$-GDD of type $1^7 7^1$ from Appendix C. A $(K_3 + e, 2)$-GDD of type $1^{15} 7^1$ comes from Appendix C. ◇

Lemma 3.8 *Let $t \geq 3$ be odd, $u \equiv 1 \pmod 2$ and $u < 3(t-1)$. Then there is a $(K_3+e,2)$-GDD of type $2^t u^1$.*

Proof Let $t = 4s + i$ where $i = 1, 3$. Then $2t = 8s + 2i$. We repeat each base block in Lemma 3.6 twice and denote the resultant base blocks as $\mathcal{B}$.

Case 1: $t = 4s+1$. When $s > 1$, we delete one base block $B = (a, b, 0)$-1 from $\mathcal{B}$ so that $\Delta B^+ = \{1, a, b, c\}$, and denote the resultant base blocks as $\mathcal{B}$ still. When $s = 1$, take $\mathcal{B} = (1, 4, 0)$-3. Then $(Z_{2t}, \{\{i, t+i\} : 0 \leq i \leq t-1\}, \mathcal{B})$ is a cyclic partial $(K_3+e,2)$-GDD of type 2^t with difference leave $L = \{1, a, b, c\}$ if $s > 1$, or $L = \{1, 2, 2, 4\}$ if $s = 1$ (in this case let $a = b = 2$ and $c = 4$). Note that there is a base block of $\mathcal{B}$ with two odd differences. By Lemmas 2.7 and 2.9 we can arrange the differences in L with w different infinite points where $w = 1, 5$.

$w = 1 :$ $(a, b, 0)$-∞_1, $\langle Z_{2t} \cup \{\infty_1\}, \{1\}\rangle$,

$w = 5 :$ $\langle Z_{2t} \cup \{\infty_1^2, \infty_2\}, \{c\}\rangle$, $\langle Z_{2t} \cup \{\infty_2, \infty_3^2\}, \{b\}\rangle$, $\langle Z_{2t} \cup \{\infty_4^2, \infty_5\}, \{a\}\rangle$, $\langle Z_{2t} \cup \{\infty_5\}, \{1\}\rangle$.

That is to say that the graph $\langle Z_{2t} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks where $w \in \{1, 5\}$. By Lemma 2.15 we get a $(K_3+e,2)$-GDD of type $2^t u^1$ for any integer $u = 2i+w$ where $0 \leq i \leq 3|\mathcal{B}|$ and $w \in \{1, 5\}$. It handles the case of $u \geq 1$.

Case 2: $t = 4s + 3$. When $s > 1$, we choose one base block $(a, b, 0)$-1 from $\mathcal{B}$ and change it into $(a, b, 0)$-$(4s + 2)$. When $s = 1$, change $\mathcal{B}$ into $(1, 3, 0)$-4, $(6, 4, 0)$-3. We denote the resultant base blocks as $\mathcal{B}$ still. It is easy to see that $(Z_{2t}, \{\{i, t+i\} : 0 \leq i \leq t-1\}, \mathcal{B})$ is a cyclic partial $(K_3+e,2)$-GDD of type 2^t with difference leave $L = \{1, 4s+1, 4s+1, 4s+2\}$, in which there is a base block having two odd differences. By Lemmas 2.7 and 2.9 we can arrange the differences in L with w different infinite points where $w = 1, 5$ as Case 1.

That is to say that the graph $\langle Z_{2t} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks where $w \in \{1, 5\}$. Then by Lemma 2.15 we get a $(K_3+e,2)$-GDD of type $2^t u^1$ for any integer $u = 2i+w$ where $0 \leq i \leq 3|\mathcal{B}|$ and $w \in \{1, 5\}$. This handles the case $u \geq 1$. For $s = 0$, a $(K_3+e,2)$-GDD of type $2^3 u^1$ is listed in Appendixes A and B. ◇

Lemma 3.9 *Let $u \equiv (t-5)/2 \pmod 4$ and $u \leq 9(t-1)/2$. Then there exists a $(K_3+e,2)$-GDD of type $3^t u^1$ for $t = 3, 5, 7, 9, 11, 13, 15, 23, 31$.*

Proof **Case 1:** $t = 9$ and $u \equiv 2 \pmod 4$. Take the base blocks $\mathcal{B}$: $2(3, 11, 0)$-1, $2(6, 13, 0)$-5, $2(10, 12, 0)$-4. Then $(Z_{27}, \{\{i, 9+i, 18+i\} : 0 \leq$

$i \leq 8\}, \mathcal{B})$ is a cyclic partial $(K_3 + e, 2)$-GDD of type 3^9. The difference leave L is $\emptyset$. By Lemma 2.15 we get a $(K_3 + e, 2)$-GDD of type $3^9 u^1$ for any integer $u = 2j$ where $0 \leq j \leq 18$. This handles the case of $u \leq 36$ and $u \equiv 2 \pmod 4$.

Case 2: $t = 3, 11$ and $u \equiv 3 \pmod 4$. For $t = 3$, take the base block $\mathcal{B}$: $(1, 2, 0)$-4. The difference leave L is $\{2, 4\}$.

For $t = 11$, take the base blocks $\mathcal{B}$: $2(1, 10, 0)$-7, $2(3, 15, 0)$-6, $2(2, 16, 0)$-4, $(5, 13, 0)$-8. The difference leave L is $\{5, 13\}$.

In each case, $(Z_{3t}, \{\{i, t + i, 2t + i\} : 0 \leq i \leq t - 1\}, \mathcal{B})$ is a cyclic partial $(K_3 + e, 2)$-GDD of type 3^t with difference leave L. By Lemma 2.13 arrange the differences in L with three different infinite points. That is to say that the graph $\langle Z_{3t} \cup \{\infty_1^2, \infty_2^2, \infty_3^2\}, L\rangle$ can be decomposed into $(K_3 + e)$-blocks. By Lemma 2.15 we get a $(K_3 + e, 2)$-GDD of type $3^t u^1$ for any integer $u = 2j + 3$ where $0 \leq j \leq 3|\mathcal{B}|$. This handles the case of $3 \leq u \leq 9(t - 1)/2$ and $u \equiv 3 \pmod 4$.

Case 3: $t = 5, 13$ and $u \equiv 0 \pmod 4$. For $t = 5$, take the base blocks $\mathcal{B}$: $(6, 2, 0)$-3, $(3, 4, 0)$-7, $(6, 7, 0)$-2. The difference leave L is $\emptyset$.

For $t = 13$, take the base blocks $\mathcal{B}$: $2(2, 9, 0)$-4, $2(15, 18, 0)$-17, $2(10, 11, 0)$-12, $(5, 19, 0)$-6, $(5, 19, 0)$-16, $(16, 8, 0)$-6. The difference leave L is $\emptyset$.

In each case, $(Z_{3t}, \{\{i, t+i, 2t+i\} : 0 \leq i \leq t - 1\}, \mathcal{B})$ is a cyclic partial $(K_3 + e, 2)$-GDD of type 3^t with difference leave L, in which there is a base block having the differences 2 and 4. By Lemma 2.15 we get a $(K_3 + e, 2)$-GDD of type $3^t u^1$ for any integer $u = 2j$ where $0 \leq j \leq 3|\mathcal{B}|$. This handles the case of $0 \leq u \leq 9(t - 1)/2$.

Case 4: $t = 7, 15, 23, 31$ and $u \equiv 1 \pmod 4$. For $t = 7$, take the base blocks $\mathcal{B}$: $2(5, 2, 0)$-4, $2(9, 10, 0)$-8. The difference leave L is $\{6, 6\}$.

For $t = 15$, take the base blocks $\mathcal{B}$: $2(8, 12, 0)$-2, $2(1, 22, 0)$-19, $2(6, 20, 0)$-18, $2(5, 16, 0)$-13, $2(7, 17, 0)$-9. The difference leave L is $\{3, 3\}$.

For $t = 23$, take the base blocks $\mathcal{B}$: $2(2, 21, 0)$-4, $2(1, 34, 0)$-25, $2(10, 32, 0)$-24, $2(11, 27, 0)$-6, $2(12, 29, 0)$-30, $2(7, 20, 0)$-15, $2(8, 26, 0)$-14, $2(3, 31, 0)$-9. The difference leave L is $\{5, 5\}$.

For $t = 31$, take the base blocks $\mathcal{B}$: $2(4, 18, 0)$-2, $2(3, 23, 0)$-36, $2(1, 46, 0)$-33, $2(21, 40, 0)$-24, $2(16, 41, 0)$-27, $2(15, 43, 0)$-44, $2(17, 30, 0)$-6, $2(12, 38, 0)$-39, $2(7, 29, 0)$-11, $2(32, 37, 0)$-10, $2(8, 42, 0)$-9. The difference leave L is $\{35, 35\}$.

In each case, $(Z_{3t}, \{\{i, t+i, 2t+i\} : 0 \leq i \leq t - 1\}, \mathcal{B})$ is a cyclic partial $(K_3 + e, 2)$-GDD of type 3^t with difference leave L, in which there is a base block having the differences 2 and 4. By Lemma 2.13 we can arrange the differences in L with three different infinite points. That is to say that the graph $\langle Z_{3t} \cup \{\infty_1^2, \infty_2^2, \infty_3^2\}, L\rangle$ can be decomposed into $(K_3 + e)$-blocks. By Lemma 2.15 we get a $(K_3 + e, 2)$-GDD of type $3^t u^1$ for any integer $u = 2j + 3$ where $0 \leq j \leq 3|\mathcal{B}|$. This handles the case of $5 \leq u \leq 9(t-1)/2$. For $u = 1$, there is a $(K_3 + e, 2)$-GDD of type $3^{t-3}10^1$ by Lemma 3.5. By Lemma 2.3 fill in the long group of the GDD with a $(K_3 + e, 2)$-GDD of type $3^3 1^1$ from Lemma 3.1. ◇

Lemma 3.10 *Let* $u \equiv (t-5)/2 \pmod 4$ *and* $u \leq 21(t-1)/2$. *Then there exists a* $(K_3 + e, 2)$-GDD *of type* $7^t u^1$ *for* $t =$ 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 29, 31.

Proof Case 1: $t = 9, 17$ and $u \equiv 2 \pmod 4$. For $t = 9$, take the base blocks $\mathcal{B}$: 2(2,15,0)-4, 2(3,29,0)-20, 2(7,24,0)-16, 2(10,21,0)-12, 2(1,31,0)-25, 2(5,28,0)-19, 2(8,22,0)-6. The difference leave L is $\emptyset$.

For $t = 17$, take the base blocks $\mathcal{B}$: 2(4,36,0)-2, 2(3,29,0)-24, 2(7,52,0)-49, 2(10,21,0)-25, 2(1,31,0)-13, 2(5,28,0)-33, 2(8,22,0)-35, 2(9,59,0)-6, 2(18,58,0)-41, 2(20,57,0)-42, 2(16, 55, 0)-43, 2(12,56,0)-47, 2(15,53,0)-48, 2(19,46,0)-54. The difference leave L is $\emptyset$.

Case 2: $t = 3, 11, 19$ and $u \equiv 3 \pmod 4$. For $t = 3$, take the base blocks $\mathcal{B}$: (2,7,0)-4, (2,10,0)-1, (1,8,0)-4. The difference leave L is $\{5, 10\}$.

For $t = 11$, take the base blocks $\mathcal{B}$: 2(2,6,0)-36, 2(3,35,0)-25, 2(5,34,0)-23, 2(7,31,0)-19, 2(8,28,0)-18, 2(9,30,0)-16, 2(10,27,0)-15, (1,38,0)-26, (12,26,0)-13, (1,14,0)-38. The difference leave L is $\{12, 37\}$.

For $t = 19$, take the base blocks $\mathcal{B}$: 2(2,14,0)-4, 2(1,61,0)-39, 2(3,35,0)-40, 2(5,34,0)-41, 2(7,31,0)-42, 2(8,28,0)-45, 2(9,30,0)-13, 2(10,37,0)-51, 2(11,66,0)-36, 2(22,65,0)-54, 2(15,64, 0)-50, 2(16,63,0)-56, 2(18,62,0)-58, 2(23,48,0)-17, 2(26,59,0)-53, (46,52,0)-6. The difference leave L is $\{46, 52\}$.

Case 3: $t = 5, 13, 21, 29$ and $u \equiv 0 \pmod 4$. For $t = 5$, take the base blocks $\mathcal{B}$: 2(1,17,0)-12, 2(3,14,0)-8, (2,6,0)-13, (2,9,0)-4, (6,13,0)-9. The difference leave L is $\emptyset$.

For $t = 13$, take the base blocks $\mathcal{B}$: 2(1,45,0)-9, 2(3,43,0)-17, 2(7,42,0)-18, 2(15,31,0)-19, 2(8,41,0)-21, 2(10,38,0)-22, 2(11,36,0)-5, 2(12,32,0)-27, 2(14,37,0)-29, (2,6,0)-24, (4,34,0)-2, (6,30,0)-34. The difference leave L is $\emptyset$.

For $t = 21$, take the base blocks $\mathcal{B}$: 2(4,34,0)-2, 2(1,45,0)-39, 2(3, 43, 0)-47, 2(15,31,0)-50, 2(8,41,0)-51, 2(10,38,0)-22, 2(11,36,0)-24, 2(12, 32, 0)-27, 2(14, 37, 0)-48, 2(5, 73, 0)-53, 2(9, 71, 0)-56, 2(7, 72, 0)-58, 2(13, 70, 0)-59, 2(17, 69, 0)-61, 2(19, 54, 0)-64, (66, 60, 0)-46, (26, 55, 0)-66, (18, 67, 0)-60, (26, 55, 0)-46, (18, 67, 0)-6. The difference leave L is $\emptyset$.

For $t = 29$, take the base blocks $\mathcal{B}$: 2(2, 6, 0)-74, 2(1, 45, 0)-75, 2(15, 31, 0)-81, 2(8, 41, 0)-82, 2(10, 38, 0)-83, 2(11, 36, 0)-84, 2(12, 32, 0)-88, 2(5, 73, 0)-42, 2(7, 72, 0)-90, 2(13, 70, 0)-59, 2(18, 67, 0)-91, 2(17, 69, 0)-61, 2(19, 54, 0)-64, 2(22, 101, 0)-92, 2(24, 100, 0)-63, 2(27, 80, 0)-94, 2(39, 89, 0)-95, 2(26, 86, 0)-96, 2(46, 93, 0)-97, 2(48, 78, 0)-98, 2(34, 85, 0)-99, 2(3, 43, 0)-55, 2(9, 71, 0)-66, (56, 77, 0)-21, (14, 37, 0)-77, (14, 37, 0)-56. The difference leave L is $\emptyset$.

Case 4: $t = 7, 15, 23, 31$ and $u \equiv 1 \pmod 4$. For $t = 7$, take the base blocks $\mathcal{B}$: 2(2, 15, 0)-4, 2(1, 24, 0)-10, 2(3, 22, 0)-9, 2(8, 20, 0)-6, 2(11, 16, 0)-17. The difference leave L is $\{18, 18\}$.

For $t = 15$, take the base blocks $\mathcal{B}$: 2(2, 34, 0)-4, 2(1, 24, 0)-28, 2(3, 22, 0)-29, 2(8, 20, 0)-31, 2(11, 16, 0)-17, 2(9, 52, 0)-27, 2(10, 51, 0)-6, 2(7, 49, 0)-36, 2(13, 48, 0)-37, 2(14, 47, 0)-38, 2(21, 46, 0)-39, 2(18, 44, 0)-40. The difference leave L is $\{50, 50\}$.

For $t = 23$, take the base blocks $\mathcal{B}$: 2(2, 64, 0)-4, 2(1, 80, 0)-68, 2(3, 22, 0)-29, 2(8, 20, 0)-54, 2(11, 16, 0)-55, 2(9, 52, 0)-53, 2(10, 51, 0)-6, 2(7, 49, 0)-65, 2(13, 48, 0)-66, 2(14, 47, 0)-38, 2(18, 44, 0)-67, 2(15, 78, 0)-70, 2(21, 77, 0)-71, 2(27, 58, 0)-72, 2(17, 57, 0)-73, 2(28, 60, 0)-74, 2(34, 59, 0)-76, (36, 75, 0)-30, 2(37, 61, 0)-45. The difference leave L is $\{50, 50\}$.

For $t = 31$, take the base blocks $\mathcal{B}$: 2(2, 84, 0)-4, 2(1, 80, 0)-81, 2(3, 22, 0)-83, 2(8, 20, 0)-6, 2(11, 16, 0)-86, 2(9, 52, 0)-87, 2(10, 51, 0)-64, 2(7, 49, 0)-88, 2(13, 48, 0)-67, 2(14, 47, 0)-90, 2(18, 44, 0)-89, 2(15, 78, 0)-70, 2(21, 77, 0)-71, 2(17, 57, 0)-73, 2(28, 60, 0)-74, 2(34, 59, 0)-76, 2(36, 75, 0)-94, 2(37, 61, 0)-95, 2(23, 108, 0)-96, 2(30, 99, 0)-98, 2(46, 91, 0)-100, 2(29, 97, 0)-101, 2(27, 92, 0)-102, 2(53, 107, 0)-103, 2(38, 104, 0)-106, 2(50, 105, 0)-58. The difference leave L is $\{72, 72\}$.

In each case, $(Z_{7t}, \{\{i, t+i, \ldots, 6t+i\} : 0 \le i \le t-1\}, \mathcal{B})$ is a cyclic partial $(K_3+e, 2)$-*GDD* of type 7^t with difference leave L, in which there is a base block having the differences 2 and 4. For Cases 2 and 4, we need to deal with the differences in L. By Lemma 2.13 arrange the differences in L with three different infinite points. That is to say that the $\langle Z_{7t} \cup \{\infty_1^2, \infty_2^2, \infty_3^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.15 we get a $(K_3+e, 2)$-*GDD* of type $7^t u^1$. This handles all the case except for $(K_3+e, 2)$-*GDD*s of types $7^t 1^1$ where $t = 7, 15, 23, 31$. By Lemma 2.3 fill in the long group of a $(K_3+e, 2)$-*GDD* of type $7^{t-3} 22^1$ from Lemma 3.1 with a $(K_3+e, 2)$-*GDD* of type $7^3 1^1$ from Lemma 3.1. $\diamond$

Lemma 3.11 *Let $t \geq 3$ be odd, $u \equiv (t-5)/2 \pmod 4$ and $u \leq 3g(t-1)/2$. Then there exists a $(K_3+e,2)$-GDD of type $g^t u^1$ for $g = 3,7$.*

Proof The conclusion follows by Lemmas 3.9 and 3.10 when $3 \leq t \leq 15$ if $g = 3$, or when $3 \leq t \leq 23$ if $g = 7$. Next we consider the case of $t \geq 17$ if $g = 3$, or $t \geq 25$ if $g = 7$. Let $t = 8m + i$ and $s = gm$ where $i = 1,3,5,7$. Then $gt = 8s + gi$ and $s \geq 6$. We repeat the following base blocks twice and denote the resultant base blocks as $\mathcal{B}$:

$(4s-j, 2s+j+1, 0)$-$(2s-2j)$ for $j = 2,3,\ldots,s-3$; $(4s, 2s+1, 0)$-$3s$, $(4s-1, 2s+2, 0)$-$(3s+1)$, $(3s+2, 3s-1, 0)$-1, $(2s, 2s-2, 0)$-4.

Delete the base blocks having the differences congruent to 0 (mod t) from $\mathcal{B}$ and denote the resultant base blocks as $\mathcal{B}_0$. Noting that the last base block can not been deleted, then $(Z_{gt}, \{\{i, t+i, \ldots, (g-1)t+i\} : 0 \leq i \leq t-1\}, \mathcal{B}_0)$ is a cyclic partial $(K_3+e,2)$-GDD of type g^t satisfying the condition in Lemma 2.15. The difference leave L is $D_0 \cup 2D_1$, where D_0 contains the differences not congruent to 0 (mod t) and appearing in the deleted base blocks of $\mathcal{B}$ and $D_1 = \{4s+1, \ldots, 4s+(gi-1)/2\} \setminus \{0, t, \ldots, (g-1)t\}$. Noting that $\Delta\mathcal{B}$ contains at most $g-1$ differences congruent to 0 (mod t) which appear in different base blocks, then we have $|D_0| \leq 3(g-1)$. By Lemma 2.13 arrange each two differences in L with three different infinite points respectively. That is to say the graph $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks where $w \leq (3gi+9g-12)/2$. Then by Lemma 2.15 we get a $(K_3+e,2)$-GDD of type $g^t u^1$ for any integer $u = 2n+w$ where $0 \leq n \leq 3|\mathcal{B}|$. This handles the case of $u \geq (3gi+9g-12)/2$.

Next we prove the case of $u^\star \leq (3gi+9g-16)/2$ and $u^\star \equiv (t-5)/2 \pmod 4$ inductively.

Case 1. $t \equiv 1,3,5 \pmod 8$ if $g = 3$, or $t \equiv 1,3 \pmod 8$ if $g = 7$.

For $g = 3$, there are $(K_3+e,2)$-GDDs of types $g^t u^1$ for $t = 3,5,9,11,13$ by Lemma 3.9. For $g = 7$, there are $(K_3+e,2)$-GDDs of types $g^t u^1$ for $t = 3,9,11,17,19$ by Lemma 3.10.

Suppose that there exists a $(K_3+e,2)$-GDD of type $g^{t-8}u^1$ for $u \leq (3gi+9g-16)/2$ and $u \equiv (t-5)/2 \pmod 4$ where $t \geq 17$ if $g = 3$, or $t \geq 25$ if $g = 7$. Then there is a $(K_3+e,2)$-GDD of type $g^{t-8}u^1$ for any integer $u \leq 3g(t-9)/2$ and $u \equiv (t-5)/2 \pmod 4$. It is not difficult but tedious to check that $u^\star + 8g < 3g(t-9)/2$ and $u^\star \leq \lfloor 21g/2 \rfloor$. By Lemma 2.3 fill in the long group of a $(K_3+e,2)$-GDD of type $g^{t-8}(8g+u^\star)^1$ with a $(K_3+e,2)$-GDD of type $g^8(u^\star)^1$ from Lemma 3.1. We obtain a $(K_3+e,2)$-GDD of type $g^t(u^\star)^1$, as required.

Case 2. $t \equiv 7 \pmod 8$ if $g = 3$, or $t \equiv 5,7 \pmod 8$ if $g = 7$.

For $g = 3$, there are $(K_3+e,2)$-GDDs of types $g^t u^1$ for $t = 7, 15, 23, 31$ by Lemma 3.9. For $g = 7$, there are $(K_3+e,2)$-GDDs of types $g^t u^1$ for $t = 5, 7, 13, 15, 21, 23, 29, 31$ by Lemma 3.10.

Suppose that there exists a $(K_3+e,2)$-GDD of type $g^{t-16}u^1$ for $u \leq (3gi+9g-16)/2$ and $u \equiv (t-5)/2 \pmod 4$ where $t \geq 37$. Then there is a $(K_3+e,2)$-GDD of type $g^{t-16}u^1$ for any integer $u \leq 3g(t-17)/2$ and $u \equiv (t-5)/2 \pmod 4$. It is not difficult but tedious to check that $u^\star + 16g \leq 3g(t-17)/2$ and $u^\star \leq \lfloor 45g/2 \rfloor$. By Lemma 2.3 fill in the long group of a $(K_3+e,2)$-GDD of type $g^{t-16}(16g+u^\star)^1$ with a $(K_3+e,2)$-GDD of type $g^{16}(u^\star)^1$ from Lemma 3.1. We obtain a $(K_3+e,2)$-GDD of type $g^t(u^\star)^1$, as required. ◇

Lemma 3.12 *Let $t \geq 3$ be odd and $g \equiv l \pmod 4$ where $l = 1,2,3$ and $g \geq 5$. Then there is a cyclic partial $(K_3+e,2)$-GDD of type g^t satisfying the conditions in Lemma 2.15 with difference leave L, in which there is an odd difference in L and $|L| = l(t-1)$.*

Proof Let $g = 4k+l$ where $l = 1,2,3$. Define $\delta_i = 0$ if $1 \leq i < (t+1)/2$, or 2 if $(t+1)/2 \leq i \leq t-1$. Let $\mathcal{G} = \{\{i, t+i, \ldots, (g-1)t+i\} : 0 \leq i \leq t-1\}$.

When k is even and $t \geq 5$, we repeat the following base blocks twice and denote the resultant base blocks as $\mathcal{B}$:

$(2kt-i-jt, kt+i+jt, 0)$-$(kt-2i-2jt+1-\delta_i)$ where $1 \leq i \leq t-1$ and $0 \leq j \leq (k-4)/2$ (note that the number of the base blocks in this part is $(t-1)(k-2)/2$ if $k \geq 4$, or 0 if $1 \leq k \leq 3$);

$((3k/2+1)t-i, (3k/2-1)t+i, 0)$-$(2t-2i+1-\delta_i)$ where $1 \leq i \leq t-3$;

$(3kt/2+2, 3kt/2-2, 0)$-2, $(2kt+1, 2kt+2, 0)$-$(3kt/2+1)$.

When $t = 3$, let $\mathcal{B}$ consist of the following base blocks:

$2(6k-i-3j, 3k+i+3j, 0)$-$(3k-2i-6j+1-\delta_i)$ where $i = 1,2$ and $0 \leq j \leq (k-4)/2$ (note that the number of the base blocks in this part is $k-2$ if $k \geq 4$, or 0 if $1 \leq k \leq 3$);

$(9k/2+2, 9k/2-2, 0)$-2; $(9k/2+1, 9k/2+2, 0)$-$(9k/2-2)$; $(9k/2+1, 9k/2-1, 0)$-5; $(5, 4, 0)$-$(9k/2-1)$.

Then $(Z_{gt}, \mathcal{G}, \mathcal{B})$ is a cyclic partial $(K_3+e,2)$-GDD of type g^t satisfying the conditions in Lemma 2.15. When $t \geq 5$, the difference leave L is $2\{3, 3kt/2-1, 2kt+3, \ldots, 2kt+(lt-1)/2\} \backslash 2\{2kt+t\}$ if $l = 1,3$; or $2\{3, 3kt/2-1, 2kt+3, \ldots, 2kt+t-1\}$ if $l = 2$. When $t = 3$, the difference leave L is $2\{6k+1, 6k+2\}$ if $l = 2$; or $L = 2\{6k+1, 6k+2, 6k+4\}$ if $l = 3$; or $L = 2\{6k+1\}$ if $l = 1$.

When k is odd and $t \geq 5$, we repeat the following base blocks twice and denote the resultant base blocks as $\mathcal{B}$:

$(2kt-i-jt, kt+i+jt, 0)$-$(kt-2i-2jt+1-\delta_i)$ where $1 \leq i \leq t-1$ and $0 \leq j \leq (k-3)/2$ (the number of the base blocks in this part is $(t-1)(k-1)/2$ if $k \geq 3$, or 0 if $k = 1, 2$);

$((3k+1)t/2-i, (3k-1)t/2+i, 0)$-$(t-2i+1)$ where $1 \leq i \leq (t-5)/2$ (the number of the base blocks in this part is $(t-5)/2$);

$((3kt+3)/2, (3kt-1)/2, 0)$-4, $(2kt+1, 2kt+2, 0)$-$(3kt+1)/2$.

When $t = 3$ and $k \geq 3$, take $\mathcal{B}$ as the following base blocks:

$2(6k-i-3j, 3k+i+3j, 0)$-$(3k-2i-6j+1-\delta_i)$ where $i = 1, 2$ and $0 \leq j \leq (k-5)/2$ (the number of the base blocks in this part is $k-3$ if $k \geq 5$, or 0 if $1 \leq k \leq 4$);

$2((9k+7)/2, (9k-7)/2, 0)$-8, $((9k+5)/2, (9k+1)/2, 0)$-4, $((9k-1)/2, (9k-5)/2, 0)$-4; $((9k+1)/2, (9k-1)/2, 0)$-5; $((9k+5)/2, (9k-5)/2, 0)$-1.

When $t = 3$ and $k = 1$, take $\mathcal{B}=\{(5,4,0)\text{-}2, (5,4,0)\text{-}2\}$.

Then $(Z_{gt}, \mathcal{G}, \mathcal{B})$ is a cyclic partial $(K_3+e, 2)$-GDD of type g^t satisfying the conditions in Lemma 2.15. When $t \geq 5$, the difference leave L is $2\{3, (3kt-3)/2, 2kt+3, \ldots, 2kt+(lt-1)/2\} \backslash 2\{2kt+t\}$ if $l = 1, 3$; or $2\{3, (3kt-3)/2, 2kt+3, \ldots, 2kt+t-1\}$ if $l = 2$. When $t = 3$, the difference leave L is $2\{6k+1, 6k+2\}$ if $l = 2$; or $L = 2\{6k+1, 6k+2, 6k+4\}$ if $l = 3$; or $L = 2\{6k+1\}$ if $l = 1$. ◇

Lemma 3.13 *Suppose that t is odd and $g \equiv l$ (mod 4) where $l = 1, 2, 3$ and $g \geq 5$. Let $3l(t-1)/2 \leq u \leq 3g(t-1)/2$ such that $u \equiv 3l(t-5)/2$ (mod 4) if $l = 1, 3$, or $u \equiv 1$ (mod 2) if $l = 2$. Then there is a $(K_3+e, 2)$-GDD of type $g^t u^1$.*

Proof Let $g = 4k + l$ where $l = 1, 2, 3$. By Lemma 3.12 there is a cyclic partial $(K_3+e, 2)$-GDD of type g^t $(Z_{gt}, \mathcal{G}, \mathcal{B})$, with the difference leave L such that $|L| = l(t-1)$ and L contains an odd difference.

When $g = 4k + l$ and $u \equiv 3l(t-5)/2$ (mod 4) where $l = 1, 3$: by Lemma 2.13 arrange each two differences in L with three different infinite points respectively. That is to say that the graph $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks where $w = 3l(t-1)/2$. Then by Lemma 2.15 we obtain a $(K_3+e, 2)$-GDD of type $g^t u^1$ for any integer $u = 2i + 3l(t-1)/2$ where $0 \leq i \leq 3|\mathcal{B}|$. It handles the case of $u \geq 3l(t-1)/2$.

When $g = 4k+2$ and $u \equiv 1 \pmod 2$: choose four differences d_1, d_2, d_3, d_4 from L such that d_1 is odd. By Lemmas 2.9 and 2.13 the graphs $\langle Z_{gt} \cup \{\infty_1\}, \{d_1\}\rangle$, $\langle Z_{gt} \cup \{\infty_1, \infty_2^2\}, \{d_2\}\rangle$, $\langle Z_{gt} \cup \{\infty_3, \infty_4^2\}, \{d_3\}\rangle$, $\langle Z_{gt} \cup \{\infty_3, \infty_5^2\}, \{d_4\}\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.13 arrange each two differences in $L \backslash \{d_1, d_2, d_3, d_4\}$ with three different infinite points. That is to say that the graph $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks for $w = 3t-4$. Then by Lemma 2.15 we obtain a $(K_3+e, 2)$-*GDD* of type $g^t u^1$ for any integer $u = 2i+3t-4$ where $0 \le i \le 3|\mathcal{B}|$. This handles the case of $u \ge 3t-4$. ◇

Lemma 3.14 *Let g, t and u be positive integers satisfying Condition* (II). *Then there exists a $(K_3+e, 2)$-GDD of type $g^t u^1$.*

Proof The conclusion follows when $g = 1, 2, 3, 7$ by Lemmas 3.7, 3.8, 3.11. Next we divide the problem with $g \ge 5$ and $g \ne 7$ into three cases.

Case 1: $g = 4k+1$, $u \equiv 3(t-5)/2 \pmod 4$ and $u \le 3g(t-1)/2$. By Lemma 3.13 we can restrict our attention to $u \le 3(t-1)/2 - 2$. There are $(K_3+e, 2)$-*GDD*s of types $2^{2k}1^1$ ($k \ge 1$) and $2^t u^1$ by Lemmas 3.1 and 3.8. Apply Lemma 2.4 to a $(K_3+e, 2)$-*GDD* of type $2^{2k}1^1$ by using $(K_3+e, 2)$-*GDD*s of types $2^t u^1$ and $1^t u^1$ from Lemma 3.7. We then obtain a $(K_3+e, 2)$-*GDD* of type $g^t u^1$.

Case 2: $g = 4k+2$, $u \equiv 1 \pmod 2$ and $u \le 3g(t-1)/2$. By Lemma 3.13 we can restrict attention to $u < 3(t-1)$. There are $(K_3+e, 2)$-*GDD*s of types $2^{2k}2^1$ ($k \ge 1$) by Lemma 3.1 and $2^t u^1$ by Lemma 3.8 for any integer $u \le 3(t-1)$ and $u \equiv 1 \pmod 2$. Apply Lemma 2.4 to a $(K_3+e, 2)$-*GDD* of type $2^{2k}2^1$ with a $(K_3+e, 2)$-*GDD* of type $2^t u^1$. We then get a $(K_3+e, 2)$-*GDD* of type $g^t u^1$.

Case 3: $g = 4k+3$, $u \equiv (t-5)/2 \pmod 4$ and $u \le 3g(t-1)/2$. By Lemma 3.13 we can restrict attention to $u < 9(t-1)/2$. There are $(K_3+e, 2)$-*GDD*s of types $4^k 3^1$ ($k \ge 2$) and $4^t u^1$ by Lemma 3.1. Apply Lemma 2.4 to a $(K_3+e, 2)$-*GDD* of type $4^k 3^1$ by using $(K_3+e, 2)$-*GDD*s of types $4^t u^1$ and $3^t u^1$ from Lemma 3.11. We then get a $(K_3+e, 2)$-*GDD* of type $g^t u^1$. ◇

From Lemmas 3.1, 3.5 and 3.14, we can obtain the following theorem.

Theorem 3.15 *The necessary conditions as in Lemma 1.1 for the existence of a $(K_3+e, 2)$-GDD of type $g^t u^1$ are also sufficient.*

4 The existence of a $(K_3+e,4)$-GDD of type g^tu^1

In this section, we will deal with the existence of a $(K_3+e,4)$-GDD of type g^tu^1. First by Lemma 1.1, we know that the necessary conditions for the existence of a $(K_3+e,4)$-GDD of type g^tu^1 are equivalent to one of the following conditions:

Case $1'$: $g^2t(t-1)/2+gtu\equiv 0 \pmod 2$ and $u\le\lfloor 3g(t-1)/2\rfloor$, and when $t=2$, $u\ge\lceil g/2\rceil$;

Case $2'$: $g^2t(t-1)/2+gtu\equiv 1 \pmod 2$ and $u\le\lfloor 3g(t-1)/2\rfloor$, and when $t=2$, $u\ge\lceil g/2\rceil$.

Lemma 4.1 *Let $g^2t(t-1)/2+gtu\equiv 0 \pmod 2$, $u\le\lfloor 3g(t-1)/2\rfloor$, and when $t=2$, $u\ge\lceil g/2\rceil$. Then there exists a $(K_3+e,4)$-GDD of type g^tu^1.*

Proof It follows immediately from Lemma 2.6 and Theorem 3.15. $\diamond$

Next we mainly deal with Case $2'$. For the sake of convenience, we classify it as follows when g, t and u are all positive and $t\ge 2$.

(I$'$) $g\equiv 1 \pmod 2$, $t\equiv 2 \pmod 4$ and $u\le\lfloor 3g(t-1)/2\rfloor$, and when $t=2$, $u\ge\lceil g/2\rceil$;

(II$'$) $g\equiv 1 \pmod 2$, $t\equiv 1 \pmod 2$, $u\equiv (t+1)/2 \pmod 2$ and $u\le\lfloor 3g(t-1)/2\rfloor$.

First observe that any (K_3+e)-block contains four different points, it follows that there is not a (K_3+e,λ)-GDD of type 1^21^1 or 1^3 for $\lambda\equiv 0 \pmod 4$.

Lemma 4.2 *Let $g\equiv 1 \pmod 2$, $\lceil g/2\rceil\le u\le\lfloor 3g/2\rfloor$ and $g>1$. Then there is a $(K_3+e,4)$-GDD of type g^2u^1.*

Proof We form the required GDD on point set $X=Z_{2g}\cup\{\infty_1,\dots,\infty_u\}$ and group set $\mathcal{G}=\{\{0,2,\dots,2g-2\},\{1,3,\dots,2g-1\},\{\infty_1,\dots,\infty_u\}\}$. Let $E=D_{2g}\cap\{0,2,\dots,2g-2\}$. By Lemma 2.12 we can decompose $\langle Z_{2g}\cup\{\infty_1^2,\infty_2\},\{g,g\}\rangle$ and $\langle Z_{2g}\cup\{\infty_1^2\},\{g,g,d\}\rangle$, $d\in 4(D_{2g}\setminus(E\cup\{g\}))$, into (K_3+e)-blocks, and say K_1. By Lemma 2.7 we can decompose $\langle Z_{2g}\cup\{\infty_{i_1}^2,\infty_{i_2}\},\{d_i\}\rangle$ into (K_3+e)-blocks, where $d_i\in[4(D_{2g}\setminus(E\cup\{g,d\}))]\cup\{d,d,d\}$ for $1\le i\le 2m$ where $m\in\{0,1,\dots,g-2\}$, and say K_2 (note that $K_2=\emptyset$ if $m=0$). By Lemma 2.9 $\langle Z_{2g}\cup\{\infty_{j_1}\},\{d_j\}\rangle$ can be decomposed into (K_3+e)-blocks, where d_j is the remaining difference in $4(D_{2g}\setminus E)$, and say K_3. It is not difficult to assure that each infinite point appears four times in all those graphs. We can calculate out that the total number of different infinite points is $m+(g+1)/2$ where $m \in \{0,1,\ldots,g-2\}$. Then $(X,\mathcal{G},K_1\cup K_2\cup K_3)$ is a $(K_3+e,4)$-GDD of type g^2u^1 where $\lceil g/2\rceil \le u \le \lfloor 3g/2\rfloor - 1$. For $u=\lfloor 3g/2\rfloor$, we only need change $\langle Z_{2g}\cup\{\infty_1^2\},\{g,g,d\}\rangle$ into $\langle Z_{2g}\cup\{\infty_1^2,\infty_2\},\{g,g\}\rangle$ and $\langle Z_{2g}\cup\{\infty_2^2,\infty_3\},\{d\}\rangle$ and let $m=g-2$, then proceed as above. ◇

Lemma 4.3 *Let $t\equiv 2 \pmod 4$, $4\le u\le \lfloor 3(t-1)/2\rfloor$ and $t>2$. Then there exists a $(K_3+e,4)$-GDD of type 1^tu^1.*

Proof Let $t=8k+j$ for $j=2,6$, $X=Z_t$, $\mathcal{G}=\{\{i\}:0\le i\le t-1\}$.

Suppose that $(X,\mathcal{G},\mathcal{B})$ is a cyclic partial $(K_3+e,4)$-GDD of type 1^{8k+j} with the difference leave L. For each base block B, the graphs $\langle Z_t\cup\{\infty_1^2,\infty_2^2\},\Delta B^+\rangle$, $\langle Z_t\cup\{\infty_1^2,\ldots,\infty_6^2\},\Delta B^+\rangle$ can be decomposed into (K_3+e)-blocks by Lemma 2.14. Hence for any $B_1,B_2\in\mathcal{B}$, the graphs $\langle Z_t\cup\{\infty_1^4,\infty_2^4\},\Delta B_1^+\cup\Delta B_2^+\rangle$, $\langle Z_t\cup\{\infty_1^4,\ldots,\infty_6^4\},\Delta B_1^+\cup\Delta B_2^+\rangle$ can also be decomposed into (K_3+e)-blocks. Similarly, if ΔB_1^+ and ΔB_2^+ contain two odd differences, the graph $\langle Z_t\cup\{\infty_1^4,\ldots,\infty_4^4\},\Delta B_1^+\cup\Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks. So it is not difficult to obtain a $(K_3+e,4)$-GDD of type 1^tu^1 where $u=2i+w$, $0\le i\le 3\lfloor|\mathcal{B}|/2\rfloor$, as long as $\langle Z_t\cup\{\infty_1^4,\ldots,\infty_w^4\},L\rangle$ can be decomposed into (K_3+e)-blocks, too.

We repeat the following base blocks four times and denote the resultant base blocks as $\mathcal{B}$: $(4k-i,2k+i+1,0)$-$(2k-2i)$ for $i=0,1,\ldots,k-1$ (note that $\mathcal{B}=\emptyset$ if $k=0$).

When $j=2$, we delete two base blocks B_1,B_2 from $\mathcal{B}$ and denote the resultant base blocks as $\mathcal{B}$ still. Then $(X,\mathcal{G},\mathcal{B})$ is a cyclic partial $(K_3+e,4)$-GDD of type 1^{8k+j} with the difference leave $L=\Delta B_1^+\cup\Delta B_2^+\cup 4\{4k+1\}$ if $j=2$, or $L=4\{4k+1,4k+2,4k+3\}$ if $j=6$. It is not difficult but tedious to check that $\langle Z_t\cup\{\infty_1^4,\ldots,\infty_w^4\},L\rangle$ can be decomposed into (K_3+e)-blocks where $w=4,5,6,7$ by Lemmas 2.7, 2.9 and 2.12. From the above conclusion, we get a $(K_3+e,4)$-GDD of type 1^tu^1 where $u\ge 4$. ◇

Lemma 4.4 *Let $t\equiv 2 \pmod 4$ and $0\le u\le\lfloor 3(t-1)/2\rfloor$, $t>2$. Then there exists a $(K_3+e,4)$-GDD of type 1^tu^1.*

Proof Use induction on t. When $t=6$, the lemma follows by Lemma 4.3 and Appendixes A and B. Suppose that the lemma is true for $t-4$

where $t \geq 10$. We then know that there is a $(K_3+e,4)$-GDD of type $1^{t-4}(4+u)^1$ for $u \leq 3$. By Lemma 2.3 fill in the long group of the GDD with a $(K_3+e,4)$-GDD of type 1^4u^1 from Lemma 4.1. We obtain a $(K_3+e,4)$-GDD of type 1^tu^1 for $u \leq 3$. By Lemma 4.3 the lemma is true for integer t. ◇

Lemma 4.5 *Let $g \equiv 1 \pmod 2$, $t \equiv 2 \pmod 4$, $0 \leq u \leq \lfloor 3g(t-1)/2 \rfloor$, and if $t=2$ then $u \geq \lceil g/2 \rceil$ where $(g,t,u) \neq (1,2,1)$. Then there exists a $(K_3+e,4)$-GDD of type g^tu^1.*

Proof The conclusion follows by Lemma 4.4 when $g=1$. We will deal with the case of $g \geq 3$.

Let $t = 4l+2$. The conclusion follows by Lemma 4.2 when $l=0$. Next we consider $l>0$. We first deal with the case of $\lceil g/2 \rceil \leq u \leq \lfloor 3g(t-1)/2 \rfloor$. By Lemma 4.1 there is a $(K_3+e,4)$-GDD of type $(2g)^{2l+1}x^1$ where $0 \leq x \leq 6gl$. By Lemma 4.2 there is a $(K_3+e,4)$-GDD of type g^2w^1 where $\lceil g/2 \rceil \leq w \leq \lfloor 3g/2 \rfloor$. Then apply Lemma 2.2 to get a $(K_3+e,4)$-GDD of type g^tu^1 where $\lceil g/2 \rceil \leq u \leq \lfloor 3g(t-1)/2 \rfloor$.

Next we consider the case of $0 \leq u^* \leq \lfloor g/2 \rfloor$. When $t>6$, there are $(K_3+e,4)$-GDDs of types $g^{t-4}(4g+u^*)^1$ and $g^4(u^*)^1$ from Lemma 4.1. By Lemma 2.3 fill in the long group of the first GDD with a $(K_3+e,4)$-GDD of type $g^4(u^*)^1$.

When $t=6$ and $g \geq 63$, it is well known that there exists a $TD(7,g)$ (for example, see [6]). Give weight 1 to the points of the first six groups and a weight 0 or 1 to the points of the last group. Apply Fundamental Construction to get a $(K_3+e,4)$-GDD of type $g^6(u^*)^1$ where $0 \leq u^* \leq \lfloor g/2 \rfloor$. The input $(K_3+e,4)$-GDDs of types 1^6 and 1^61^1 are from Lemma 4.4.

When $t=6$ and $g \leq 61$, we prove it inductively. For $g=1$, there is a $(K_3+e,4)$-GDD of type $1^6(u^*)^1$ by Lemma 4.4. For $g=3$, apply Lemma 2.5 to a $(K_3+e,4)$-GDD of type 1^6 to get a $(K_3+e,4)$-GDD of type 3^6. A $(K_3+e,4)$-GDD of type 3^61^1 comes from Appendix B. Then there is a $(K_3+e,4)$-GDD of type $3^6(u^*)^1$ for $0 \leq u^* \leq \lfloor g/2 \rfloor$. Suppose that the lemma is true for the case $g<g'$, that is to say that there is a $(K_3+e,4)$-GDD of type g^6x^1 for admissible x. Next we deal with the case of group type $(g')^6(u^*)^1$. We can choose a, b, k so that $g' = ak+b$ and $(K_3+e,4)$-GDDs of types a^kb^1, $a^6(u^*)^1$ and $b^6(u^*)^1$ exist by Lemma 4.1 and by induction. This can be done: when $5 \leq g' \leq 15$, take $a=2$, $k=[g'/2]$ and $b=1$; when $17 \leq g' \leq 61$, take $a=4$, $k=[(g'-5)/4]$ and $b=5$ if $g' \equiv 1 \pmod 4$, or $b=7$ if $g' \equiv 3 \pmod 4$)). Then apply Lemma

2.4 to get a $(K_3+e,4)$-GDD of type $(g')^t(u^*)^1$. ◇

Lemma 4.6 *Let n be odd, B_1 and B_2 be (K_3+e)-blocks so that $\Delta B_1^+ = \{a,b,2,4\}$ and $n/2 \notin \Delta B_1^+ \cup \Delta B_2^+$. Then the graph $\langle Z_n \cup \{\infty_1^4, \infty_2^4, \ldots, \infty_w^4\}, \Delta B_1^+ \cup \Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks for $w = 1,3,5$.*

Proof For $w=1$, let $\Delta B_2 = \{c,d,e,f\}$, change B_1 and B_2 into $(\infty_1,2,0)$-a, $(\infty_1,4,0)$-b, $(\infty_1,c,0)$-d, $(\infty_1,e,0)$-f.

For $w=3$, by Lemma 2.10 $\langle Z_n \cup \{\infty_1,\infty_2\},\{2,4\}\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.13 $\langle Z_n \cup \{\infty_1^2, \infty_2^2, \infty_3^2\}, \{a,b\}\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.14 make a change to the infinite points of $\langle Z_n \cup \{\infty_1^2, \infty_2^2\}, \Delta B_2^+\rangle$ so that $\langle Z_n \cup \{\infty_1, \infty_2, \infty_3^2\}, \Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks. That is to say that $\langle Z_n \cup \{\infty_1^4, \infty_2^4, \infty_3^4\}, \Delta B_1^+ \cup \Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks.

For $w=5$, by Lemma 2.10 $\langle Z_n \cup \{\infty_1,\infty_2\},\{2,4\}\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.13 make a change to the infinite points of $\langle Z_n \cup \{\infty_3^2, \infty_4^2, \infty_5^2\}, \{a,b\}\rangle$ so that $\langle Z_n \cup \{\infty_3^2, \infty_4^2, \infty_2, \infty_5\}, \{a,b\}\rangle$ can be decomposed into (K_3+e)-blocks. By Lemma 2.14 make a change to the infinite points of $\langle Z_n \cup \{\infty_1^2, \ldots, \infty_6^2\}, \Delta B_2^+\rangle$ so that $\langle Z_n \cup \{\infty_1^3, \infty_2^2, \infty_3^2, \infty_4^2, \infty_5^3\}, \Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks. That is to say that $\langle Z_n \cup \{\infty_1^4, \infty_2^4, \infty_3^4, \infty_4^4, \infty_5^4\}, \Delta B_1^+ \cup \Delta B_2^+\rangle$ can be decomposed into (K_3+e)-blocks. ◇

Lemma 4.7 *Let gt be odd, $(Z_{gt}, \mathcal{G}, \mathcal{B})$ be a cyclic partial $(K_3+e,4)$-GDD of type g^t with difference leave L where $\mathcal{G} = \{\{i, t+i, \ldots, (g-1)t+i\} : 0 \le i \le t-1\}$, in which there exists one base block $B \in \mathcal{B}$ such that $2,4 \in \Delta B^+$. If the graph $\langle Z_{gt} \cup \{\infty_1^4, \ldots, \infty_w^4\}, L\rangle$ can be decomposed into (K_3+e)-blocks, then there exists a $(K_3+e,4)$-GDD of type $g^t u^1$ for any integer $u = 2l+1+w$ where $0 \le l \le 3\lfloor|\mathcal{B}|/2\rfloor - 1$.*

Proof Let $l = 3k+j$ where $j=0,1,2$ and $0 \le k \le \lfloor|\mathcal{B}/2|\rfloor - 1$. Without loss of generality, let $\Delta B^+ = \{a,b,2,4\}$.

For $j=0$, choose $2k+2$ base blocks from $\mathcal{B}$, say $B, B_1, \ldots, B_{2k+1}$. By Lemma 4.6 arrange the differences of B and B_1 with 1 (or 3 if $j=1$; or 5 if $j=2$) different infinite points, saying the resultant collection of (K_3+e)-blocks, K_1. By Lemma 2.14 $\langle Z_{gt} \cup \{\infty_{i_1}^4, \ldots, \infty_{i_6}^4\}, \Delta B_i^+ \cup \Delta B_{i+1}^+\rangle$, $i = 2,4,\ldots,2k$ can be decomposed into (K_3+e)-blocks, say K_2. Denote the (K_3+e)-blocks generated by other base blocks and $\langle Z_{gt} \cup \{\infty_1^4, \ldots, \infty_w^4\}, L\rangle$ as K_3. All infinite points form a group R_u. It is easy to see $(Z_{gt} \cup R_u, \mathcal{G} \cup R_u, K_1 \cup K_2 \cup K_3)$ is a $(K_3+e,4)$-GDD of type $g^t u^1$ for any integer $u = 2l+1+w$ where $0 \le l \le 3\lfloor|\mathcal{B}|/2\rfloor - 1$. ◇

Lemma 4.8 *Let gt be odd, $(Z_{gt}, \mathcal{G}, \mathcal{B})$ be a cyclic partial $(K_3+e,2)$-GDD of type g^t with difference leave L where $\mathcal{G} = \{\{i, t+i, \ldots, (g-1)t+i\} : 0 \le i \le t-1\}$, in which there exists one base block $B \in \mathcal{B}$ such that $2, 4 \in \Delta B^+$. If the graph $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks, then there exists a $(K_3+e,4)$-GDD of type $g^t u^1$ for any integer $u = 2l+1+w$ where $0 \le l \le 3|\mathcal{B}|-1$.*

Proof It is easy to see that $(Z_{gt}, \mathcal{G}, \mathcal{B} \cup \mathcal{B})$ is a cyclic partial $(K_3+e,4)$-*GDD* of type g^t with difference leave $L \cup L$. Since $\langle Z_{gt} \cup \{\infty_1^2, \ldots, \infty_w^2\}, L\rangle$ can be decomposed into (K_3+e)-blocks, $\langle Z_{gt} \cup \{\infty_1^4, \ldots, \infty_w^4\}, L \cup L\rangle$ can also be decomposed into (K_3+e)-blocks. Then by Lemma 4.7 we obtain a $(K_3+e,4)$-*GDD* of type $g^t u^1$ for any integer $u = 2l+1+w$ where $0 \le l \le 3|\mathcal{B}|$-1. ◇

Lemma 4.9 *Let t be odd, $u \equiv (t+1)/2 \pmod 2$ and $u \le 3(t-1)/2$. Then there exists a $(K_3+e,4)$-GDD of type $1^t u^1$.*

Proof Let $t = 8s+i$ where $i = 1,3,5,7$. By the proof of Lemma 3.7 and Lemma 4.8 we can handle the case of $u \ge 3(i-1)/2+1$.

For $t = 8s+1$, it handles the case of $1 \le u \le 3(t-1)/2$ and $u \equiv 1 \pmod 2$.

For $t = 8s+3$, it handles the case of $4 \le u \le 3(t-1)/2$ and $u \equiv 0 \pmod 2$. For $u = 0$, it follows by Lemma 4.4. For $u = 2$, by Lemma 2.3 fill in the long group of a $(K_3+e,4)$-*GDD* of type $1^{8s}5^1$ from Lemma 4.1 with a $(K_3+e,4)$-*GDD* of type $1^3 2^1$ from Appendix A.

For $t = 8s+5$, it handles the case of $7 \le u \le 3(t-1)/2$ and $u \equiv 1 \pmod 2$. For $u = 1$, it follows by Lemma 4.4. For $u = 3,5$, by Lemma 2.3 fill in the long group of a $(K_3+e,4)$-*GDD* of type $1^{8s}(5+u)^1$ from Lemma 4.1 with a $(K_3+e,4)$-*GDD* of type $1^5 u^1$ from Appendix B.

For $t = 8s+7$, it handles the case of $10 \le u \le 3(t-1)/2$ and $u \equiv 0 \pmod 2$. For $u = 0$, it follows by Lemma 4.4. For $u = 2,4,6,8$, by Lemma 2.3 fill in the long group of a $(K_3+e,4)$-*GDD* of type $1^{8s}(7+u)^1$ from Lemma 4.1 with a $(K_3+e,4)$-*GDD* of type $1^7 u^1$ from Appendix C.
◇

Lemma 4.10 *Let $u \equiv (t+1)/2 \pmod 2$ and $u \le 9(t-1)/2$. Then there exists a $(K_3+e,4)$-GDD of type $3^t u^1$ for t = 3, 5, 7, 9, 11, 13, 15, 23, 31.*

Proof **Case 1:** $t \equiv 1 \pmod 4$. By the proof of Lemma 3.9 and Lemma 4.8, we can handle the case of $u \ge 1$ and $u \equiv 1 \pmod 2$.

Case 2: $t \equiv 3 \pmod 4$. By the proof of Lemma 3.9 and Lemma 4.8, we can handle the case of $u \geq 4$ and $u \equiv 0 \pmod 2$. For $u = 0$, it follows by Lemma 4.5. For $u = 2$, by Lemma 2.3 fill in the long group of a $(K_3+e,4)$-*GDD* of type $3^{t-3}11^1$ from Lemma 4.1 with a $(K_3+e,4)$-*GDD* of type $3^3 2^1$ from Appendix B. ◇

Lemma 4.11 *Let* $u \equiv (t+1)/2 \pmod 2$ *and* $u \leq 21(t-1)/2$. *Then there exists a* $(K_3+e,4)$-*GDD of type* $7^t u^1$ *for* $t = 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 29, 31$.

Proof **Case 1:** $t \equiv 1 \pmod 4$. By the proof of Lemma 3.10 and Lemma 4.8, we can handle the case of $u \geq 1$ and $u \equiv 1 \pmod 2$.

Case 2: $t \equiv 3 \pmod 4$. By the proof of Lemma 3.10 and Lemma 4.8, we can handle the case of $u \geq 4$ and $u \equiv 0 \pmod 2$. For $u = 0$, it follows by Lemma 4.5. For $u = 2$, by Lemma 2.3 fill in the long group of a $(K_3+e,4)$-*GDD* of type $7^{t-3}23^1$ from Lemma 4.1 with a $(K_3+e,4)$-*GDD* of type $7^3 2^1$ from Appendix B. ◇

Lemma 4.12 *Let* $t \geq 3$ *be odd,* $u \equiv (t+1)/2 \pmod 2$ *and* $u \leq 3g(t-1)/2$. *Then there exists a* $(K_3+e,4)$-*GDD of type* $g^t u^1$ *for* $g = 3, 7$.

Proof The conclusion follows by Lemmas 4.10 and 4.11 when $3 \leq t \leq 15$ if $g = 3$, or when $3 \leq t \leq 23$ if $g = 7$. Next we consider the case of $t \geq 17$ if $g = 3$ or $t \geq 25$ if $g = 7$. Let $t = 8m + i$ and $s = gm$ where $i = 1, 3, 5, 7$. Then $3t = 8s + gi$ and $s \geq 6$. By the proof of Lemma 3.11 and Lemma 4.8, it handles the case of $u \geq (3gi + 9g - 10)/2$. For $u \leq (3gi + 9g - 14)/2$, we prove it inductively as in Lemma 3.11. ◇

Lemma 4.13 *Let* g *and* t *be odd,* $u \equiv (t+1)/2 \pmod 2$ *and* $u \leq 3g(t-1)/2$. *Then there exists a* $(K_3+e,4)$-*GDD of type* $g^t u^1$.

Proof Let $g = 4k + l$ where $l = 1, 3$. By the proof of Lemma 3.13 and Lemma 4.8, it handles the case of $u \geq 3l(t-1)/2 + 1$. A similar arguments as in Lemma 3.14 can deal with the case of $u \leq 3l(t-1)/2 - 1$. ◇

From Lemmas 4.1, 4.5 and 4.13 we obtain the following theorem.

Theorem 4.14 *The necessary conditions as in Lemma 1.1 for the existence of a* $(K_3+e,4)$-*GDD of type* $g^t u^1$ *are also sufficient except* $(g, t, u) = (1, 2, 1)$ *and* $(1, 3, 0)$.

5 Conclusion

By Theorems 1.2, 3.15 and 4.14, we obtain the following theorem.

Theorem 5.1 *The necessary conditions as in Lemma 1.1 for the existence of a* $(K_3 + e, \lambda)$*-GDD of type* $g^t u^1$ *are also sufficient except* $(g, t, u, \lambda) = (1, 2, 1, \lambda)$ *and* $(1, 3, 0, \lambda)$ *where* $\lambda \equiv 0 \pmod 4$.

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Appendix A

Let $X = Z_{gt} \cup \{\infty_1, \ldots, \infty_u\}$ and $\mathcal{G}=\{\{i, t+i, \ldots, (g-1)t+i\} : 0 \leq i \leq t-1\} \cup \{\{\infty_1, \ldots, \infty_u\}\}$. A (K_3+e, λ)-GDD of type $g^t u^1$ $(X, \mathcal{G}, \mathcal{B})$ is constructed by listing its blocks $\mathcal{B}$ as below.

$\lambda = 2 \Rightarrow$

$1^5 4^1$:	$(1, \infty_1, 0)$-2	$(3, \infty_1, 2)$-∞_4	$(3, 4, \infty_3)$-0	$(4, 2, 0)$-∞_2	$(1, 4, \infty_4)$-2
	$(2, 3, \infty_1)$-4	$(1, 3, \infty_4)$-0	$(0, 3, \infty_3)$-4	$(1, \infty_2, 4)$-2	$(3, 4, \infty_2)$-0
	$(1, 3, \infty_2)$-2	$(0, 3, \infty_4)$-4	$(1, \infty_3, 2)$-∞_2	$(4, \infty_1, 0)$-1	$(2, \infty_3, 1)$-∞_1

$2^3 1^1$:	$(0, 1, 2)$-3	$(3, 5, 1)$-2	$(0, 2, 4)$-3	$(0, 4, 5)$-1	$(\infty_1, 4, 3)$-1
	$(4, 5, \infty_1)$-1	$(\infty_1, 3, 2)$-4	$(\infty_1, 0, 5)$-3	$(0, 1, \infty_1)$-2	

$2^3 3^1$:	$(\infty_1, 1, 0)$-∞_3	$(\infty_2, 5, 4)$-0	$(3, \infty_3, 2)$-1	$(2, \infty_2, 0)$-1	$(4, \infty_1, 3)$-∞_2
	$(1, 5, \infty_2)$-2	$(3, \infty_2, 1)$-∞_3	$(3, 5, \infty_3)$-1	$(4, \infty_1, 2)$-3	$(\infty_3, 4, 2)$-0
	$(0, 5, \infty_3)$-4	$(0, \infty_2, 4)$-3	$(3, \infty_1, 5)$-1	$(2, \infty_1, 1)$-3	$(0, \infty_1, 5)$-4

$\lambda = 4 \Rightarrow$

1^6 :	$(1, 2, 3)$-4	$(3, 0, 5)$-1	$(4, 0, 2)$-1	$(1, 3, 0)$-5	$(1, 4, 5)$-3
	$(2, 1, 5)$-4	$(4, 5, 3)$-1	$(2, 4, 0)$-1	$(1, 4, 3)$-0	$(5, 0, 2)$-1
	$(1, 0, 4)$-2	$(2, 3, 5)$-4	$(5, 0, 1)$-4	$(3, 4, 2)$-5	$(3, 2, 0)$-4

1^7 :	$(2, 0, 1)$-6	$(3, 5, 4)$-2	$(3, 4, 0)$-6	$(2, 6, 5)$-1	$(3, 6, 1)$-4
	$(2, 5, 3)$-0	$(2, 6, 4)$-1	$(0, 5, 6)$-2	$(1, 5, 4)$-6	$(0, 6, 3)$-1
	$(5, 4, 0)$-2	$(1, 2, 0)$-4	$(3, 6, 2)$-1	$(1, 5, 3)$-4	$(1, 4, 6)$-5
	$(0, 4, 2)$-3	$(0, 1, 3)$-5	$(1, 2, 5)$-0	$(3, 4, 2)$-5	$(4, 5, 6)$-3
	$(1, 6, 0)$-5				

$1^3 2^1$:	$(\infty_1, 2, 0)$-∞_2	$(0, \infty_1, 2)$-1	$(1, \infty_2, 0)$-∞_1	$(0, \infty_2, 1)$-∞_1	$(2, \infty_2, 1)$-0
	$(1, \infty_1, 2)$-∞_2	$(1, \infty_1, 0)$-2	$(2, \infty_2, 1)$-∞_1	$(0, \infty_2, 2)$-∞_1	

Appendix B

A (K_3+e, λ)-GDD of type $g^t u^1$ is constructed by listing its some blocks and some base blocks as below.

$\lambda = 2 \Rightarrow$

$2^3 5^1$:	$(\infty_1, 1, 0)$-∞_5	$(\infty_2, 2, 0)$-∞_5	$(\infty_3, 2, 0)$-∞_4	(mod 6)	
	$(\infty_4, 0, 1)$-2	$(\infty_4, 2, 3)$-4	$(\infty_4, 4, 5)$-0		

$3^4 1^1$:	$(1, 3, 0)$-∞	$(5, 3, 0)$-∞	(mod 12)		
	$(0, 5, 6)$-7	$(2, 8, 7)$-1	$(4, 10, 9)$-3	$(10, 11, 5)$-4	$(0, 6, 11)$-5
	$(0, 7, 1)$-6	$(1, 2, 8)$-3	$(2, 3, 9)$-8	$(3, 10, 4)$-11	

$3^4 2^1$:	$(\infty_1, 1, 0)$-2	$(\infty_2, 5, 0)$-1	(mod 12)		
	$(0, 3, 6)$-9	$(1, 4, 7)$-10	$(2, 5, 8)$-11	$(2, 5, 0)$-9	$(3, 6, 1)$-10
	$(4, 7, 2)$-11	$(8, 5, 3)$-9	$(6, 9, 4)$-10	$(7, 10, 5)$-11	$(8, 11, 6)$-0
	$(9, 0, 7)$-1	$(10, 1, 8)$-2	$(11, 2, 9)$-3	$(0, 3, 10)$-4	$(1, 4, 11)$-5

$\lambda = 4 \Rightarrow$

$1^5 3^1$:	$(\infty_1, 1, 0)$-2	$(\infty_1, 2, 0)$-∞_3	$(\infty_2, 1, 0)$-∞_3	$(\infty_2, 1, 0)$-2	$(\infty_3, 2, 0)$-1 (mod 5)

$1^5 5^1$:	$(\infty_1, 1, 0)$-2	$(\infty_2, 1, 0)$-∞_4	$(\infty_1, 2, 0)$-∞_4	$(\infty_2, 2, 0)$-∞_5	$(\infty_3, 1, 0)$-∞_5
	$(\infty_4, 1, 0)$-∞_5	$(\infty_3, 2, 0)$-∞_5	(mod 5)		

$1^6 2^1$:	$(\infty_1, 2, 0)$-1	$(1, 2, 0)$-∞_2	$(3, \infty_1, 0)$-∞_2	(mod 6)	
	$(2, \infty_2, 0)$-3	$(3, \infty_2, 1)$-2	$(4, \infty_2, 2)$-0	$(5, \infty_2, 3)$-4	$(0, \infty_2, 4)$-1
	$(1, \infty_2, 5)$-0	$(0, 3, 1)$-4	$(2, 3, 5)$-1	$(2, 5, 4)$-0	

$1^6 3^1$:	$(\infty_1, 2, 0)$-1	$(\infty_3, 2, 0)$-∞_2	$(\infty_3, 1, 0)$-∞_2	$(3, \infty_1, 0)$-1	(mod 6)
	$(2, \infty_2, 0)$-3	$(3, \infty_2, 1)$-2	$(4, \infty_2, 2)$-0	$(5, \infty_2, 3)$-4	$(0, \infty_2, 4)$-1
	$(1, \infty_2, 5)$-0	$(0, 3, 1)$-4	$(2, 3, 5)$-1	$(2, 5, 4)$-0	

$3^3 2^1$:	$(4, 2, 0)$-∞_1	$(1, 2, 0)$-∞_2	$(\infty_1, 4, 0)$-∞_2	$(\infty_2, 4, 0)$-∞_1	$(1, 2, 0)$-4 (mod 9)

$3^5 1^1$:	$(6, 7, 0)$-∞	$(2, 4, 0)$-∞	$(3, 4, 0)$-∞	$(3, 7, 0)$-∞	$(1, 3, 0)$-7
	$(6, 7, 0)$-6	$(2, 6, 0)$-3	(mod 15)		

$3^6 1^1$:	$(7,8,0)$-1	$(2,4,0)$-1	$(7,8,0)$-∞	$(7,4,0)$-3	$(8,4,0)$-∞
	$(5,2,0)$-9	$(\infty,9,0)$-5	(mod 18)		
	$(0,5,7)$-2	$(1,6,8)$-3	$(11,9,4)$-17	$(5,12,10)$-3	$(6,13,11)$-16
	$(7,12,14)$-9	$(8,15,13)$-0	$(14,1,3)$-16	$(4,2,15)$-10	$(3,8,0)$-2
	$(4,9,1)$-12	$(5,10,2)$-9	$(6,11,3)$-5	$(7,12,4)$-6	$(8,13,5)$-16
	$(9,14,6)$-17	$(10,15,7)$-9	$(11,16,8)$-10	$(12,17,9)$-16	$(13,0,10)$-17
	$(14,1,11)$-0	$(15,2,12)$-17	$(16,3,13)$-2	$(17,4,14)$-16	$(0,5,15)$-17
	$(1,6,16)$-0	$(2,7,17)$-1			
$7^3 2^1$:	$(4,2,0)$-∞_1	$(1,2,0)$-∞_2	$(\infty_1,4,0)$-∞_2	$(\infty_2,4,0)$-∞_1	$(5,10,0)$-7
	$(7,8,0)$-4	$(2,10,0)$-8	$(3,10,0)$-8	$(5,10,0)$-7	(mod 9)

Appendix C

$(K_3 + e, \lambda)$-$GDDs$ of types $g^t u^1$ are constructed by Lemmas 2.7-2.12.

$\lambda = 2 \Rightarrow$

$1^7 3^1$:	$(1,2,0)$-3 (mod 7)		
	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{2\}\rangle$	$\langle Z_7 \cup \{\infty_2, \infty_3^2\}, \{3\}\rangle$	
$1^7 7^1$:	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_2, \infty_3^2\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_4^2, \infty_5\}, \{2\}\rangle$
	$\langle Z_7 \cup \{\infty_6, \infty_7^2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_5, \infty_6\}, \{2,4\}\rangle$	
$1^{15} 7^1$:	$(7,6,0)$-5	$(7,6,0)$-5 (mod 15)	
	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{3\}\rangle$	$\langle Z_7 \cup \{\infty_2, \infty_3^2\}, \{3\}\rangle$	$\langle Z_7 \cup \{\infty_4^2, \infty_5\}, \{2\}\rangle$
	$\langle Z_7 \cup \{\infty_6, \infty_7^2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_5, \infty_6\}, \{2,4\}\rangle$	

$\lambda = 4 \Rightarrow$

$1^7 2^1$:	$(1,2,0)$-4	$(1,2,0)$-4 (mod 7)	
	$\langle Z_7 \cup \{\infty_1, \infty_2\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_2^2, \infty_1\}, \{2\}\rangle$
$1^7 4^1$:	$(1,2,0)$-4 (mod 7)		
	$\langle Z_7 \cup \{\infty_1, \infty_2\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_2^2, \infty_1\}, \{2\}\rangle$
	$\langle Z_7 \cup \{\infty_3, \infty_4\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_3^2, \infty_4\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_4^2, \infty_3\}, \{1\}\rangle$
$1^7 6^1$:	$\langle Z_7 \cup \{\infty_1, \infty_2\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_2^2, \infty_1\}, \{2\}\rangle$
	$\langle Z_7 \cup \{\infty_3, \infty_4\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_3^2, \infty_4\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_4^2, \infty_3\}, \{1\}\rangle$
	$\langle Z_7 \cup \{\infty_5, \infty_6\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_5^2, \infty_6\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_6^2, \infty_5\}, \{1\}\rangle$
$1^7 8^1$:	$\langle Z_7 \cup \{\infty_1, \infty_2\}, \{2,4\}\rangle$	$\langle Z_7 \cup \{\infty_1^2, \infty_2\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_2^2, \infty_1\}, \{2\}\rangle$
	$\langle Z_7 \cup \{\infty_7^2, \infty_4\}, \{2\}\rangle$	$\langle Z_7 \cup \{\infty_3^2, \infty_4\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_4^2, \infty_3\}, \{1\}\rangle$
	$\langle Z_7 \cup \{\infty_8^2, \infty_6\}, \{2\}\rangle$	$\langle Z_7 \cup \{\infty_5^2, \infty_6\}, \{1\}\rangle$	$\langle Z_7 \cup \{\infty_6^2, \infty_5\}, \{1\}\rangle$
	$\langle Z_7 \cup \{\infty_8^2, \infty_3\}, \{4\}\rangle$	$\langle Z_7 \cup \{\infty_7^2, \infty_5\}, \{4\}\rangle$	