Weakly Edge-Pancyclicity of Locally Twisted Cubes*

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Abstract The locally twisted cube LTQ_n is a newly introduced interconnection network for parallel computing. As a variant of the hypercube Q_n, LTQ_n has better properties than Q_n with the same number of links and processors. Yang, Megson and Evans [Locally twisted cubes are 4-pancyclic, Applied Mathematics Letters, 17 (2004), 919-925] showed that LTQ_n contains a cycle of every length from 4 to 2^n. In this note, we improve this result by showing that every edge of LTQ_n lies on a cycle of every length from 4 to 2^n inclusive.

Keywords Cycle, Locally twisted cubes, Pancyclicity, Edge-pancyclicity

AMS Subject Classification: 05C38 90B10

1 Introduction

The architecture of an interconnection work is usually represented by a connected simple graph $G = (V, E)$, where the vertex-set V is the set of processors and the edge-set E is the set of communication links in the network. The edge connecting two vertices x and y is denoted by (x, y). A graph G is weakly pancyclic if it contains cycles of all lengths from 4 to $|V|$ in G. The pancyclicity is an important measurement to determine if a topology of network is suitable for an application where mapping rings of any length into the topology of network is required. Large amount of related work appeared in the literature [3, 4, 5]. Vertex-pancyclicity and edge-

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pancyclicity are two stronger pancyclic properties. A graph G is weakly vertex(edge)-pancyclic if, for each vertex(edge) and any integer l ranging from 4 to $|V|$, there is a cycle of length l containing the vertex(edge). It is clear that a weakly edge-pancyclic graph is also a weakly vertex-pancyclic graph. Vertex-pancyclicity and edge-pancyclicity of some graphs have been discussed [6, 7, 8].

The hypercube network Q_n has proved to be one of the most popular interconnection networks since it has a simple structure and is easy to implement. As a variant of Q_n, the locally twisted cube LTQ_n, proposed by Yang *et al* [1], has many properties superior to Q_n. For example, LTQ_n has about half of the diameter of Q_n [1]. In particular, Yang *et al* [2] proved that LTQ_n contains a cycle of length from 4 to 2^n. In this note, we improve this result by showing the following theorem.

Theorem Every edge of LTQ_n lies on a cycle of every length from 4 to 2^n inclusive for $n \geq 2$.

The proof of the theorem is in Section 3. In Section 2, the definition and basic properties of LTQ_n are given.

2 Locally Twisted Cubes

An n-dimensional locally twisted cube $LTQ_n (n \geq 2)$, proposed first by Yang *et al* [1], has 2^n vertices. Each vertex is an n-string on $\{0, 1\}$. Two vertices $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ are adjacent if and only if one of the following conditions are satisfied.

(1) There is an integer $1 \leq k \leq n-2$ such that

(a) $x_k = \bar{y}_k$ ($\bar{y}_k$ is the complement of y_k in $\{0, 1\}$),

(b) $x_{k+1} = y_{k+1} + x_n$, and

(c) all the remaining bits of x and y are identical.

(2) There is an integer $k \in \{n-1, n\}$ such that x and y differ only in the k^{th} bit.

According to the above definition, it is not difficult to see that LTQ_n can be recursively defined as follows. LTQ_2 is a graph consisting of four vertices labelled with $00, 01, 10$, and 11, respectively, connected by four edges $(00, 01), (00, 10), (10, 11)$ and $(01, 11)$. For $n \geq 3$, LTQ_n is constructed from two disjoint copies of LTQ_{n-1} by adding 2^{n-1} edges as follows. Let LTQ^0_{n-1} denote the graph obtained by prefixing the label of each vertex of one copy of LTQ_{n-1} with 0, let LTQ^1_{n-1} denote the graph obtained by prefixing the label of each vertex of the other copy of LTQ_{n-1} with 1, and connect each vertex $x = 0x_2x_3 \ldots x_n$ of LTQ^0_{n-1} with the vertex $1(x_2+x_n)x_3 \ldots x_n$ of LTQ^1_{n-1} by an edge, where '+' represents the modulo 2 addition. For short, we denote $LTQ_n = L \oplus R$, where $L \cong LTQ^0_{n-1}$ and

$R \cong LTQ^1_{n-1}$ and call edges between L and R cross edges. Moreover, we write a cross edge as (u_L, u_R), where $u_L \in L$ and $u_R \in R$.

We use LTQ^{ij}_{n-2} to denote the $(n-2)$-dimensional locally twisted cube which is a subgraph of LTQ_n induced by the vertices labelled $ijx_3 \dots x_n$. We say an edge of LTQ_n to be critical if it is an edge in LTQ^i_{n-1} with one endpoint in LTQ^{i0}_{n-2} and the other in LTQ^{i1}_{n-2} for $i \in \{0, 1\}$.

Lemma 1 Let $LTQ_n = L \oplus R$ with $n \geq 3$. If $(u_L, v_L) \in E(L)$ is a critical edge of LTQ_n, then $(u_R, v_R) \in E(R)$ is also a critical edge of LTQ_n, where u_R and v_R are the neighbors of u_L and v_L in R, respectively.

Proof Suppose that $(u_L, v_L) \in E(L)$ is a critical edge of LTQ_n and, without loss of generality, assume that $u_L = 00u_3u_4 \dots u_n$. We have $v_L = 01(u_3 + u_n)u_4 \dots u_n$, $u_R = 1(0 + u_n)u_3u_4 \dots u_n$ and $v_R = 1(1 + u_n)(u_3 + u_n)u_4 \dots u_n$. By definition of LTQ_n, u_R and v_R are adjacent, hence (u_R, v_R) is a critical edge in R. ∎

Note that if $LTQ_n = L \oplus R$, then, for any two adjacent u_L and v_L in L, their neighbors u_R and v_R in R are not always adjacent in R, and vice versa. However, it is clear from Lemma 1 that if (u_L, v_L) is a critical edge, then their neighbors u_R and v_R in R must be adjacent in R, and vice versa. Thus, the vertices u_L, v_L, v_R, u_R, u_L form a 4-cycle. Critical edges play an important role in the proof of our theorem. A cycle in LTQ_n is called a 2-critical if it contains at least two critical edges. It is easy to see that every vertex in LTQ_n is incident with a critical edge and every cross edge lies on a 2-critical cycle of length four.

Lemma 2 If the length of a cycle is greater than 2^{n-2} in the subgraph LTQ^0_{n-1} of LTQ_n for $n \geq 4$, then it must be a 2-critical cycle.

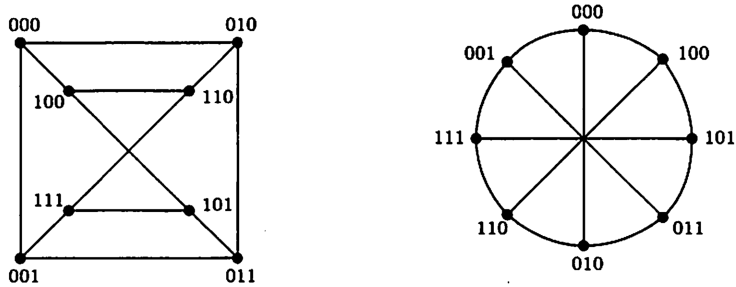
Proof Note that the $(n-2)$-dimensional crossed cube LTQ^{0j}_{n-2} for $j \in \{0, 1\}$ has only 2^{n-2} vertices. Since any cycle in LTQ^{00}_{n-2} or in LTQ^{01}_{n-2} has length at most 2^{n-2}, any cycle of length greater than 2^{n-2} in LTQ^0_{n-1} must contain vertices in both LTQ^{00}_{n-2} and LTQ^{01}_{n-2} and so contain at least two critical edges between LTQ^{00}_{n-2} and LTQ^{01}_{n-2}. ∎

3 Proof of Theorem

In this section, we give the proof of Theorem stated in Introduction.

Proof We prove the theorem by induction on $n \geq 2$. The theorem is true for $n = 2$.

For $n = 3$, by the symmetric property of LTQ_3 (see Fig. 1), we only need to show that the theorem holds for the edge $e \in \{(000, 001), (010, 000)\}$.



(a) (b)

Figure 1: (a)An ordinary drawing of LTQ_3 (b) a symmetric drawing of LTQ_3

All cycles of lengths from 4 to 8 containing (000, 001)(labelled by underlines) and (010, 000)(labelled by overlines) are as follows.

(1) length 4: $\langle \underline{000, 001}, 011, \overline{010, 000} \rangle$;

(2) length 5: $\langle \underline{000, 001}, 111, 110, \overline{010, 000} \rangle$;

(3) length 6: $\langle \underline{000, 001}, 111, 101, 011, \overline{010, 000} \rangle$;

(4) length 7: $\langle \underline{000, 001}, 111, 101, 100, 110, \overline{010, 000} \rangle$;

(5) length 8: $\langle \underline{000, 001}, 111, 110, 100, 101, 011, \overline{010, 000} \rangle$;

Hence, the theorem is true for $n = 3$.

Assume now that the theorem is true for all $3 \leq k < n$. Let e be any edge of LTQ_n and let ℓ be any integer with $4 \leq \ell \leq 2^n$, where $n \geq 4$. To complete the proof of the theorem, we need to show that e is contained in a cycle of length ℓ by considering two cases according as e is a cross edge or not.

Case 1. The edge e is not a cross edge. Then the edge e is in L or R. Without loss of generality, we may assume e is in L.

If $4 \leq \ell \leq 2^{n-1}$, by the induction hypothesis, there exists a cycle of length ℓ in $L \subset LTQ_n$ that contains e.

Suppose that $2^{n-1} + 1 \leq \ell \leq 2^{n-1} + 3$. By the induction hypothesis, there exists a cycle C of length $2^{n-1} - 3$ in L containing e. For $n \geq 4$, we have $2^{n-1} - 3 > 2^{n-2}$, and so C is a 2-critical cycle by Lemma 2. Thus, we can choose a critical edge (u_L, v_L) in C different from e. Then the neighbors of u_L, v_L are u_R, v_R in R with $(u_R, v_R) \in E(R)$ by Lemma 1. By the induction hypothesis, there exists a cycle C' of length $4 \leq \ell' \leq 6$ in R containing (u_R, v_R). Thus $P' = C' - (u_R, v_R)$ is a path between v_R and u_R in R. Let $P = C - (u_L, v_L)$. Then P contains e and $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in LTQ_n containing e (see Fig. 2 (a)).

Suppose that $2^{n-1} + 4 \leq \ell \leq 2^n$. Let $\ell' = \ell - 2^{n-1}$. Then $4 \leq \ell' \leq$

2^{n-1}. By the induction hypothesis and Lemma 2, there exists a 2-critical cycle C of length 2^{n-1} in L containing e. We can choose a critical edge (u_L, v_L) different from e. Without loss of generality, let u_R and v_R be the neighbors of u_L and v_L, respectively. Then u_R and v_R are adjacent in R. Let $P = C - (u_L, v_L)$. Obviously e lies on P. By the induction hypothesis there exists a cycle C' of length ℓ' in R that contains (u_R, v_R). Let $P' = C' - (v_R, u_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in LTQ_n and contains e (see Fig. 2 (b)).

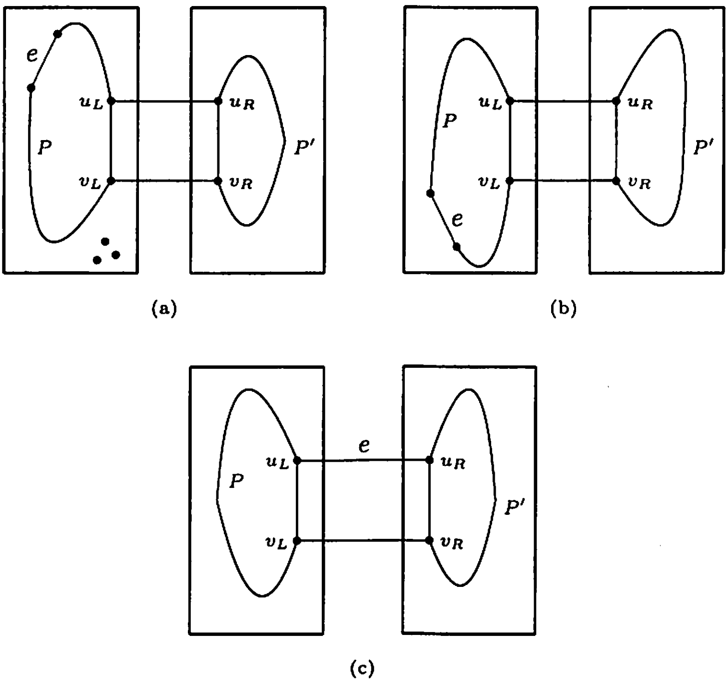


Figure 2: Illustrations for the proof of Theorem

Case 2. The edge e is a cross edge between L and R. We may assume $e = (u_L, u_R)$ and $u_L = 0u_2u_3\ldots u_n$. Then $u_R = 1(u_2 + u_n)u_3 \ldots u_n$

The cycles of length 4 and 5 containing e are as follows.

Let $v_L = 0\bar{u}_2(u_3 + u_n)\ldots u_n$, and $v_R = 1(\bar{u}_2 + u_n)(u_3 + u_n)\ldots u_n$, then (u_L, v_L) and (u_R, v_R) are critical edges and $\langle u_L, v_L, v_R, u_R, u_L\rangle$ is a cycle of length four in LTQ_n containing e. And if $u_n = 0$

$$\langle 0u_2u_3\ldots u_n,\ 0u_2u_3\ldots\bar{u}_n,\ 1(u_2+\bar{u}_n)u_3\ldots\bar{u}_n,$$
$$1(u_2+\bar{u}_n)u_3\ldots u_n,\ 1(u_2+u_n)u_3\ldots u_n,\ 0u_2u_3\ldots u_n\rangle.$$

is a cycle of length five in LTQ_n containing e.

If $u_n = 1$

$$\langle 0u_2u_3\dots u_n,\ 0u_2u_3\dots \bar{u}_n,\ 1(u_2+\bar{u}_n)u_3\dots \bar{u}_n,$$
$$1(u_2+u_n)u_3\dots \bar{u}_n,\ 1(u_2+u_n)u_3\dots u_n,\ 0u_2u_3\dots u_n\rangle.$$

is a cycle of length five in LTQ_n containing e.

For $\ell \geq 6$,we can write $\ell = \ell_1 + \ell_2$ where $\ell_1 = 2$, $\ell_2 \geq 4$ or $\ell_1 \geq 4$, $\ell_2 \geq 4$. Consider the cycle $\langle u_L, v_L, v_R, u_R, u_L\rangle$ of length four in LTQ_n containing e. By the induction hypothesis, there exists a cycle C of length ℓ_1 in L containing (u_L, v_L) if $\ell_1 \geq 4$ and exists a cycle C' of length ℓ_2 in R containing (u_R, v_R). Let $P = (u_L, v_L)$ if $\ell_1 = 2$ or $P = C - (u_L, v_L)$ if $\ell_1 \geq 4$; $P' = C' - (v_R, u_R)$. Then $P + (v_L, v_R) + P' + (u_R, u_L)$ is a cycle of length ℓ in LTQ_n and contains e (see Fig. 2 (c)).

By the induction principle, the theorem follows. ∎

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