PACKING THREE-VERTEX PATHS IN 2-CONNECTED CUBIC GRAPHS

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ABSTRACT. We show that every 2-connected cubic graph of order $n > 8$ admits a P_3-packing of at least $\lceil \frac{9n}{11} \rceil$ vertices. The proof is constructive, implying an $O(M(n))$ time algorithm for constructing such a packing, where $M(n)$ is the time complexity of the perfect matching problem for 2-connected cubic graphs.

1. INTRODUCTION

Generalized matching problems have been studied in a wide variety of contexts [1, 3, 4, 6, 11]. For a fixed graph H, an H-packing in a graph G is defined as a subgraph $F \subseteq G$ such that each connected component of F is isomorphic to G. The *maximum H-packing* problem consists in determining an H-packing of maximum order in the input graph. In particular, in the *maximum P_3-packing* problem we seek a cover of as many vertices of a graph as possible using vertex-disjoint copies of the 3-vertex path.

A lot of attention has been given to the maximum P_3-packing problem in different subclasses of cubic (3-regular) graphs. Let us recall that a graph G is said to be *k-connected* if there does not exist a set of $k - 1$ vertices whose removal disconnects the graph. In 1985, Akiyama and Kano made a conjecture concerning P_3-packing in 3-connected cubic graphs, which still remains unproved.

Conjecture (Akiyama and Kano [1]). *Every 3-connected cubic graph of order divisible by three admits a perfect P_3-packing, i.e. a P_3-packing on all its vertices.*

A partial positive answer to this problem was obtained by Kaneko et al. [5] who established that every connected n-vertex claw-free graph having at most two end-blocks (in particular, a 2-connected claw-free graph) has a maximum P_3-packing on $3\lfloor \frac{n}{3} \rfloor$ vertices. Another result related to Akiyama and Kano's conjecture was obtained in 2004 by Kelmans and Mubayi [7] who established the existence of a P_3-packing of at least $\lceil \frac{3n}{4} \rceil$ vertices for all cubic graphs. In this paper our main result is as follows.

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Theorem 1. *Every 2-connected cubic graph of order $n > 8$ has a P_3-packing of at least $\lceil \frac{9n}{11} \rceil$ vertices.*

The proof is based upon the idea of adding edges from G to a C_3-free 2-factor of G until each of the connected components of the obtained subgraph of G has a P_3-packing of the required cardinality — this we shall discuss in the next section. Since the proof is constructive, it directly implies an $\frac{11}{9}$-approximation algorithm for the maximum P_3-packing problem, which is discussed in Section 3.

2. Proof of Theorem 1

2.1. **Notation and definitions.** Given a graph G, its vertex set is denoted by $V(G)$, and its edge set by $E(G)$. By $\deg_G(v)$ we denote the *degree* of a vertex $v \in V(G)$ in graph G. The k-vertex cycle is denoted by C_k, while the k-vertex path is denoted by P_k. For a given set of integers S, the notation $C_{k \in S}$ describes a graph isomorphic to any k-vertex cycle such that $k \in S$.

The symbol $\varrho(G)$ is used to denote the size of the maximum P_3-packing in G relative to the order of G, that is:

$$\varrho(G) = \max_M \frac{|V(M)|}{|V(G)|},$$

where M spans the set of all P_3-packings in G. Hence, the claim of Theorem 1 can clearly be restated as follows: *For every 2-connected cubic graph G with* $|V(G)| > 8$, $\varrho(G) \geq \frac{9}{11}$.

Let A and B be vertex-disjoint connected subgraphs of graph G. We say that A and B are *adjacent* in G if there exist vertices $v_1 \in V(A)$ and $v_2 \in V(B)$ such that $e = \{v_1, v_2\}$ is an edge of G, $e \in E(G)$. For adjacent subgraphs we define the operation of *connection* of subgraph A to subgraph B by means of edge e, which leads to the new subgraph $A \sim B$ of G, formally defined as follows:

$$V(A \sim B) = V(A) \cup V(B);$$
$$E(A \sim B) = E(A) \cup E(B) \cup \{e\}.$$

A connected component of a graph which is isomorphic to a cycle is called a *cycle component*. Let us recall that a spanning subgraph F of a given graph G is called a *2-factor* if $\deg_F(v) = 2$ for all $v \in V(F)$, or equivalently, if all connected components of F are cycles. A special case holds if a 2-factor is restricted to have no cycle components C_3 — then such a factor is called C_3*-free*. A graph H is called $(2,3)$*-regular* if $\deg_H(v) \in \{2,3\}$ for all $v \in V(H)$. A spanning $(2,3)$-regular subgraph is called a $(2,3)$*-factor*. The subgraph of G induced by set of vertices $U \subseteq V(G)$ is denoted by the symbol $G[U]$.

2.2. **Outline of approach.** In our considerations we will make use of a result of Kawarabayashi *et al.* [6] who proved that every 2-connected cubic graph G has a C_3-free 2-factor. Given 2-connected cubic graph G of order at least $n > 8$, let F be such a C_3-free 2-factor. Bearing in mind the ratio $\frac{9}{11}$, we observe that only the connected components of F isomorphic to cycles C_4, C_5 or C_8 are problematic;

clearly, all other cycles $C_{x \notin \{4,5,8\}}$ of F admit a P_3-packing of at least $\lceil \frac{9|V(C_x)|}{11} \rceil$ vertices. For the proof we will therefore use a modification of factor F. Such a spanning subgraph $H \subseteq G$ will be the $(2,3)$-factor described by Theorem 2 (see Fig. 1(c) for an illustration). In Subsection 2.4 we will then show that $\varrho(H) \geq \frac{9}{11}$.

2.3. Construction of $(2,3)$-factor $H \subseteq G$.

Theorem 2. *Every 2-connected 3-regular graph G has a $(2,3)$-factor $H \subseteq G$ and a vertex partition $V(G) = V_S \cup V_P$, $V_S \cap V_P = \emptyset$, such that:*

(1) *All the connected components of $H[V_P]$ are isomorphic to paths from the set $\{P_4, P_5, P_8\}$; these are known as* pendant paths.

(2) *All the connected components of $H[V_S]$ are isomorphic to graphs from the set $\{C_{t \notin \{3,4,5,8\}}, C_5 \sim C_{x \in \{4,5,8\}}, C_8 \sim C_{y \in \{4,8\}}, C_4 \sim C_4 \sim C_4\}$; these are known as* supporting components,

(3) *The set of all remaining edges, $E_C = E(H) \setminus (E(H[V_S]) \cup E(H[V_P]))$, connects each end-vertex of a pendant path to a vertex of some supporting component in such a way that for every vertex of any supporting component, there is at most one such edge incident to it.*

Proof. First, consider the following approach which creates an auxiliary factor $F \subseteq G$ (an example is shown in Fig. 1(b)). Initially F, is an arbitrary C_3-free 2-factor of G, and throughout the process F remains a spanning subgraph of G.

Step (1). Let C_5 be an arbitrarily chosen 5-vertex cycle component of $F \subseteq G$.

(a) If C_5 is adjacent in G to a cycle component $C_{x \in \{4,5,8\}}$ of F, then connect C_5 to C_x, obtaining a new component $C_5 \sim C_x$ of F.

(b) Otherwise, the considered cycle C_5 is called *pendant*, and graph F is not modified at this point.

Step (2). Let C_8 be an arbitrarily chosen 8-vertex cycle component of $F \subseteq G$.

(c) If C_8 is adjacent in G to a cycle component $C_{x \in \{4,8\}}$ of F, then connect C_8 to C_x, obtaining a new component $C_8 \sim C_x$ of F.

(d) Otherwise, the considered cycle C_8 is called *pendant*, and graph F is not modified at this point.

Step (3). Let C_4 be an arbitrarily chosen 4-vertex cycle component of $F \subseteq G$.

(e) If C_4 is adjacent in G to another 4-vertex cycle component of F, then connect C_4 to this component, thus obtaining a new 8-vertex component of F denoted by $C_4 \sim C_4$.

 (e.1) If there exists a cycle component C of F isomorphic to C_4 and adjacent to $C_4 \sim C_4$ in G, then connect $C_4 \sim C_4$ to C, obtaining component $C_4 \sim C_4 \sim C_4$ of F (notice that this notation disregards which of the two 4-vertex cycles is connected to C).

 (e.2) Otherwise, the considered component $C_4 \sim C_4$ is called *pendant*, and graph F is not modified at this point.

(f) If (e) does not hold, then the considered cycle C_4 is called *pendant*, and graph F is not modified at this point.

The above procedure is executed as follows. First, Step (1) is iterated until all cycle components C_5 in F disappear or become pendant; next, Step (2) is iterated until all components C_8 in F disappear or become pendant; finally, Step (3) is iterated until all components C_4 in F disappear or become pendant. All connected components of F created in Steps (1)-(3) which are not pendant are called *supporting*. Thus, all components of F belong to one of the groups:

- pendant components isomorphic to graphs from the set

$$\{C_4, C_5, C_8, C_4 \sim C_4\},$$

- supporting cycle components $C_{t \notin \{3,4,5,8\}}$,
- supporting components isomorphic to graphs from the set

$$\{C_5 \sim C_{x \in \{4,5,8\}}, C_8 \sim C_{y \in \{4,8\}}, C_4 \sim C_4 \sim C_4\}.$$

Now, we perform an additional step of the algorithm to obtain the desired factor H from the factor F. This is achieved by transforming pendant components of F into pendant paths in H. Note that by definition of Steps (1)-(3) of the procedure, *if an edge of G has exactly one end-vertex in some pendant component of F, then it must have its other end-vertex in a supporting component of F.* Moreover, we make the following simple observation.

Proposition 2.1. *Let $C \in \{C_4, C_5, C_8, C_4 \sim C_4\}$ be a subgraph of 2-connected 3-regular graph $G \supseteq C$, $|V(G)| > 8$. Then there exists a Hamiltonian path $(v_1 v_2 \dots v_{|V(C)|})$ in $G[V(C)]$, and some two vertices $x_1, x_2 \in V(G) \setminus V(C)$, such that the path $P_C = (x_1 v_1 v_2 \dots v_{|V(C)|} x_2)$ belongs to graph G, $P_C \subseteq G$.*

Step (4). In the final step of the algorithm, each pendant component C of F is replaced by the corresponding path P_C in accordance with Proposition 2.1. Since pendant components are connected by edges of G to supporting components only, this step can be performed simultaneously for all pendant components of F. The newly obtained spanning subgraph of G is called H.

After Step (4), all connected components of H are composed of supporting components of F possibly interconnected by pendant paths of 4, 5 or 8 vertices. This naturally implies a definition the partition $V = V_S \cup V_P$, and the sought claims (1), (2) and (3) clearly hold. □

2.4. Proof of bound $\varrho(H) \geq \frac{9}{11}$.

Theorem 3. *Let H be any $(2,3)$-regular graph on vertex set $V = V_S \cup V_P$ fulfilling claims (1), (2) and (3) of Theorem 2. Then $\varrho(H) \geq \frac{9}{11}$.*

Without loss of generality we will assume that H is a connected graph (otherwise our proof that $\varrho(H') \geq \frac{9}{11}$ holds for each connected component H' of H, hence also $\varrho(H) \geq \frac{9}{11}$).

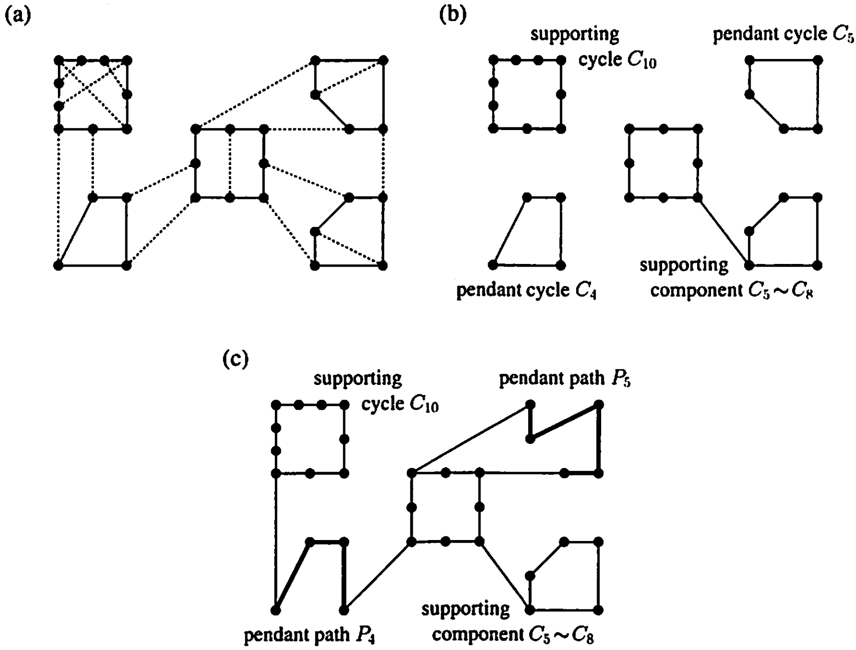

Fig. 1. (a) Solid and dashed lines: graph G. Solid lines: triangle-free 2-factor $F = \{C_4, C_5, C_5, C_8, C_{10}\}$ of G. (b) Auxiliary spanning subgraph F after Steps (1)-(3): we have two supporting components, C_{10} and $C_5 \sim C_8$, and two pendant cycles, C_4 and C_5. (c) The final graph $H \subseteq G$ after Step (4): cycles C_4 and C_5 are replaced with the relevant Hamiltonian paths (in G) and double-connected to supporting components; the resulting vertex partition $V(G) = V_S \cup V_P$, where $V_S = V(C_{10}) \cup V(C_5 \sim C_8)$ and $V_P = V(P_4) \cup V(P_5)$.

Now, for each supporting component $S \subseteq H[V_S]$, define $k(S)$ as the number of edges from E_C connecting S to end-vertices of pendant paths from $H[V_P]$. If for some S we have $k(S) = 0$, then clearly

$$H = S \in \{C_{t \notin \{3,4,5,8\}}, C_5 \sim C_{x \in \{4,5,8\}}, C_8 \sim C_{y \in \{4,8\}}, C_4 \sim C_4 \sim C_4\}$$

and the bound $\varrho(H) \geq \frac{9}{11}$ is easy to verify. Moreover, notice that the value of ϱ cannot increase if we replace a pendant path P_8 by a pendant path P_5. Thus we may assume that each supporting component in graph H is adjacent to at least one pendant path, and all pendant paths are $P_{z \in \{4,5\}}$.

A supporting component $S \subseteq H[V_S]$ will be called *deficient* if it has few connecting edges, namely, $3k(S) < |V(S)|$. As will be shown later, proving that $\varrho(H) \geq \frac{9}{11}$ is easy if H has no deficient components. Suppose however that H has deficient components $\{S_1, S_2, \ldots, S_l\}$. With each such component we associate a positive number called its *deficiency* $d_i = |V(S_i)| - 3k(S_i)$. We will eliminate the deficiencies one-by-one through local modification of edges, forming a sequence of graphs $H = H^0, H^1, H^2, \ldots, H^l$ fulfilling the following formal characterisation.

Property. *Graph H^i is a graph with vertex set $V = V(H)$ and an associated partition $V = V_1^i \cup V_3^i \cup V_S^i \cup V_P^i$, such that:*

(1) *For all $1 \leq i \leq l$, $\varrho(H^i) \leq \varrho(H^{i-1})$.*

(2) *H^i is the disconnected union of three graphs:*

$$H^i = H^i[V_1^i] \cup H^i[V_3] \cup H^i[V_S^i \cup V_P^i].$$

(3) *Graph $H^i[V_1^i]$ is a set of isolated vertices.*

(4) *Graph $H^i[V_3^i]$ is a set of connected components isomorphic to the path P_3.*

(5) *Graph $H^i[V_S^i \cup V_P^i]$ is $(2,3)$-regular and has no connected components of order* 8.

(6) *All the connected components of $H^i[V_P^i]$ are isomorphic to paths from the set $\{P_4, P_5\}$; these are known as* pendant paths.

(7) *All the connected components of $H^i[V_S^i]$ have at least* 2 *vertices; these are known as* supporting components.

(8) *The set of edges $E_C^i = E(H^i) \backslash (E(H^i[V_S^i]) \cup E(H^i[V_P^i]))$ connects each end-vertex of a pendant path with exactly one vertex from some supporting component; moreover, the two ends of a path P_4 are never connected to the same vertex of some supporting component.*

(9) *Each connected component of $H^i[V_S^i \cup V_P^i]$ contains at least one pendant path.*

(10) *The following condition is fulfilled:*

$$6p^i - |V_S^i| + \frac{8}{3}|V_3^i| - 12|V_1^i| \geq -\sum_{j=i+1}^{l} d_j,$$

where p^i is the number of pendant paths in H^i.

2.4.1. *Example.* Before we proceed any further with the details of the proof, let us provide some intuition and an outline of our technique using an example. Consider the initial graph $H = H^0$ shown in Fig. 2(a). In the first step, we replace some edges of the supporting cycle C_{10}, obtaining the new graph $H = H^1$ presented in Fig. 2(b). Likewise, the supporting component $C_5 \sim C_8$ is replaced in H^1 to obtain the new graph H^2, Fig. 2(c). Note that any P_3-packing on α vertices in H^2 (see e.g. Fig. 2(f)) can be easily converted into a P_3-packing on α vertices in H^1, and a P_3 packing on α vertices in H^1 can be converted into a P_3-packing on α vertices in H^0 (Fig. 3), only through the local replacement of the arrangement of paths.

Looking at graph H^2, we see that it has a 12-vertex subset V_3^2 consisting of 4 connected components isomorphic to P_3, a single isolated vertex forming subset

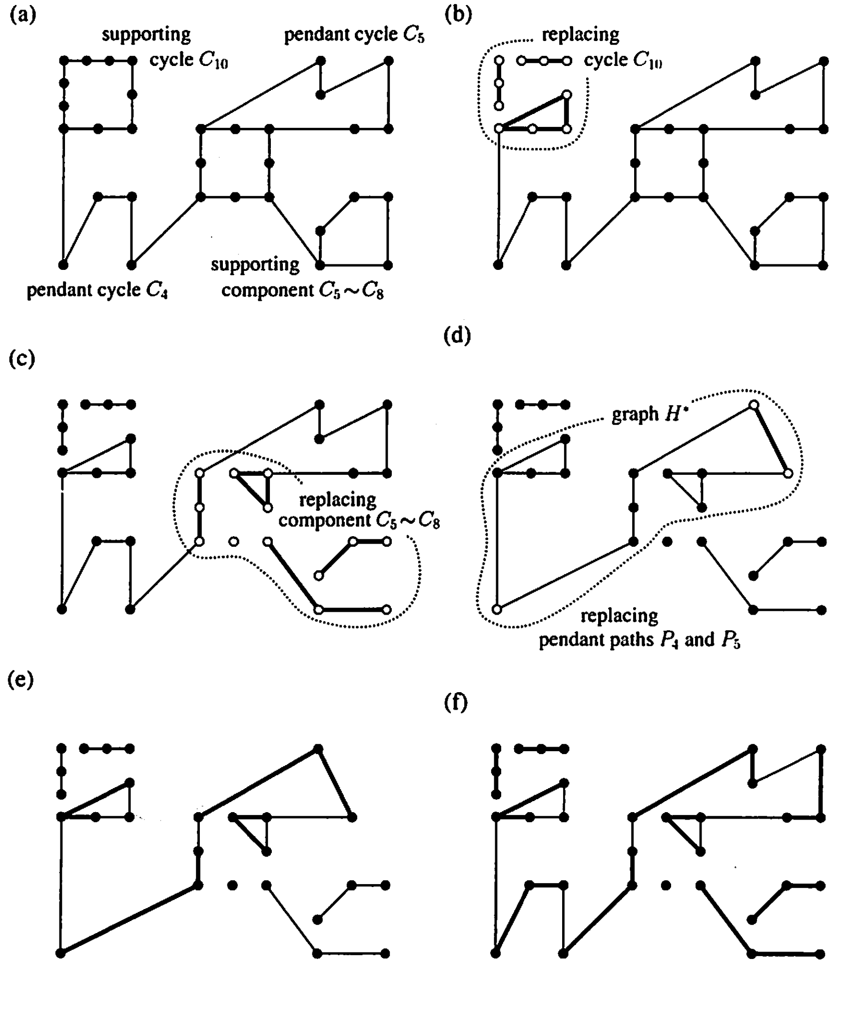

Fig. 2. (a) The original graph $H = H^0$. (b) Replacement of edge set of supporting graph $S_1 = C_{10}$ with graph $S_1' = C_4$ and two P_3-components results in graph H^1. (c) Replacement of edge set of supporting graph $S_2 = C_5 \sim C_8$ with graphs $S_2' = C_3, S_2'' = P_3$, two P_3-components and one isolated vertex results in graph H^2. (d) Replacement of pendant paths P_4 and path P_5 with 1-vertex and 2-vertex paths, respectively, results in graph H^*. (e) In graph H^*, there exists a P_3-packing with at least $\lceil \frac{3|V(H^*)|}{4} \rceil = 10$ vertices. (f) In graph H^2, there exists a P_3-packing with at least $10 + 6 + 12 = 28$ vertices.

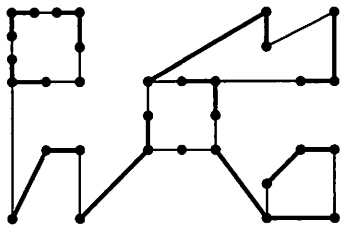

Fig. 3. P_3-packing with at least 28 vertices in graph H.

V_1, a 9-vertex subset V_P^2 (in the form $p^2 = 2$ paths, a path P_4 and a path P_5), and finally 10 vertices in the set V_S^2. The graph $H^2[V_S^2 \cup V_P^2]$ is clearly a $(2,3)$-regular graph on $10 + 9 = 19$ vertices. It can be turned into the $(2,3)$-regular graph 13-vertex graph H^* by removing exactly 3 vertices from both of its pendant paths (Fig. 2(d)). However, we know [8] that any $(2,3)$-regular graph without connected components of order 5 admits a P_3-packing on at least a $\frac{3}{4}$ part of its vertices, hence H^* has a P_3-packing on at least $\lceil \frac{3}{4} \cdot 13 \rceil = 10$ vertices, Fig. 2(e); the packing shown here is in fact slightly larger. From this, by replacing the 3 vertices previously removed from the pendant paths, we obtain a packing on at least $10 + 6 = 16$ vertices in $H^2[V_S^2 \cup V_P^2]$. Augmenting this with a perfect packing on $H^2[V_3^2]$ gives a packing on at least $16 + 12 = 28$ vertices in H^2, Fig. 2(f). As stated earlier, this can easily be transformed into a P_3-packing on at least 28 vertices in graph H, Fig. 3. Since graph H has 32 vertices, we obtain $\varrho(H) \geq \frac{28}{32} \geq \frac{9}{11}$, as expected.

2.4.2. *General approach.* We need to show three facts: that graph $H^0 = H$ fulfills the Property, that graphs H^i, $1 \leq i \leq l$, can be constructed in accordance with the Property, and that $\varrho(H^l) \geq \frac{9}{11}$. Then the proof of the Theorem is complete, since we obtain:

$$\varrho(H) = \varrho(H^0) \geq \varrho(H^1) \geq \ldots \geq \varrho(H^l) \geq \frac{9}{11}.$$

Lemma 2.2. *Graph $H^0 = H$ fulfills the Property.*

Proof. Putting $V_S^0 = V_S$, $V_P^0 = V_P$, $V_1^0 = V_3^0 = \emptyset$, we see that clauses (2)-(9) of the Property are immediately satisfied by definition of H and by the earlier assumptions. Only clause (10) remains to be shown; that is, for graph H we need to show that:

$$\sum_{j=1}^{l} d_j \geq |V_S| - 6p,$$

where p is the number of pendant paths of H. Note that by definition, for a supporting component $S \subseteq H[V_S]$ we have $|V(S)| - 3k(S) < 0$ if S is not deficient, and $|V(S)| - 3k(S) = d_j$ if $S \equiv S_j$ is a deficient component. Thus, we may write:

$$\sum_{j=1}^{l} d_j = \sum_{j=1}^{l}(|V(S_j)| - 3k(S_j))$$
$$\geq \sum_{S \subseteq H[V_S]} (|V(S)| - 3k(S)) = |V_S| - 3 \sum_{S \subseteq H[V_S]} k(S),$$

where S spans all supporting components in $H[V_S]$. Since $2p = \sum_{S \subseteq H[V_S]} k(S)$ by a trivial analogue of the handshaking lemma, the claim follows directly. □

Lemma 2.3. *For all $1 \leq i \leq l$, graphs H^i can be constructed in accordance with the Property.*

Proof. First, using the incremental notation $\delta x^i \equiv x^i - x^{i-1}$, the required clause (10) of the Property may be rewritten as follows:

(10') $\quad 6\delta p^i - \delta|V_S^i| + \frac{8}{3}\delta|V_3^i| - 12\delta|V_1^i| \geq d_i = k(S_i) - 3|V(S_i)|.$

In the proposed construction each of the graphs H^i will always be formed from H^{i-1} by a local replacement of edges within supporting component S_i and pendant paths adjacent to it, only. This means that the number of edges from E_C^{i-1} adjacent to S_i in H^{i-1} is still equal to $k(S_i)$. The modification of H^{i-1} used to obtain H^i depends on the structure of component S_i and the arrangement of adjacent edges in H^{i-1}. We need to consider the following cases, remembering that $d_i = |V(S_i)| - 3k(S_i) \geq 0$:

- (A) $S_i = C_{t \notin \{3,4,5,8\}}$,
- (B) $S_i = C_5 \sim C_4$ and $k(S_i) \in \{1, 2\}$,
- (C) $S_i = C_5 \sim C_5$ and $k(S_i) \in \{1, 2, 3\}$,
- (D) $S_i = C_5 \sim C_8$ and $k(S_i) \in \{1, 2, 3, 4\}$,
- (E) $S_i = C_8 \sim C_4$ and $k(S_i) \in \{1, 2, 3\}$,
- (F) $S_i = C_8 \sim C_8$ and $k(S_i) \in \{1, 2, 3, 4, 5\}$,
- (G) $S_i = C_4 \sim C_4 \sim C_4$ and $k(S_i) \in \{1, 2, 3\}$.

First, consider Case (A), $S_i = C_{t \notin \{3,4,5,8\}}$. Suppose for a moment that $d_i = 1$; then $t = 3k(S_i) + 1$, which means that by the pigeon-hole principle some three successive vertices v_1, v_2, v_3 of the cycle S_i are not connected to pendant paths, and thus of degree 2 in H^{i-1}. We will remove these vertices from S_i, replacing it by a cycle C_{t-3} and an isolated path P_3. Let v_0 and v_4 be their neighbours in S_i, i.e. $(v_0 v_1 v_2 v_3 v_4)$ form a path. Graph H^i is now formally defined with edge set $E(H^i) = E(H^{i-1}) \cup \{\{v_0, v_4\}\} \setminus \{\{v_0, v_1\}, \{v_3, v_4\}\}$, and vertex subsets $V_S^i = V_S^{i-1} \setminus \{v_1, v_2, v_3\}$, $V_3^i = V_3^{i-1} \cup \{v_1, v_2, v_3\}$, $V_1^i = V_1^{i-1}$, $V_P^i = V_P^{i-1}$. Graph H^i clearly fulfills clauses (2)-(9) of the Property. It is easy to see that clause

(1) is also fulfilled, actually as the equality $\varrho(H^i) = \varrho(H^{i-1})$. To write clause (10'), it suffices to observe that $\delta p^i = \delta|V_1^i| = 0$, $\delta|V_3^i| = 3$, and $\delta|V_S^i| = -3$.

The more general case of arbitrary $d_i > 0$ is handled similarly, only then we have to repeat the procedure $\lceil \frac{d_i}{3} \rceil$ times, obtaining a new cycle $C_{t-3\lceil \frac{d_i}{3} \rceil}$ and $\lceil \frac{d_i}{3} \rceil$ new isolated paths P_3. The values in clause (10') are then as follows: $\delta p^i = \delta|V_1^i| = 0$, $\delta|V_3^i| = 3\lceil \frac{d_i}{3} \rceil$, and $\delta|V_S^i| = -3\lceil \frac{d_i}{3} \rceil$, giving:

$$6\delta p^i - \delta|V_S^i| + \frac{8}{3}\delta|V_3^i| - 12\delta|V_1^i| = 11\lceil \frac{d_i}{3} \rceil \geq d_i,$$

which completes the proof of Case (A).

Cases (B)-(G) are solved using similar, somewhat more complex transformations which sometimes also modify sets V_P^i and V_1^i; some of them are discussed in the Appendix. □

Lemma 2.4. $\varrho(H^l) \geq \frac{9}{11}$.

Proof. Consider a P_3-packing $M = M_a \cup M_b$ in graph H^l formed as follows. Graph $H^l[V_3^l]$ is a set of paths P_3, so it admits a perfect packing M_a on all its vertices,

$$V(M_a) = V_3^l.$$

To obtain a P_3-packing M_b in $H^l[V_S^l \cup V_P^l]$, consider the graph H^* formed by removing exactly 3 vertices from each pendant path (that is, replacing all pendant P_4 and P_5 by P_1 and P_2, respectively). Note that H^* is a well defined graph by the additional condition in clause (8) of the Property. Since $H^l[V_S^l \cup V_P^l]$ was $(2,3)$-regular by clause (5), H^* is also clearly $(2,3)$-regular. Moreover, by clauses (5), (7) and (9), it is easy to see that no connected component of H^* has exactly 5 vertices. The authors [8] have shown that a $(2,3)$-regular graph without connected components of order 5 always fulfills $\varrho(H^*) \geq \frac{3}{4}$, so H^* admits a P_3-packing M^* such that

$$|V(M^*)| \geq \frac{3}{4}|V(H^*)| = \frac{3}{4}(|V_S^l| + |V_P^l| - 3p^l).$$

Using packing M^*, by appropriately reinserting a path P_3 into each of the p^l pendant paths of H^*, we obtain an appropriate P_3-packing M_b in $H^l[V_S^l \cup V_P^l]$:

$$|V(M_b)| = |V(M^*)| + 3p^l \geq \frac{3}{4}(|V_S^l| + |V_P^l| + p^l).$$

For the whole packing $M = M_a \cup M_b$ in H^l, we obtain:

$$|V(M)| = |V(M_a)| + |V(M_b)| \geq |V_3^l| + \frac{3}{4}(|V_S^l| + |V_P^l| + p^l).$$

This gives a lower bound on the ratio $\varrho(H^l)$:

$$\varrho(H^l) \geq \frac{|V(M)|}{|V(H^l)|} \geq \frac{|V_3^l| + \frac{3}{4}(|V_S^l| + |V_P^l| + p^l)}{|V_1^l| + |V_3^l| + |V_S^l| + |V_P^l|}$$
$$= \frac{9}{11} + \frac{11p^l - |V_S^l| + \frac{8}{3}|V_3^l| - 12|V_1^l| - |V_P^l|}{\frac{44}{3}(|V_1^l| + |V_3^l| + |V_S^l| + |V_P^l|)}.$$

So, it remains to be shown that $11p^l - |V_S^l| + \frac{8}{3}|V_3^l| - 12|V_1^l| - |V_P^l| \geq 0$. However, by clause (10) of the Property we have:

$$6p^l - |V_S^l| + \frac{8}{3}|V_3^l| - 12|V_1^l| \geq 0,$$

and since no pendant path can have more than 5 vertices:

$$5p^l - |V_P^l| \geq 0.$$

Adding together the last two inequalities completes the proof. □

3. Notes on the Algorithm

Let us recall that Kirkpatrick and Hell [3] have established that the perfect H-packing problem is NP-complete for any connected graph H with at least three vertices (thus implying the NP-hardness of the maximum P_3-packing problem). Recently the authors [9] have shown the NP-hardness of the problem even for the class of planar bipartite cubic graphs (which also implies 2-connectivity). Therefore, finding the maximum P_3-packing for the class of graphs considered in this paper may be considered computationally hard, which justifies a search for approximation algorithms.

The algorithm implied by the proof presented in Section 2 clearly provides an $\frac{11}{9}$-approximation of the optimal P_3-packing in the considered graph. It can roughly be divided into the following steps:

(1) Find a C_3-free 2-factor $F \subseteq G$;
(2) Construct $H \subseteq G$ using factor F;
(3) Construct graphs $H = H^0, H^1, \ldots, H^l$ from graph H;
(4) Construct graph H^* from H^l;
(5) Find a P_3-packing in H^* on at least $\frac{3}{4}|V(H^*)|$ vertices;
(6) Convert the packing in H^* into an appropriate packing in $H \subseteq G$.

The procedures used in Steps (2)-(4) and (6) are explicitly described in the paper and may all be implemented with a runtime of $O(n)$ for a graph G on n vertices. Step (1), i.e. determining a C_3-free 2-factor in G using the approach from [6], requires that a perfect matching be found in G; this can be achieved in $O(n \log^4 n)$ time [2]. Finally, Step (5) can also be performed using an $O(n)$ procedure [8]. Summing up, we obtain the following theorem.

Theorem 4. *There exists an $O(n \log^4 n)$ time $\frac{11}{9}$-approximation algorithm for the maximum P_3-packing problem in an n-vertex 2-connected cubic graph.*

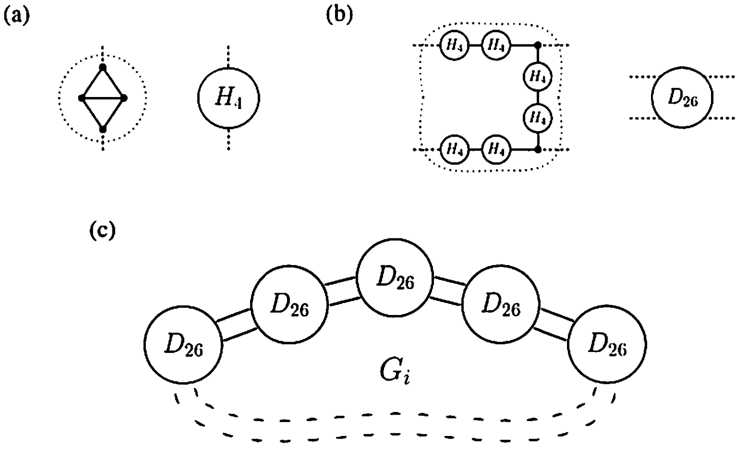

Fig. 4. The construction of graph G_i from i copies of graph D_{26}.

4. Final remarks

Let parameter $\varrho(\mathcal{G})$ describe the best possible asymptotical lower bound on the size of a P_3-packing for a graph family $\mathcal{G}$, i.e.

$$\varrho(\mathcal{G}) = \lim_{n\to\infty} \min_{G\in\mathcal{G}:|V(G)|\geq n} \varrho(G).$$

The authors [10] established that $\frac{117}{152} \leq \varrho(\mathcal{G}) \leq \frac{4}{5}$ for the family $\mathcal{G}$ of connected cubic graphs. When the family $\mathcal{G}$ is restricted to 2-connected graphs, then Theorem 1 gives $\varrho(\mathcal{G}) \geq \frac{9}{11}$. Consider now the 2-connected cubic graph G_i shown in Fig. 4(c), for any $i \geq 3$. One can check that it has $|V(G_i)| = 26i$ vertices, and that there are at most $24i$ vertex-disjoint 3-vertex paths in G_i, thus $\varrho(G_i) \leq \frac{12}{13}$. Consequently, we may write the following theorem.

Theorem 5. *For the family $\mathcal{G}$ of all 2-connected cubic graphs, we have*

$$\frac{9}{11} \leq \varrho(\mathcal{G}) \leq \frac{12}{13}.$$

This means that although we do not as yet know the precise values of parameter $\varrho(\mathcal{G})$ either for connected cubic graphs or for 2-connected cubic graphs, these two numbers are most certainly different. The value of $\varrho(\mathcal{G})$ for 3-connected cubic graphs is probably still another number; note that a weaker version of Akiyama and Kano's Conjecture can be stated as follows: *for the family $\mathcal{G}$ of all 3-connected cubic graphs, we have $\varrho(\mathcal{G}) = 1$.*

A survey of known results concerning the maximum P_3-packing problem in different classes of subcubic graphs is given in Table 1.

Graph class	*connected*	ref.	2-*connected*	ref.	3-*connected*	ref.
subcubic	$3/5$	*				
(2, 3)-*regular*	$3/4$	[8]	$3/4$	*		
cubic	$\in \langle 117/152;\ 4/5 \rangle$	[10]	$\in \langle 9/11;\ 12/13 \rangle$	Thm. 5	$\in \langle 9/11;\ 1 \rangle$	*

Table 1. Asymptotic lower bounds on the ratio of vertices contained in a maximum P_3-packing, for different classes of subcubic graphs. Simple corollaries and observations are marked with asterisks.

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APPENDIX — PROOF OF LEMMA 2.3, CASES (B)-(G)

Lemma 2.3. *For all $1 \le i \le l$, graphs H^i can be constructed in accordance with the Property.*

Proof. Let us recall that the considered cases are as follows; for simplicity of notation, we simply write $S \equiv S_i$ and $k \equiv k(S_i)$:

(A) $S = C_{t \notin \{3,4,5,8\}}$,
(B) $S = C_5 \sim C_4$ and $k \in \{1,2\}$,
(C) $S = C_5 \sim C_5$ and $k \in \{1,2,3\}$,
(D) $S = C_5 \sim C_8$ and $k \in \{1,2,3,4\}$,
(E) $S = C_8 \sim C_4$ and $k \in \{1,2,3\}$,
(F) $S = C_8 \sim C_8$ and $k \in \{1,2,3,4,5\}$,
(G) $S = C_4 \sim C_4 \sim C_4$ and $k \in \{1,2,3\}$.

Cases (B)-(G) may be solved using the approach approach from Section 2.4. For improved clarity, the relevant replacements are illustrated in figures. Their analysis is very straightforward and will only be presented in detail for chosen exemplary cases.

Case (B): $S = C_5 \sim C_4$. If $k = 1$ then the relevant replacements are illustrated in Fig. 5, while for the case $k = 2$ the relevant replacements are illustrated in Fig. 6. The applied transformation depends on the arrangement of the k connecting edges; we confine ourselves to the discussion of cases from Fig. 5(a) and Fig. 6(c).

As the first selected case, we shall discuss in detail the replacement in Fig. 5(a). W.l.o.g. assume $C_5 = (v_1 v_2 v_3 v_4 v_5)$, $C_4 = (v_6 v_7 v_8 v_9)$, $E(S) = E(C_5) \cup E(C_4) \cup \{\{v_5, v_6\}\}$, and let $e = \{x, v_1\}$ be an edge from E_C^{i-1} incident to vertex $v_1 \in V(C_5)$ (see Fig. 7). In order to obtain H^i from H^{i-1}, the edge set of S is replaced by the following:

- the edge set of the new supporting subgraph S', where $V(S') = \{v_1, v_2, v_3\}$, and $E(S') = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$;
- two isolated paths P_3, $(v_4 v_5 v_6)$ and $(v_7 v_8 v_9)$.

It is easy to see that clauses (2)-(9) of the Property are satisfied by the new graph. For clause (10'), note that $\delta p^i = \delta|V_1^i| = 0$, $\delta|V_3^i| = 6$, $\delta|V_S^i| = -6$, $d_i = 6$. Thus all we need to prove is clause (1), i.e. $\varrho(H^{i-1}) \ge \varrho(H^i)$.

(a) (b)

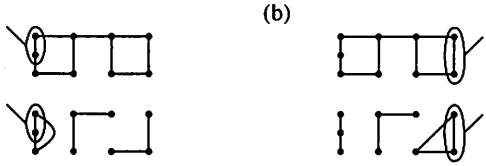

Fig. 5. $C_5 \sim C_4$. The case of one connecting edge.

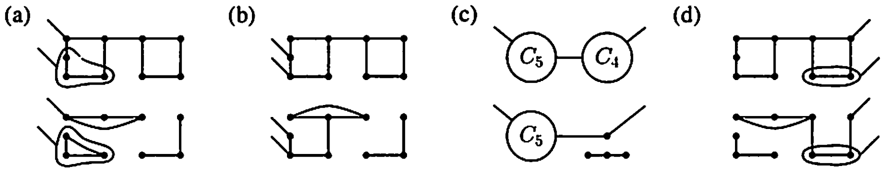

Fig. 6. $C_5 \sim C_4$. The case of two pendant edges.

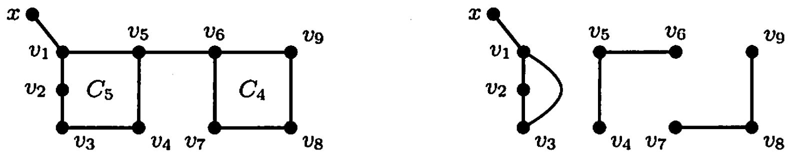

Fig. 7. $C_5 \sim C_4$ with one connecting edge — the detailed analysis.

Let M be a maximum P_3-packing in H^i. Clearly, if $\{v_1, v_3\} \notin E(M)$ then M is a P_3-packing in H^{i-1}, as $E(H^i) \setminus \{v_1, v_3\} \subset E(H^{i-1})$. Otherwise, keeping in mind the definition of graph S', there are three subcases to consider:

- if $\{v_2, v_3\} \in E(M)$, then $\{v_1, v_2\} \notin E(M)$, and $(M \setminus \{v_3v_1v_2\}) \cup \{v_1v_2v_3\}$ is a P_3-packing in H^{i-1};
- if $\{v_1, v_2\} \in E(M)$, then $\{v_2, v_3\} \notin E(M)$, and $(M \setminus \{v_1v_3v_2\}) \cup \{v_1v_2v_3\}$ is a P_3-packing in H^{i-1};
- if $e \in E(M)$, then $\{v_2, v_3\} \notin E(M)$, and $(M \setminus \{v_3v_1x\}) \cup \{v_2v_1x\}$ is a P_3-packing in H^{i-1}.

Consequently, $\varrho(H^{i-1}) \geq \varrho(H^i)$.

As the second example, consider the replacement from Fig. 6(c). W.l.o.g. assume $C_5 = (v_1v_2v_3v_4v_5)$, $C_4 = (v_6v_7v_8v_9)$, $E(S) = E(C_5) \cup E(C_4) \cup \{\{v_5, v_6\}\}$, and let $\{x, v\}$ be an edge from E_C^{i-1} incident to vertex $v \in V(C_5)$, and let $\{y, v_8\}$ be an edge from E_C^{i-1} incident to vertex $v_8 \in V(C_4)$ (see Fig. 8). In order to obtain H^i from H^{i-1}, the edge set of S is replaced by the following:

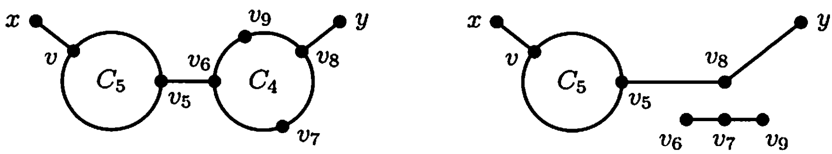

Fig. 8. $C_5 \sim C_4$ with two connecting edges — the detailed analysis.

- the new supporting subgraph S', where $V(S') = V(C_5) \cup \{v_8\}$, and $E(S') = E(C_5) \cup \{\{v_5, v_8\}\}$;
- an isolated path P_3 on vertices $(v_6 v_7 v_9)$.

Once again, it is easy to see that clauses (2)-(9) of the Property are satisfied by the new graph. For clause (10'), we observe that $\delta p^i = \delta|V_1^i| = 0$, $\delta|V_3^i| = 3$, $\delta|V_S^i| = -3$, $d_i = 3$. Thus only clause (1) remains to be shown.

Let M be a maximum P_3-packing in H^i. Clearly, if $\{v_5, v_8\} \notin E(M)$ then M is a P_3-packing in H^{i-1}, as $E(H^i) \setminus \{v_5, v_8\} \subset E(H^{i-1})$. Otherwise, keeping in mind the definition of graph S', there are three subcases to consider:

- if $\{y, v_8\} \in E(M)$, then $(M \setminus \{v_5 v_8 y, v_6 v_7 v_9\}) \cup \{v_5 v_6 v_7, v_9 v_8 y\}$ is a P_3-packing in H^{i-1};
- if $\{v_1, v_5\} \in E(M)$ (and $\{y, v_8\} \notin E(M)$), then $(M \setminus \{v_1 v_5 v_8, v_6 v_7 v_9\}) \cup \{v_1 v_5 v_6, v_7 v_8 v_9\}$ is a P_3-packing in H^{i-1};
- if $\{v_4, v_5\} \in E(M)$ (and $\{y, v_8\} \notin E(M)$), then $(M \setminus \{v_4 v_5 v_8, v_6 v_7 v_9\}) \cup \{v_4 v_5 v_6, v_7 v_8 v_9\}$ is a P_3-packing in H^{i-1}.

Consequently, $\varrho(H^{i-1}) \geq \varrho(H^i)$.

Case (C): $S = C_5 \sim C_5$. For $k = 1, 2, 3$, the relevant replacements and the argumentation for the correctness of the replacements are similar to those in the previously considered cases. We omit details and shall confine ourselves only to a brief discussion of the replacements in Fig. 9.

Let us discuss the replacement in Fig. 9(a). Assume $C_5^1 = (v_1 v_2 v_3 v_4 v_5)$, $C_5^2 = (v_6 v_7 v_8 v_9 v_{10})$, $E(S) = E(C_5^1) \cup E(C_5^2) \cup \{\{v_5, v_6\}\}$, let $\{x, v_1\}$ and $\{y, v_3\}$ be edges from E_C^{i-1} incident to vertices $v_1, v_3 \in V(C_5^1)$, respectively, and let $\{z, v_9\}$ be the edge from E_C^{i-1} incident to vertex $v_9 \in V(C_5^2)$ (see Fig. 10). Then the performed replacement consists in removing all edges of S and edge $\{z, v_9\}$, and inserting the edges of the following:

- new supporting subgraph S', where $V(S') = \{v_1, v_2, v_3\}$, and $E(S') = \{\{v_1, v_2\}, \{v_2, v_3\}\}$;
- two isolated paths P_3, $(v_4 v_5 v_6)$ and $(v_7 v_8 v_9)$;
- new edge $\{z, v_3\}$ (in E_C^i).

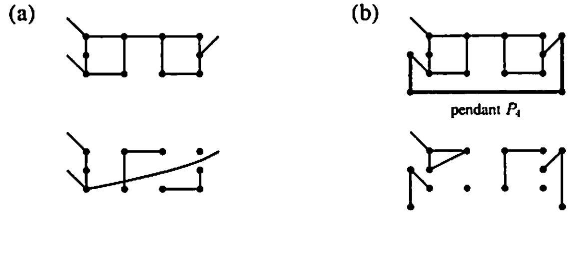

Fig. 9. (a) $C_5 \sim C_5$. One of the cases of three pendant edges. (b) The degenerate case.

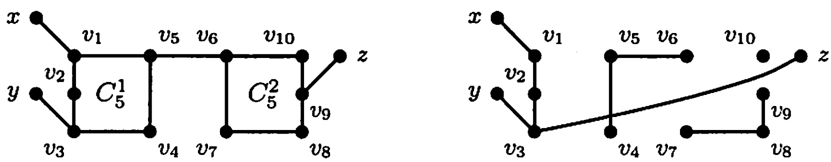

Fig. 10. $C_5{\sim}C_5$ with three pendant edges — the detailed analysis.

Again, it is easy to see that clauses (2)-(9) of the Property are always satisfied by the new graph, with the exception of clause (8) which is not fulfilled in the degenerate case when y and z belong to the same pendant path P_4 (for the applicable transformation in this case, see Fig. 9(b)). For clause (10'), note that $\delta p^i = 0$, $\delta|V_1^i| = 1$, $\delta|V_3^i| = 6$, $\delta|V_S^i| = -7$, $d_i = 1$. Thus all we need to prove is clause (1).

Let M be a maximum P_3-packing in H^i. Clearly, if $\{z, v_3\} \notin E(M)$ then M is a P_3-packing in H^{i-1}, as $E(H^i) \setminus \{z, v_3\} \subset E(H^{i-1})$. Otherwise, there are three subcases:

- if $\{v_2, v_3\} \in E(M)$, then
  $(M \setminus \{v_2v_3z, v_4v_5v_6, v_7v_8v_9\}) \cup \{v_2v_3v_4, v_5v_6v_7, v_8v_9z\}$
  is a P_3-packing in H^{i-1};
- if $\{y, v_3\} \in E(M)$, then
  $(M \setminus \{yv_3z, v_4v_5v_6, v_7v_8v_9\}) \cup \{yv_3v_4, v_5v_6v_7, v_8v_9z\}$
  is a P_3-packing in H^{i-1};
- if $\{z, w\} \in E(M)$, where $w \notin V(H^i) \setminus V(C_5{\sim}C_5)$, then
  $(M \setminus \{v_3zw, v_4v_5v_6, v_7v_8v_9\}) \cup \{v_9zw, v_3v_4v_5, v_6v_7v_8\}$
  is a P_3-packing in H^{i-1}.

Consequently, $\varrho(H^{i-1}) \geq \varrho(H^i)$.

Next, consider the replacement in Fig. 9(b). Assume $C_5^1 = (v_1v_2v_3v_4v_5)$, $C_5^2 = (v_6v_7v_8v_9v_{10})$, $E(S) = E(C_5^1) \cup E(C_5^2) \cup \{\{v_5, v_6\}\}$, let $\{x, v_1\}$ be the only edge from E_C^{i-1} adjacent to S, and let $P_4 = (v_{11}v_{12}v_{13}v_{14})$ be a pendant path (see Fig. 11). Then the edge set of $H^{i-1}[V(S) \cup V(P_4)]$ is replaced by the following:

- the new supporting subgraph S', where $V(S') = \{v_1, v_2, v_5\}$, and $E(S') = \{\{v_1, v_2\}, \{v_1, v_5\}, \{v_2, v_5\}\}$;
- three isolated paths P_3, $(v_3v_{11}v_{12})$, $(v_7v_6v_{10})$, and $(v_9v_{14}v_{13})$.

It is easy to see that clauses (2)-(9) of the Property are satisfied by the new graph. For clause (10'), note that $\delta p^i = -1$, $\delta|V_1^i| = 2$, $\delta|V_3^i| = 9$, $\delta|V_S^i| = -7$, $d_i = 1$; as a matter of fact, in this case clause (10') is an equality. Thus all we need to prove is clause (1).

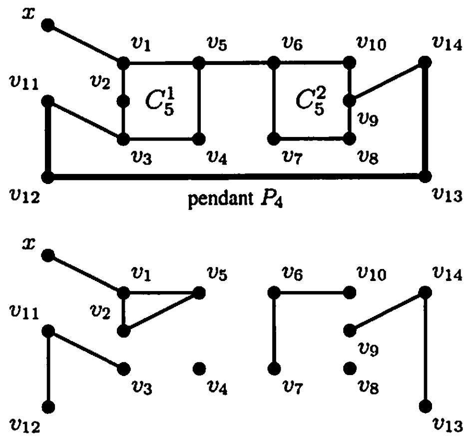

Fig. 11. The detailed analysis of the degenerate case.

Let M be a maximum P_3-packing in H^i. Clearly, if $\{v_2, v_5\} \notin E(M)$ then M is a P_3-packing in H^{i-1}, as $E(H^i) \setminus \{v_2, v_5\} \subset E(H^{i-1})$. Otherwise, there are two subcases to consider:

- if $\{v_1, v_2\} \in E(M)$, then
  $(M \setminus \{v_1v_2v_5\}) \cup \{v_2v_1v_5\}$ is a P_3-packing in H^{i-1};
- if $\{v_1, v_5\} \in E(M)$, then
  $(M \setminus \{v_1v_5v_2\}) \cup \{v_2v_1v_5\}$ is a P_3-packing in H^{i-1}.

Consequently, $\varrho(H^{i-1}) \geq \varrho(H^i)$.

Case (D): $S = C_5 \sim C_8$. For $k = 1, 2, 3, 4$, except for the cases illustrated in Fig. 12, the relevant replacements and the argumentation for the correctness of the replacements are similar to those in the previously considered cases, and thus we shall only discuss the details of the replacement from Fig. 12(a).

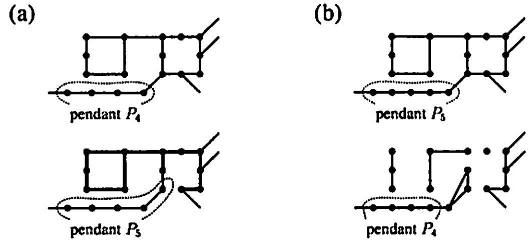

Fig. 12. $C_5 \sim C_8$. The case of four pendant edges.

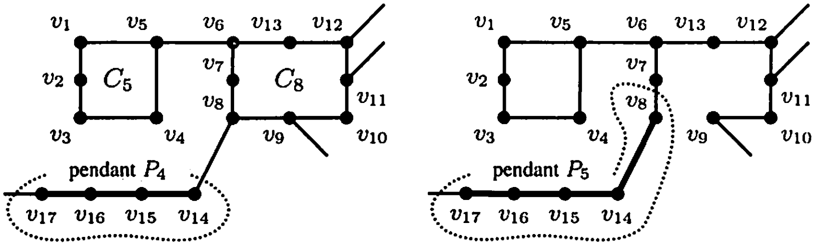

Fig. 13. $C_5 \sim C_8$ with four connecting edges — the detailed analysis.

Let $C_5 = (v_1v_2v_3v_4v_5)$, $C_8 = (v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13})$, $E(S) = E(C_5) \cup E(C_8) \cup \{\{v_5, v_6\}\}$; suppose that edges from E_C^{i-1} are incident to v_8, v_9, v_{11} and v_{12}, and let $P_4 = (v_{14}v_{15}v_{16}v_{17})$ be a pendant path connected by the edge $\{v_8, v_{14}\}$ to S (see Fig. 13). Then the edge set of graph H^i is defined as $E(H^i) = E(H^{i-1}) \setminus \{\{v_8, v_9\}\}$. Vertex $v_8 \in V_S^{i-1}$ now changes roles, moving to $v_8 \in V_P^i$.

It is easy to see that clauses (2)-(9) of the Property are satisfied by the new graph. Clause (1) is also trivially satisfied, since $E(H^i) \subset E(H^{i-1})$. To prove the final clause (10'), it is enough to put $\delta p^i = \delta|V_1^i| = \delta|V_3^i| = 0$, $\delta|V_S^i| = -1$, $d_i = 1$; note that in this case clause (10') is an equality.

For cases (E)-(G) the replacements and the argumentation for the correctness of the replacements are similar to those in the cases (B)-(D); details are omitted. □

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