ON CONNECTED COLOURINGS OF GRAPHS

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Abstract

In this paper, first we introduce the concept of a *connected* graph homomorphism as a homomorphism for which the inverse image of any edge is either empty or a connected graph, and then we concentrate on *chromatically connected* (resp. *chromatically disconnected*) graphs such as G for which any $\chi(G)$-colouring is a connected (resp. disconnected) homomorphism to $K_{\chi(G)}$.

In this regard, we consider the relationships of the new concept to some other notions as uniquely-colourability. Also, we specify some classes of chromatically disconnected graphs such as Kneser graphs $KG(m,n)$ for which m is sufficiently larger than n, and the line graphs of non-complete class II graphs.

Moreover, we prove that the existence problem for connected homomorphisms to any fixed complete graph is an NP-complete problem.

Index Words: graph colouring, connectivity, uniquely-colourable graphs.

1 Introduction

Graph homomorphism problems have been considered from many different aspects [4, 5, 6]. It is well known that in general it is a hard problem to

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decide whether there exists a homomorphism from a given graph G to a given graph H. In this paper we consider a variant of this problem in which we impose some connectivity restrictions on the inverse images of the edges (for some other variants e.g. see [2]). In this regard, we not only prove that despite this new restriction the problem is still NP-complete, but also we show that the new concept is closely related to some other well studied areas in graph theory such as uniquely-colourable graphs.

Moreover, we consider *chromatically disconnected* graphs such as G for which any $\chi(G)$-colouring is a disconnected homomorphism to $K_{\chi(G)}$, and we show that there are many interesting *chromatically disconnected* graphs as Kneser graphs $KG(m,n)$ for which m is sufficiently larger than n, and the line graphs of non-complete graphs of class II, where, the latter case can be considered as an extension of a result of S. Fiorini about uniquely-colourable graphs [1]. We also formulate a couple of problems related to the subject.

Throughout the paper the word *graph* is used for the concept of a *finite simple graph*. A *homomorphism* σ from a graph G to a graph H is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. Notations $\mathrm{Hom}(G,H)$, $\mathrm{Hom}^{\mathrm{v}}(G,H)$ and $\mathrm{Hom}^{\mathrm{e}}(G,H)$ denote the sets of ordinary, onto (vertices) and onto-edges homomorphisms from G to H, respectively. In the rest of the paper we always assume that the graph H appearing in the range of a homomorphism does not have any isolated vertex. Note that this implies $\mathrm{Hom}^{\mathrm{e}}(G,H) \subseteq \mathrm{Hom}^{\mathrm{v}}(G,H)$.

Let m,n be positive integers such that $m \geq 2n$. The notation $[m]$ stands for the set $\{1,2,\cdots,m\}$, and $\binom{[m]}{n}$ stands for the collection of all n-subsets of $[m]$. The *Kneser graph* $KG(m,n)$ has the vertex set $\binom{[m]}{n}$, in which A is connected to B if and only if $A \cap B = \emptyset$. It is well known that $\chi(KG(m,n)) = m - 2n + 2$.

2 Connected graph homomorphisms

In this section we introduce the concept of a *connected graph homomorphism* and we present some basic results.

Definition 1. A graph homomorphism $\sigma \in \mathrm{Hom}(G,H)$ is said to be *k-connected* if for any two adjacent vertices x and y in $V(H)$ the subgraph induced on the subset $\sigma^{-1}(x) \cup \sigma^{-1}(y) \subseteq V(G)$ is either empty or a k-connected graph. The space of connected graph homomorphisms $\sigma : G \longrightarrow H$ is denoted by $\mathrm{Hom}^{\mathrm{c}}_{k}(G,H)$, where we omit the subscript k when $k = 1$. ♠

It is an easy observation that for any χ-chromatic graph G, if we have $\mathrm{Hom}^{\mathrm{c}}_{k}(G,K_{\chi}) \neq \emptyset$ then $k \leq \frac{\delta(G)}{\chi-1}$.

A basic example of a connected homomorphism is the (unique) colouring of a χ-chromatic uniquely vertex colourable graph G. Moreover, since

$$\mathrm{Hom}^{c}(G, K_{\chi(G)}) \subseteq \mathrm{Hom}(G, K_{\chi(G)}) = \mathrm{Hom}^{e}(G, K_{\chi(G)})$$

holds for any graph G, we have the following generalization of a result in [10, 12] for the minimum size of uniquely vertex colourable graphs, which essentially has the same proof.

Proposition 1. *For any χ-chromatic graph G, if* $\mathrm{Hom}^{c}(G, K_{\chi}) \neq \emptyset$ *then* $|E(G)| \geq (\chi - 1)|V(G)| - \binom{\chi}{2}$.

The following definition can be considered as a generalization of the above properties of uniquely vertex colourable graphs.

Definition 2. A χ-chromatic graph G is called *chromatically k-connected* if any χ-colouring of G as a homomorphism to K_{χ} is k-connected.
Dually, a χ-chromatic graph G is called *chromatically k-disconnected* if none of the χ-colourings of G as homomorphisms to K_{χ} are k-connected. ♠

It is easy to see that if there exists a homomorphism $\sigma \in \mathrm{Hom}^{v}(G, H)$ where G is chromatically connected, then H is also a chromatically connected graph. In what follows we introduce a couple of concrete examples.

Example 1. Some chromatically connected graphs
Clearly, any uniquely vertex colourable graph is chromatically connected [10]. It is interesting to ask whether chromatically connectedness imposes any restriction on the number of colourings of a graph. To show that a chromatically connected graph may admit a relatively large number of colourings, consider the graph $K_3 \square P_n$, the cartesian product of K_3 and the path P_n on n vertices. Note that this graph is a *planar* chromatically connected graph with 2^{n-1} different 3-colourings up to permutation of the colours. ♣

3 Chromatically disconnected graphs

We begin by the following proposition as a basic result.

Proposition 2. *Let G be a χ-chromatic graph and* $\mathrm{Hom}^{c}(G, K_{\chi}) \neq \emptyset$. *If* $\chi > \frac{\Delta(G)}{2} + 1$ *then* $|V(G)| \leq \frac{\chi(\chi-1)}{2\chi - \Delta(G) - 2}$.

Proof. Let $\sigma \in \mathrm{Hom}^{c}(G, K_{\chi})$ and also let C_i's be the colour-classes of σ. Without loss of generality, assume that $|C_1| = m$ is the size of the smallest colour-class and let E_i be the edge-set of the induced subgraph on $C_1 \cup C_i$. Since, $\sigma \in \mathrm{Hom}^c(G, K_\chi)$ we have $|E_i| \geq |C_1| + |C_i| - 1$. Hence,

$$\sum_{i=2}^{\chi} |E_i| \geq (\chi - 2)m + |V(G)| - (\chi - 1).$$

On the other hand, we have $\Delta(G)m \geq \sum_{i=2}^{\chi} |E_i|$ and consequently,

$$m \geq \frac{|V(G)| - \chi + 1}{\Delta(G) - \chi + 2}.$$

Moreover, by hypothesis and the fact that $\chi m \leq |V(G)|$, we have

$$|V(G)| \leq \frac{\chi(\chi - 1)}{2\chi - \Delta(G) - 2}.$$

■

In the next theorem the graph Prism is the Cartesian product of K_3 and K_2. Also, $\mathcal{D}$ is the class of all graphs obtained by excluding one edge from the complete graph K_n for any $n \geq 3$.

Corollary 1. *Consider a graph $G \notin \mathcal{D}$ that is not isomorphic to the Prism. If $\chi(G) = \Delta(G)$ then G is chromatically disconnected.*

Proof. For any $\sigma \in \mathrm{Hom}^c(G, K_\chi)$, by $\chi(G) = \Delta(G)$ and Proposition 2 we have

$$|V(G)| \leq \frac{\Delta(G)(\Delta(G) - 1)}{\Delta(G) - 2}.$$

If $\Delta(G) \geq 5$ then we have $|V(G)| < \Delta(G) + 2$ which implies that $|V(G)| = \Delta(G) + 1$. Also, for both of the cases $\Delta(G) = 4$ and $\Delta(G) = 3$ we have $|V(G)| \leq 6$.
Note that for any connected colouring σ, the degree of any vertex that appears as a colour-class of size one must be equal to $|V(G)| - 1$. Hence, for the case $\Delta = 4$ we have $|V(G)| = \Delta(G) + 1 = 5$, and for the case $\Delta = 3$ we have either $|V(G)| = 6$ or $|V(G)| = \Delta(G) + 1 = 4$.
It is easy to check that the case $\Delta(G) = 3$ and $|V(G)| = 6$ reduces to a graph isomorphic to the Prism. Consequently, for the rest of the cases we should have $|V(G)| = \Delta(G) + 1$ when $\Delta(G) \geq 3$. But it is easy to see that in these cases one of the colour-classes is of size two and all other colour-classes must be of size one. This clearly reduces the possible cases to the graphs in $\mathcal{D}$. ■

Corollary 2. *The set $\mathcal{D}$ can be described as follows,*

$$\mathcal{D} = \{G \mid \chi(G) = \Delta(G) \ \&\ G \text{ is uniquely vertex colourable}\}.$$

In what follows we consider some other interesting classes of chromatically disconnected graphs.

Theorem 1. *If m is sufficiently larger than $n \geq 2$, then the Kneser graph $KG(m,n)$ is chromatically disconnected.*

Proof. Assume that $KG(m,n)$ admits a connected χ-colouring $\sigma \in \mathrm{Hom}^c(KG(m,n), K_\chi)$ with the set of colour-classes

$$\{C_i \mid 1 \leq i \leq m - 2n + 2 = \chi\}.$$

It was proved by Hilton and Milner [8] that if X is an independent set of $KG(m,n)$ of size greater than

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1,$$

then for some $a \in [m]$,

$$\bigcap_{A \in X} A = \{a\}.$$

Therefore, since $\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2 = O(m^{n-2})$, there exists an integer $t(n)$ such that if $m \geq t(n)$, then there are two colour-classes C_i and C_j for which the following inequalities hold,

$$|C_i| \geq \frac{|V(KG(m,n))|}{\chi} > \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1, \qquad (1)$$

$$|C_j| \geq \frac{|V(KG(m-1,n))|}{\chi - 1} > \binom{m-2}{n-1} - \binom{m-n-2}{n-1} + 1. \qquad (2)$$

On one hand, by Hilton and Milner theorem [8] and Equation 1, there exists an integer $a \in [m]$ such that,

$$\bigcap_{A \in C_i} A = \{a\}.$$

Hence, for any subset A with $a \in A \subseteq [m]$ we should have $A \in C_i$ (or otherwise σ will become a disconnected homomorphism).
Consider the graph $KG(m,n) - C_i \simeq KG(m-1,n)$ where again, Hilton and Milner theorem [8] and Equation 2, imply that there exists an integer $b \neq a$ such that,

$$\bigcap_{B \in C_j} B = \{b\}.$$

Choose $T \subset [m]$ with $|T| = n \geq 2$ and $\{a,b\} \subseteq T$. Then since T should represent a vertex of C_i and is not adjacent to any vertex in C_j, the homomorphism σ should be disconnected, which is a contradiction. ■

Existence of small chromatically disconnected Kneser graphs such as the Petersen graph and the above theorem are good motivations for the following problem.

Problem 1. *Is it true that any non complete Kneser graph is chromatically disconnected ?*

It is known that any graph of class II (i.e. $\chi'(G) = \Delta(G) + 1$) other than K_3 is not uniquely edge-colourable [1, 3]. The next theorem is a result that extends this fact in terms of connectivity of colourings.

Theorem 2. *If G is a class* II *graph and G is not a complete graph, then the line graph $L(G)$ is chromatically disconnected.*

Proof. Let G be a class II graph, $\sigma \in \mathrm{Hom}^c(L(G), K_{\Delta(G)+1})$ and let σ' be the corresponding edge-colouring of G. Note that the subgraph induced on any two colour-classes of σ' in G is either a path or a cycle.
Consider a vertex v of G with degree d_v. If A is the set of colours appearing on the edges incident to v, then the graph induced on the colours $c_1 \in A$ and $c_2 \notin A$ is a path with the end-vertex v for any such colours c_1 and c_2. Hence, the number of two-coloured paths with end-vertex v is $d_v(\Delta(G) + 1 - d_v)$. On the other hand, the number of end-vertices of all paths appearing as the induced subgraphs of pairs of colours in σ' is at most $2\binom{\Delta(G)+1}{2}$ and consequently,

$$\sum_{u \in V(G)} d_u(\Delta(G) + 1 - d_u) \leq 2\binom{\Delta(G) + 1}{2}.$$

Note that the minimum of any term in the left-hand-side is $\Delta(G)$ and this is possible only when d_u is equal to 1 or $\Delta(G)$. This shows that $|V(G)| = \Delta(G) + 1$ and since σ is connected, the degree of any vertex should be equal to 1 or $\Delta(G)$.
Also, for any vertex $v \in V(G)$ we have $d_u > 1$, since $|V(G)| = \Delta(G) + 1$ and G is a graph of class II. Therefore, G should be a complete graph on an odd number of vertices. ■

On the other hand, one can consider the case of chromatically critical graphs and note that any non-complete critical graph is not chromatically connected. Hence, we formulate the following problem.

Problem 2. *Characterize all chromatically disconnected critical graphs.*

4 Algorithmic considerations

Our main result in this section is to prove that the following decision problem is NP-complete.

Problem:	**CONnCOL**
Given:	A graph G and an integer $n \geq 3$.
Question:	Does there exist a connected homomorphism $\sigma \in \mathrm{Hom}^c(G, K_n)$?

We proceed by considering the basic case $n = 3$.

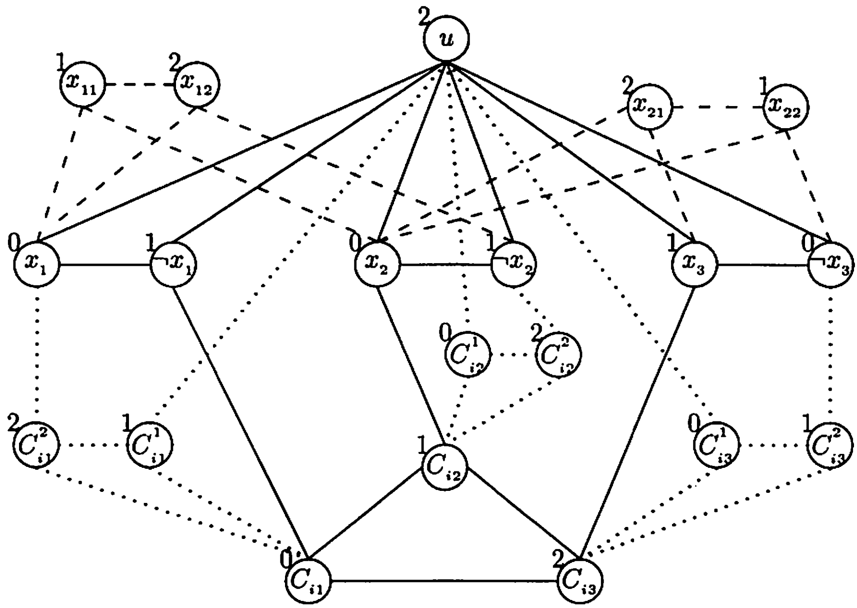

Figure 1: Portion of the graph G_ϕ for the clause $(x_1 \vee \neg x_2 \vee \neg x_3)$ (see Theorem 3).

Theorem 3. **CON3COL** *is NP-complete.*

Proof. The proof is a modified version of the standard reduction used to prove the NP-completeness of **3-COL** from **NAESAT** (e.g. see [9]). In what follows we express the details.
We are given a set of clauses $\phi = (C_1, \ldots, C_m)$, each with three literals,

involving the variables $x_1, \ldots, x_t$, and we are asked whether there is a truth assignment on the variables such that no clause has all literals *true* or all literals *false*. On the other hand, we shall construct a graph G_ϕ and argue that it admits a connected 3-colouring if and only if ϕ admits such an **NAE** truth assignment.
First, we describe the structure of G_ϕ and then we prove that such a construction is actually a polynomial time many-to-one reduction from **NAE-SAT** to **CON3COL**. The vertex set of G_ϕ is defined as follows:

$$V(G_\phi) \stackrel{\text{def}}{=} \{u\} \cup V_1 \cup V_2 \cup V_3 \cup V_4$$

where,

- $V_1 \stackrel{\text{def}}{=} \{x_i \mid 1 \leq i \leq t\} \cup \{\neg x_i \mid 1 \leq i \leq t\}$,
- $V_2 \stackrel{\text{def}}{=} \{x_{ij} \mid 1 \leq i \leq t-1 \;\&\; 1 \leq j \leq 2\}$,
- $V_3 \stackrel{\text{def}}{=} \{C_{ij} \mid 1 \leq i \leq m \;\&\; 1 \leq j \leq 3\}$,
- $V_4 \stackrel{\text{def}}{=} \{C_{ij}^k \mid 1 \leq i \leq m \;\&\; 1 \leq j \leq 3 \;\&\; 1 \leq k \leq 2\}$.

Now, we describe the edges as follows (see Figures 1).

- For each $1 \leq i \leq t$, vertices x_i and $\neg x_i$ along with the vertex u form a triangle.
- For each $1 \leq i \leq t$, vertices x_i, x_{i1} and x_{i2} form a triangle.
- For each $1 \leq i \leq t-1$, x_{i1} is adjacent to x_{i+1} and x_{i2} is adjacent to $\neg x_{i+1}$
- For each clause C_i $(1 \leq i \leq m)$, the vertices C_{i1}, C_{i2} and C_{i3} form a triangle.
- For each clause C_i $(1 \leq i \leq m)$, the vertex C_{ij} is connected to the vertex in V_1 that represents the negation of the jth literal of C_i for any $1 \leq j \leq 3$.
- For each vertex C_{ij}, corresponding to the jth literal in the ith clause, the vertices C_{ij}^1, C_{ij}^2 and C_{ij} form a triangle.
- For all $1 \leq i \leq m$ and $1 \leq j \leq 3$, the vertex C_{ij}^1 is adjacent to u and the vertex C_{ij}^2 is connected to the vertex in V_1 that represents the jth literal of C_i.

This completes the description of G_ϕ and it is easy to see that $|E(G_\phi)| = 21m + 8t - 5$ which implies that the construction can be simulated in polynomial time. In the rest of the proof we check that this construction defines a many-to-one reduction.

On one hand, we prove that if G_ϕ admits a 3-colouring then ϕ is a satisfiable instance of **NAESAT** (note that this is essentially more than what we need since we do not assume that the 3-colouring is a connected colouring of G_ϕ). To see this, assume that G_ϕ admits a 3-colouring σ with colours in $\{0,1,2\}$, where we interpret the colour 0 as the truth value *False* and the colour 1 as the truth value *True*. Without loss of generality we may assume that $\sigma(u) = 2$, and consequently, all vertices in V_1 take their colours in $\{0,1\}$. Since each pair of vertices $\{x_i, \neg x_i\}$ form a 2-clique, we may consider the truth assignment σ_ϕ induced by $\sigma|_{V_1}$. By considering the edges between V_1 and V_3, and the fact that for each clause C_i the vertices C_{i1}, C_{i2} and C_{i3} form a 3-clique, it is easy to see that σ_ϕ is a valid truth assignment for ϕ that also satisfies the **NAE** condition.

On the other hand, assume that ϕ is a satisfiable instance of **NAESAT**, and we shall show that G_ϕ admits a connected 3-colouring. For this, let σ_ϕ be a valid truth assignment for ϕ and consider the 3-colouring σ for G_ϕ defined as follows:

- $\sigma(u)$=2.
- $\forall\, v \in V_1 \quad \sigma(v) = \sigma_\phi(v)$.
- Since σ_ϕ satisfies the **NAE** condition, without loss of generality, we let C_{i1} be a *False* literal, and C_{i2} be a *True* literal in the ith clause C_i, and we define,

$$\sigma(C_{i1}) = 0, \;\; \sigma(C_{i2}) = 1 \text{ and } \sigma(C_{i3}) = 2.$$

- If $\sigma(C_{ij}) = 2$, we let $\sigma(C^1_{ij})$ be the truth value, and $\sigma(C^2_{ij})$ be the negation of the truth value, of the jth literal in C_i. Otherwise, if $\sigma(C_{ij}) \neq 2$, we let $\sigma(C^1_{ij})$ be the negation of the truth value of the jth literal in C_i and $\sigma(C^2_{ij}) = 2$.
- For each $1 \leq i \leq t-1$, if $\sigma_\phi(x_i) = \sigma_\phi(x_{i+1})$ then we let $\sigma(x_{i1})$ be the truth value of $\neg x_i$ and $\sigma(x_{i2}) = 2$. Otherwise, we let $\sigma(x_{i1}) = 2$ and we let $\sigma(x_{i2})$ be the truth value of $\neg x_i$.

It is easy to check that σ is a proper 3-colouring of G_ϕ. In the rest of the proof we show that σ is a connected 3-colouring of G_ϕ.

- *We show that the graph G_{01} induced on the set of vertices with colours in $\{0,1\}$ is connected.*

First, note that the subgraph induced on $V(G_{01}) \cap (V_1 \cup V_2)$ is connected. Also, each vertex of type C_{i1}, C_{i2} or C^2_{ij} is connected to one of the vertices in V_1 by the definition of G_ϕ.
On the other hand, a vertex of type C^1_{ij} is adjacent to both of the vertices C_{ij} and C^2_{ij}, while we know that in any case exactly one of these vertices take its colour from $\{0,1\}$ in the colouring σ.

- *We show that the graph* G_{02} *induced on the set of vertices with colours in* $\{0,2\}$ *is connected (a similar proof holds for the case* $\{1,2\}$*).*
  By the colouring procedure, it is easy to check that any vertex $v \in V_2$ with $\sigma(v) = 2$ is adjacent to two vertices in V_1 with different colours in $\{0,1\}$. Hence, the subgraph induced on $V(G_{02}) \cap (\{u\} \cup V_1 \cup V_2)$ is connected.
  Also, by the colouring procedure and the definition of G_ϕ, since $\sigma(C_{i1}) = 0$ we know that $\sigma(C^2_{i1}) = 2$ and the vertex adjacent to C^2_{i1} in V_1 has the colour 0.
  On the other hand, the 2-clique formed by C^1_{i2} and C^2_{i2} is connected to u through C^1_{i2}. Similarly, since C_{i3} is connected to both C_{i2} C_{i1}, the vertex in $\{C^1_{i3}, C^2_{i3}\}$ whose colour is 0 is connected to the rest of this subgraph.

■

As a corollary we have,

Theorem 4. *For any* $n \geq 3$ **CONnCOL** *is NP-complete.*

Proof. Construct a new graph G'_ϕ by considering the same graph G_ϕ of Theorem 3 and adding a $(n-3)$-clique that is joined to G_ϕ, i.e. each vertex of this clique is adjacent to all vertices of G_ϕ. It is easy to see that G'_ϕ introduces the necessary polynomial reduction. ■

We naturally formulate the following problem in connection with the main result of [7].

Problem 3. *Is the following problem NP-complete ?*

Problem:	**CONHCOL**
Constant:	A non-bipartite simple graph H.
Given:	A graph G.
Question:	Does there exist a connected homomorphism $\sigma \in \mathrm{Hom}^c(G,H)$?

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References

[1] S. FIORINI, *On the chromatic index of a graph, III: Uniquely edge-colourable graphs*, Quart. J. Math. Oxford, **26** (1975), 129–140.

[2] T. FEDER, P. HELL AND B. MOHAR, *Acyclic homomorphisms and circular colorings of digraphs*, SIAM J. Discrete Math., **17** (2003), 161–169.

[3] D. L. GREENWELL AND H. V. KRONK, *Uniquely line colorable graphs*, Canad. Math. Bull., **16** (1973), 525–529.

[4] G. HAHN AND C. TARDIF, *Graph homomorphisms: structure and symmetry*, in Graph Symmetry, G. Hahn and G. Sabidussi, eds., no. 497 in NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer, Dordrecht, (1997), 107–167.

[5] P. HELL, *Algorithmic aspects of graph homomorphisms*, in Surveys in Combinatorics (2003), C.D. Wensley, ed., no. 307 in London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003, 239–276.

[6] P. HELL AND J. NEŠETŘIL, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, (2004).

[7] P. HELL AND J. NEŠETŘIL, *On the complexity of H-coloring*, J. Combin. Theory (B), **48** (1990), 92–110.

[8] A.J.W. HILTON AND E.C.MILNER, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford (2), **18** (1967), 369-384.

[9] C. H. PAPADIMITRIOU, *Computational Complexity*, Addison-Wesley Publishing Company Inc., (1994).

[10] M. TRUSZCZYŃSKI, *Some results on uniquely colourable graphs*, Colloquia Math. Soc. János Bolyai, **37** (1981), 733–746.

[11] A. VINCE, *Star chromatic number*, J. Graph Theory, **12**(1988), 551-559.

[12] S. J. XU, *On the size of uniquely colorable graphs*, J. Comb. Theory B, 50 (1990), 319–320.

[13] X. ZHU, *Circular chromatic number, a survey*, Disc. Math. **229** (2001), 371-410.