On Some Metric Properties of the Sierpiński Graphs $S(n, k)$

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Abstract

Sierpiński graphs $S(n, k)$, $n, k \in \mathbb{N}$, can be interpreted as graphs of a variant of the Tower of Hanoi with $k \geq 3$ pegs and $n \geq 1$ discs. In particular, it has been proved that for $k = 3$ the graphs $S(n, 3)$ are isomorphic to the Hanoi graphs H_3^n. In this paper the chromatic number, the diameter, the eccentricity of a vertex, the radius and the centre of $S(n, k)$ will be determined. Moreover, an important invariant and a number-theoretical characterization of $S(n, k)$ will be derived. By means of these results the complexity of Problem 1, that is the complexity to get from an arbitrary vertex $v \in S(n, k)$ to the nearest and to the most distant extreme vertex, will be given. For the Hanoi graphs H_3^n some of these results are new.

Key words: Sierpiński Graphs, Tower of Hanoi, Graph Diameter, Graph Radius, Graph Centre, Chromatic Number, Linear Diophantine Equation

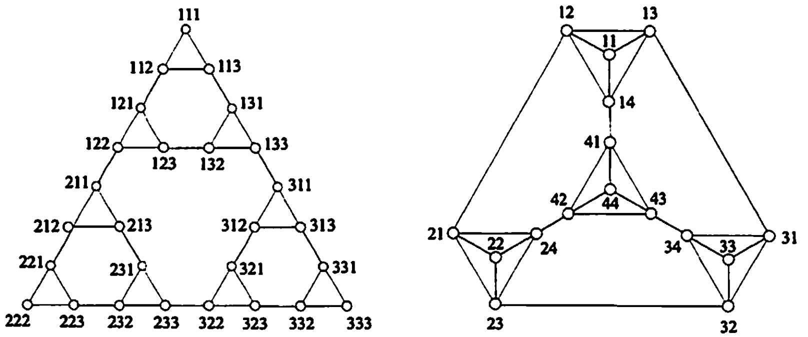
AMS subject classification (2000): 05C12, 05C15, 11D04

1 Introduction

Graphs $S(n, k)$, $n, k \in \mathbb{N}$, have been introduced for the first time in [8] as a two parametric generalization of the Hanoi graphs and named *Sierpiński graphs* in [9], since their introduction was motivated by topological studies of certain generalizations of the Sierpiński gasket [11, 12, 13]. The graphs $S(n, k)$, $n, k \in \mathbb{N}$, are defined as follows: The vertex set is $\{1, 2, \dots, k\}^n$ and two different vertices $u = u_1 u_2 \dots u_n := (u_1, u_2, \dots, u_n)$ and $v = v_1 v_2 \dots v_n := (v_1, v_2, \dots, v_n)$ are adjacent if and only if there exists an index $h \in \{1, 2, \dots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \dots, h-1$;
(ii) $u_h \neq v_h$;
(iii) $u_t = v_h$ and $v_t = u_h$, for $t = h+1, \dots, n$.

As an example the Sierpiński graphs $S(3,3)$ and $S(2,4)$ together with their corresponding vertex labelings, are shown in Fig. 1.



S(3,3) S(2,4)

Figure 1: Sierpiński graphs $S(3,3)$ and $S(2,4)$

For any $n \in \mathbb{N}$, $S(n,1)$ is isomorphic to the one vertex graph K_1 and $S(n,2)$ is isomorphic to the path graph P_{2^n} on 2^n vertices. As pointed out by Hinz [6] $S(n,2)$ is also isomorphic to the state graph of the *Chinese rings* puzzle with n rings (also known by the French word *baguenaudier*). Further, it has been proved that, for any $n \in \mathbb{N}$, $S(n,3)$ is isomorphic to the Hanoi graph H_3^n with three pegs and $n \in \mathbb{N}$ discs (cf. [8, Theorem 2]) and, more generally, that $S(n,k)$ is isomorphic to the graph of a variant of the Tower of Hanoi called *switching Tower of Hanoi* (cf. [8, Theorem 1]). Finally, for any $k \in \mathbb{N}$, $S(1,k)$ is the complete graph K_k on k vertices. (Alternatively, the edge-sets of $S(n,k)$, $k \in \mathbb{N}$, can be defined recursively as follows:

$$E(S(1,k)) := \Big\{\{i,j\} \mid i,j \in \{1,2,\dots,k\},\ i \neq j\Big\},$$

and for all $n \in \mathbb{N}$

$$E(S(n+1,k)) := \Big\{\{iu, iv\} \mid i \in \{1,2,\dots,k\},\ u, v \,\text{adjacent in}\, S(n,k)\Big\}$$
$$\cup \Big\{\{ij\dots j, ji\dots i\} \subset \{1,2,\dots,k\}^{n+1} \mid i,j \in \{1,2,\dots,k\}, i \neq j\Big\}.)$$

A vertex of $S(n,k)$ of the form $ii\dots i$, $i \in \{1,2,\dots,k\}$, will be called an *extreme vertex* of $S(n,k)$, the other vertices will be called *inner*. The degree of the extreme vertices is $k-1$, while the degree of the inner vertices is k. In $S(n,k)$ there are exactly k extreme vertices (in $S(n,3)$ they correspond to the perfect states of the Tower of Hanoi with n discs) and, since $|S(n,k)| = |\{1,2,\ldots,k\}^n| = k^n$, $k^n - k$ inner vertices. Therefore, $S(n,k)$ has exactly $\frac{1}{2}\big(k(k-1) + (k^n - k)k\big) = \frac{k}{2}(k^n - 1)$ edges. (Note that since $S(n+1,k)$ consists of k copies of $S(n,k)$ connected by only an edge each two copies, that is the k copies of $S(n,k)$ are connected by $\binom{k}{2}$ edges, and since $S(1,k) \cong K_k$ the number of edges $|E(S(n,k))|$ of $S(n,k)$ can also be derived by the recurrence relation $|E(S(n+1,k))| = k|E(S(n,k))| + \binom{k}{2}$, $n \geq 1$, and $|E(S(1,k))| = \binom{k}{2}$ with the solution $|E(S(n,k))| = \frac{k}{2}(k^n - 1)$ for all $n, k \in \mathbb{N}$.)

Some properties of the graphs $S(n,k)$ have been established in [8]. So it has been proved that there are at most two shortest paths between any two vertices of $S(n,k)$ [8, Theorem 6] and a formula for the distance of two arbitrary vertices of $S(n,k)$ has been given ([8, Theorem 5]). Further, it has also been shown that for any $n \geq 1$ and any $k \geq 3$ the graphs $S(n,k)$ are Hamiltonian [8, Proposition 3] and in a recent paper [10] Klavžar and Mohar have studied the crossing numbers of the Sierpiński graphs $S(n,k)$ and their regularizations $S^{+}(n,k)$ and $S^{++}(n,k)$. Finally, since the Sierpiński graphs $S(n,k)$ are basically iterated complete graphs on k vertices with n iterations they have been used to create a perfect one error correcting code [2, 7].

An interesting conclusion due to an observation by Danielle Arett (cf. [1] and [3, Theorem 4]) in connection with the Hanoi graphs H_p^n with $p \geq 3$ pegs and $n \geq 1$ discs can be drawn from the vertex labeling, namely that for two adjacent vertices $u = u_1u_2\ldots u_n$ and $v = v_1v_2\ldots v_n$ it is $u_n \neq v_n$. Defining the function

$$c\colon \{1,2,\ldots,k\}^n \longrightarrow \{1,2,\ldots,k\}, \quad u_1u_2\ldots u_n \longmapsto u_n,$$

we obtain a vertex colouring for $S(n,k)$. This means, that the chromatic number of $S(n,k)$ is equal to k, since the complete graph $S(1,k) \cong K_k$ is a subgraph of $S(n,k)$.

In this paper we study several additional metric properties of the graphs $S(n,k)$. First of all we will determine the diameter, the eccentricity of a vertex, the radius and the centre of $S(n,k)$. Further, we will derive an important invariant and a number-theoretical characterization of the graphs $S(n,k)$. In particular, for $k=3$, we will obtain some old and new properties of the Hanoi graphs H_3^n. Finally, a finer analysis of the complexity of Problem 1 will be given. (As for the Tower of Hanoi, Problem 1 is to get in the least possible number of moves from an arbitrary vertex to an extreme vertex, while Problem 0 is to get in the least possible number of moves from an extreme vertex to another extreme vertex.)

2 Characterization of $S(n,k)$

Before proceeding let's recall some definitions from graph theory. Let G be a simple connected graph, then for all $u, v \in V(G)$ the *distance* $d_G(u,v)$ between u and v denotes the minimum number of edges for paths joining u and v. For a fixed vertex $v \in V(G)$ the integer

$$e_G(v) := \max\{d_G(u,v) \mid u \in V(G)\} \tag{2.1}$$

measures the distance from v to the vertex (or the vertices) most remote from v and is called the *eccentricity* of the vertex v. The integer

$$rad(G) := \min\{e_G(v) \mid v \in V(G)\} \tag{2.2}$$

is called the *radius* of the graph G and

$$diam(G) := \max\{e_G(v) \mid v \in V(G)\} = \max\{d_G(u,v) \mid u,v \in V(G)\} \tag{2.3}$$

is called the *diameter* of the graph G. Further, a vertex is called a *central vertex* of G if

$$e_G(v) = rad(G) \tag{2.4}$$

and the subgraph induced by all central vertices

$$C(G) := \{v \in V(G) \mid e_G(v) = rad(G)\} \tag{2.5}$$

is called the *centre* of G. In the sequel we will apply repeatedly the following result proved in [8, Lemma 4]. (Note that from now on, $v \in S(n,k)$ stands for $v \in V(S(n,k))$, $d(u,v)$ for $d_{S(n,k)}(u,v)$ and $e_{n,k}(v)$ for $e_{S(n,k)}(v)$.)

Lemma 2.1 *Let $v = v_1v_2\dots v_n \in S(n,k)$, $n,k \in \mathbb{N}$ and $i \in \{1,2,\dots,k\}$, then*

$$d(v, ii\dots i) = (\rho_{v_1,i}\rho_{v_2,i}\cdots\rho_{v_n,i})_2, \tag{2.6}$$

where

$$\rho_{i,j} := \begin{cases} 1, & i \neq j, \\ 0, & i = j, \end{cases}$$

and the right-hand side is a binary number, rhos representing its digits.

Using Lemma 2.1 we can determine immediately the diameter of $S(n,k)$.

Corollary 2.2 *Let $v := lv_2\dots v_n$, $w := lw_2\dots w_n$ be vertices of $S(n,k)$ and $l \in \{1,2,\dots,k\}$. Then*

(i) $d(lv_2\dots v_n, lw_2\dots w_n) = d(v_2\dots v_n, w_2\dots w_n)$, $n > 1$.

(ii)

$$\forall n \in \mathbb{N}: \quad diam(S(n,k)) = 2^n - 1, \quad k > 1. \tag{2.7}$$

Proof. Obviously, $d(lv_2 \ldots v_n, lw_2 \ldots w_n) \leq d(v_2 \ldots v_n, w_2 \ldots w_n)$, since a path from $v_2 \ldots v_n$ to $w_2 \ldots w_n$ is also a path from $lv_2 \ldots v_n$ to $lw_2 \ldots w_n$.

By repeated use of this inequality the formula (2.7) can be proved by induction on n. (Note that $diam(S(n,1)) = 0$, $n \geq 1$, since $S(n,1)$ is isomorphic to the one vertex graph K_1.)

The case $n = 1$, $k > 1$, is obvious, since $S(1,k)$ is isomorphic to the complete graph K_k with $diam(K_k) = 1$.

Let (2.7) be true for $n \geq 1$ and let $v, w \in S(n+1,k)$. If $v_1 = w_1$, i.e. if v and w lie in the same copy of $S(n,k)$, we obtain by induction assumption $d(v,w) \leq d(v_2 \ldots v_{n+1}, w_2 \ldots w_{n+1}) \leq 2^n - 1$ and, therefore, $d(v,w) \leq 2^{n+1} - 1$. If $v_1 \neq w_1$, i.e. if v and w lie in two different copies of $S(n,k)$, we obtain by induction assumption

$$\begin{aligned} d(v,w) &\leq d(v, v_1w_1 \ldots w_1) + 1 + d(w_1v_1 \ldots v_1, w) \\ &\leq d(v_2 \ldots v_{n+1}, w_1 \ldots w_1) + 1 + d(v_1 \ldots v_1, w_2 \ldots w_{n+1}) \\ &\leq (2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1. \end{aligned}$$

Hence, $diam(S(n,k)) \leq 2^n - 1$, for all $n \in \mathbb{N}$ and $k > 1$. Choosing for v and w two different extreme vertices, we obtain by Lemma 2.1 that the upper bound $2^n - 1$ will be attained and this proves (2.7).

It remains to prove part "$\geq$" of (i). This is a consequence of the fact, that any path from $lv_2 \ldots v_n$ to $lw_2 \ldots w_n$ which is not completely inside the copy of $S(n-1,k)$ consisting of all vertices beginning with l, $l \in \{1, 2, \ldots, k\}$, contains at least a subpath between two different extreme vertices of a copy of $S(n-1,k)$. By Lemma 2.1 such a subpath has a length equal to $2^{n-1} - 1$ and, consequently, $d(lv_2 \ldots v_n, lw_2 \ldots w_n) \geq 2^{n-1} - 1 \geq d(v_2 \ldots v_n, w_2 \ldots w_n)$, where the last inequality follows by (2.7). □

We shall now determine the eccentricity of an arbitrary $v \in S(n,k)$. It turns out that it is sufficient to consider only the extreme vertices.

Lemma 2.3 *Let $n, k \in \mathbb{N}$ and $v \in S(n,k)$, then*

$$e_{n,k}(v) = \max\{d(v, ii \ldots i) \mid i \in \{1, 2, \ldots, k\}\}, \tag{2.8}$$

where $ii \ldots i$, $i \in \{1, 2, \ldots, k\}$, are the k extreme vertices of $S(n,k)$.

Proof. Let $v \in S(n,k)$, then we have to show that for all $w \in S(n,k)$ there is an extreme vertex $ii \ldots i \in S(n,k)$ with $d(v,w) \leq d(v, ii \ldots i)$. This will be proved by induction on n.

The case $n = 1$ is clear, since $S(1,k)$ is isomorphic to the complete graph K_k with $e_{1,k}(v) = 1$ for each $v \in S(1,k)$.

Now let the assertion be true for $n \geq 1$ and let $v, w \in S(n+1,k)$. Then either v and w lie in the same copy of $S(n,k)$, i.e. $v_1 = w_1$, or v and w lie

in two different copies of $S(n,k)$, i.e. $v_1 \neq w_1$.
If $v_1 = w_1$, then there is by induction assumption an extreme vertex $ii\dots i \in S(n,k)$, $i \in \{1,2,\dots,k\}$, such that by Corollary 2.2 we have $d(v,w) = d(v_2 \dots v_{n+1}, w_2 \dots w_{n+1}) \leq d(v_2 \dots v_{n+1}, ii\dots i) \leq d(v, ii\dots i)$, where the last inequality follows by Lemma 2.1 and the last $ii\dots i$ is an extreme vertex of $S(n+1,k)$.
If $v_1 \neq w_1$, then it is $d(v,w) \leq d(v, v_1 w_1 \dots w_1) + 1 + d(w_1 v_1 \dots v_1, w)$. By Corollary 2.2 and by induction assumption we obtain

$$\begin{aligned} d(w_1 v_1 \dots v_1, w) &= d(v_1 \dots v_1, w_2 \dots w_{n+1}) \leq d(v_1 \dots v_1, w_1 \dots w_1) \\ &= d(w_1 v_1 \dots v_1, w_1 w_1 \dots w_1) \end{aligned}$$

and, therefore, $d(v,w) \leq d(v, v_1 w_1 \dots w_1) + 1 + d(w_1 v_1 \dots v_1, w_1 w_1 \dots w_1)$. The right-hand side is equal to $d(v, w_1 w_1 \dots w_1)$ (see [8, Lemma 4] and its proof) and this completes the proof. □

Note that the diameter is the same for all $k > 1$ and that it will be attained not only by two extreme vertices, but also by an extreme vertex $ii\dots i$, $i \in \{1,2,\dots,k\}$ and the vertices $v = v_1 v_2 \dots v_n$ with $v_j \neq i$, $j \in \{1,2,\dots,n\}$. The next corollary shows that there are in all $(k-1)^n$ vertices with $d(v, ii\dots i) = 2^n - 1$.

Corollary 2.4 *Let $n,k \in \mathbb{N}$, $i \in \{1,2,\dots,k\}$ and $l \in \{0,1,\dots,2^n-1\}$, then*

$$|\{v \in S(n,k) \mid d(v, ii\dots i) = l\}| = (k-1)^{\beta(l)} \tag{2.9}$$

and

$$\sum_{l=0}^{2^n-1} (k-1)^{\beta(l)} = k^n, \tag{2.10}$$

where $\beta(l)$ is the number of non-zero binary digits of l.

Proof. By (2.7) $d(v, ii\dots i) = l \in \{0,1,\dots,2^n-1\}$ and by Lemma 2.1 $d(v, ii\dots i) = (\rho_{v_1,i}\rho_{v_2,i}\dots\rho_{v_n,i})_2$. Hence, we have exactly $k-1$ possibilities for each digit 1 in the binary representation of l. The total number is therefore $(k-1)^{\beta(l)}$, thus proving (2.9). Statement (2.10) now follows from (2.9) by summing up all l from 0 to $2^n - 1$ and noting that $S(n,k)$ has exactly k^n vertices. □

Property (2.9) is well-known for the Tower of Hanoi ($k = 3$) (cf. [5, Proposition 5]) and goes back to Glaisher [4], while property (2.10) shows that $(k-1)^{\beta(l)}/k^n$, $l \in \{0,1,\dots,2^n-1\}$, is a discrete (probability) distribution, which may be called in his honour *Glaisher distribution*.

An interesting invariant of $S(n,k)$ is a generalization of the invariant of the Hanoi graphs $H_3^n \cong S(n,3)$, namely

$$\forall v \in H_3^n: \quad d(v,11\ldots1)+d(v,22\ldots2)+d(v,33\ldots3)=2\cdot(2^n-1),$$

where $11\ldots1$, $22\ldots2$ and $33\ldots3$ are the three perfect states of H_3^n [14, equation (2.1)].

Proposition 2.5 *Let $n,k \in \mathbb{N}$, $v \in S(n,k)$ and $11\ldots1$, $22\ldots2$, $\ldots$, $kk\ldots k$ be the extreme vertices, then*

$$\sum_{i=1}^{k} d(v,ii\ldots i)=(k-1)\cdot(2^n-1). \tag{2.11}$$

Proof. Since for each $j \in \{1,2,\ldots,n\}$ it is $v_j \in \{1,2,\ldots,k\}$, only one of the k digits $\rho_{v_j,1}$, $\rho_{v_j,2}$, $\ldots$, $\rho_{v_j,k}$ is 0 and the others $k-1$ are equal to 1. Hence, we have to sum $k-1$ times the binary number $(11\ldots1)_2=2^n-1$ and this is exactly the claimed assertion. □

An immediate consequence of (2.11) is the determination of the average complexity of Problem 1, i.e. the determination of the *mean vertex deviation* of an extreme vertex defined for all $i \in \{1,2,\ldots,k\}$

$$\mu_{n,k}(ii\ldots i):=\frac{1}{|S(n,k)|}\sum_{v\in S(n,k)} d(v,ii\ldots i).$$

Corollary 2.6 *Let $n,k \in \mathbb{N}$ and $ii\ldots i$, $i \in \{1,2,\ldots,k\}$, be an extreme vertex of $S(n,k)$, then*

$$\mu_{n,k}(ii\ldots i)=\frac{k-1}{k}\cdot(2^n-1) \tag{2.12}$$

with the standard deviation

$$\sigma_{n,k}(ii\ldots i)=\frac{1}{k}\sqrt{\frac{k-1}{3}\cdot(4^n-1)}. \tag{2.13}$$

Proof. By Proposition 2.5 we obtain by summing up all $v \in S(n,k)$ and noting that by symmetry the k sums are all equal

$$\begin{aligned} k\sum_{v\in S(n,k)} d(v,ii\ldots i) &= \sum_{v\in S(n,k)}\left(\sum_{j=1}^{k} d(v,jj\ldots j)\right)\\ &= \sum_{v\in S(n,k)}(k-1)(2^n-1)=(k-1)\cdot(2^n-1)\cdot k^n \end{aligned}$$

and hence

$$\mu_{n,k}(ii\dots i) = \frac{1}{k^n} \sum_{v \in S(n,k)} d(v, ii\dots i) = \frac{k-1}{k} \cdot (2^n - 1).$$

To prove (2.13) we note that the standard deviation is by definition

$$\sigma_{n,k}(ii\dots i) = \sqrt{V_{n,k}(ii\dots i)}, \tag{2.14}$$

where

$$\begin{aligned} V_{n,k}(ii\dots i) &:= \frac{1}{|S(n,k)|} \sum_{v \in S(n,k)} \left(d(v, ii\dots i) - \mu_{n,k}(ii\dots i)\right)^2 \\ &= \frac{1}{k^n} \sum_{v \in S(n,k)} d^2(v, ii\dots i) - \left(\mu_{n,k}(ii\dots i)\right)^2 \end{aligned}$$

is the variance of the distance to an extreme vertex. Hence, it remains to determine the *mean square vertex deviation* $\frac{1}{k^n} \sum_{v \in S(n,k)} d^2(v, ii\dots i)$ of an extreme vertex.

Let $a_k(n) := \sum_{v \in S(n,k)} d^2(v, ii\dots i)$, then by (2.12)

$$\begin{aligned} a_k(n+1) &= \sum_{v \in S(n+1,k)} d^2(v, ii\dots i) \\ &= a_k(n) + (k-1) \cdot \sum_{v \in S(n,k)} \left(d(v, ii\dots i) + 2^n\right)^2 \\ &= a_k(n) + (k-1) \cdot \Big(\sum_{v \in S(n,k)} d^2(v, ii\dots i) + 2 \cdot 2^n \cdot \\ &\quad \sum_{v \in S(n,k)} d(v, ii\dots i) + \sum_{v \in S(n,k)} 4^n \Big) \\ &= k a_k(n) + 2^{n+1} \cdot (2^n - 1) \cdot (k-1)^2 \cdot k^{n-1} + 4^n \cdot (k-1) \cdot k^n \end{aligned}$$

with the initial value $a_k(1) = k - 1$, since $S(1,k)$ is isomorphic to K_k. The solution of this recurrence relation is given for all $n, k \in \mathbb{N}$ by

$$\begin{aligned} a_k(n) &= \sum_{i=0}^{n-1} k^i \Big\{ 2^{n-i} \cdot (2^{n-1-i} - 1) \cdot (k-1)^2 \cdot k^{n-2-i} + \\ &\quad 4^{n-1-i} \cdot (k-1) \cdot k^{n-1-i} \Big\} \\ &= 2 \cdot (k-1)^2 \cdot k^{n-2} \cdot \sum_{i=0}^{n-1} 2^{n-1-i} \cdot (2^{n-1-i} - 1) + \\ &\quad (k-1) \cdot k^{n-1} \cdot \sum_{i=0}^{n-1} 4^{n-1-i}, \end{aligned}$$

that is,

$$a_k(n) = \frac{k-1}{3} \cdot k^{n-2} \cdot \Big\{(3k-2) \cdot 4^n - 6 \cdot (k-1) \cdot 2^n + 3k - 4\Big\} \quad (2.15)$$

and therefore by (2.12)

$$V_{n,k}(ii\dots i) = \frac{a_k(n)}{k^n} - \Big\{\frac{k-1}{k} \cdot (2^n-1)\Big\}^2 = \frac{k-1}{3k^2} \cdot (4^n - 1).$$

Finally, by (2.14) we obtain (2.13). □

Note that in view of Corollary 2.4 formula (2.12) also gives the first moment

$$\sum_{l=0}^{2^n-1} l \cdot \frac{(k-1)^{\beta(l)}}{k^n} = \frac{k-1}{k} \cdot (2^n - 1), \quad (2.16)$$

whereas (2.15) gives the second moment

$$\sum_{l=0}^{2^n-1} l^2 \cdot \frac{(k-1)^{\beta(l)}}{k^n} = \frac{k-1}{3k^2} \cdot \{(3k-2) \cdot 4^n - 6 \cdot (k-1) \cdot 2^n + 3k - 4\} \quad (2.17)$$

of the Glaisher distribution.

For $k = 3$ formula (2.12) gives the average length of shortest paths from an arbitrary regular state to a perfect state (cf. [5, Corollary 1]) of the Hanoi graphs H_3^n and (2.13) gives its standard deviation, a result that has been given for the first time by Scarioni and Speranza [15].

By Proposition 2.5 and Corollary 2.2 we see that for each $v \in S(n,k)$ the k-tuple $(d(v, 11\dots1), d(v, 22\dots2), \dots, d(v, kk\dots k))$ is a solution of the linear Diophantine equation

$$x_1 + x_2 + \cdots + x_k = (k-1) \cdot (2^n - 1)$$

in $k \geq 1$ variables $x_1, x_2, \dots, x_k \in \{0, 1, \dots, 2^n - 1\}$. The converse is not true, since for instance for $k = 3$ and $n = 3$ the triple $(2, 6, 6)$ is a solution of the above equation, but there is no $v \in S(3,3)$ such that $d(v, i_1 i_1 \dots i_1) = 2$, $d(v, i_2 i_2 \dots i_2) = 6$, $d(v, i_3 i_3 \dots i_3) = 6$, $i_1 i_2 i_3$ a permutation of $\{1, 2, 3\}$, as you can see in Fig. 1. Moreover, the above equation has the same number of solutions as

$$t_1 + t_2 + \cdots + t_k = 2^n - 1,$$

with $t_1, t_2, \dots, t_k \in \{0, 1, \dots, 2^n - 1\}$, (consider the transformation $t_i := 2^n - 1 - x_i$, $i \in \{1, 2, \dots, k\}$), namely $\binom{2^n-1+k-1}{k-1} = \binom{2^n+k-2}{k-1} > k^n = |S(n,k)|$ for all $k \geq 3$ and $n \geq 2$.

A necessary and sufficient condition for the case $k = 3$, i.e. for the Tower of Hanoi with 3 pegs and n discs, has been communicated some years ago to the author by D. Singmaster [16]. Its formulation can be found in [14, Theorem A8].

Theorem 2.7 *Let $n, k \in \mathbb{N}$ and $x_i := d(v, ii\dots i) = (\rho_{v_1,i}\rho_{v_2,i}\dots\rho_{v_n,i})_2$, $i \in \{1,2,\dots,k\}$, $v \in S(n,k)$. Then $(x_1, x_2, \dots, x_k) \in \{0,1,\dots,2^n-1\}^k$ is a solution of*

$$x_1 + x_2 + \cdots + x_k = (k-1)\cdot(2^n-1) \tag{2.18}$$

and

$$\rho_{v_j,1} + \rho_{v_j,2} + \cdots + \rho_{v_j,k} = k-1 \tag{2.19}$$

for all $j \in \{1,2,\dots,n\}$.

Conversely, let $(x_1, x_2, \dots, x_k) \in \{0,1,2,\dots,2^n-1\}^k$ be a solution of (2.18) with the binary expressions $x_1 = (x_{1,1}x_{2,1}\dots x_{k,1})_2$, …, $x_k = (x_{1,k}x_{2,k}\dots x_{k,k})_2$ satisfying the condition (2.19), i.e.

$$x_{j,1} + x_{j,2} + \cdots + x_{j,k} = k-1$$

for all $j \in \{1,2,\dots,n\}$. Then there exists exactly one vertex $v \in S(n,k)$ with $d(v,11\dots1) = x_1, d(v,22\dots2) = x_2, \dots, d(v,kk\dots k) = x_k$. This v is given for each $j \in \{1,2,\dots,n\}$ by $v_j := i$ if $x_{j,i} = 0$, $i \in \{1,2,\dots,k\}$. □

Proof. Let $v \in S(n,k)$ and $x_1 = d(v,11\dots1), x_2 = d(v,22\dots2), \dots, x_k = d(v,kk\dots k)$, then, by Proposition 2.5, $(x_1, x_2, \dots, x_k)$ is a solution of (2.18) and since $v_j \in \{1,2,\dots,k\}$ exactly $k-1$ of the values $\rho_{v_j,1}, \rho_{v_j,2}, \dots, \rho_{v_j,k}$ are equal to 1 for all $j \in \{1,2,\dots,n\}$. This means that equation (2.19) is fullfilled.

Conversely, let $(x_1, x_2, \dots, x_k) \in \{0,1,\dots,2^n-1\}^k$ be a solution of (2.18) such that writing out the binary expressions $x_1 = (x_{1,1}x_{2,1}\dots x_{k,1})_2$, $x_2 = (x_{1,2}x_{2,2}\dots x_{k,2})_2$, …, $x_k = (x_{1,k}x_{2,k}\dots x_{k,k})_2$ we have $x_{j,1} + x_{j,2} + \cdots + x_{j,k} = k-1$ for all $j \in \{1,2,\dots,n\}$. This means that exactly one of the digits is equal to 0. Define now for each $j \in \{1,2,\dots,n\}$ $v_j := i$ if $x_{j,i} = 0$, $i \in \{1,2,\dots,k\}$. In this way we obtain a unique $v = v_1v_2\dots v_k \in S(n,k)$ such that $d(v,11\dots1) = x_1, d(v,22\dots2) = x_2, \dots, d(v,kk\dots k) = x_k$. □

Corollary 2.8 *Let $n, k \in \mathbb{N}$, then*

(i) Exactly one value of $x_1, x_2, \dots, x_k$ in Theorem 2.7 is even.

(ii) Two of the values $x_1, x_2, \dots, x_k$ are equal if and only if this value is $(11\dots1)_2$.

(iii) $k-1$ of the values $x_1, x_2, \dots, x_k$ are equal if and only if the remaining one is 0, i.e. the vertex $v \in S(n,k)$ is equidistant from $k-1$ extreme vertices if and only if v is itself an extreme vertex.

(iv) There is no vertex $v \in S(n,k)$ equidistant to all the extreme vertices, i.e. such that $d(v,11\dots1) = d(v,22\dots2) = \cdots = d(v,kk\dots k)$.

(v) Let $t_i := 2^n - 1 - x_i$, $i \in \{1,2,\dots,k\}$, then

$$2^{k(k-1)/2} \mid \prod_{i=1}^{k} t_i. \tag{2.20}$$

Proof. (i) - (iv) follow immediately from (2.18) and (2.19). To prove (v) let $t_i := 2^n - 1 - x_i$, $i \in \{1, 2, \ldots, k\}$, be the binary one's complement of x_i, then by Theorem 2.7 $k-1$ of the k digits $1 - \rho_{v_j,1}, 1 - \rho_{v_j,2}, \ldots, 1 - \rho_{v_j,k}$, $j \in \{1, 2, \ldots, n\}$, are equal to 0. Hence, in the binary representation of the k numbers $t_1, t_2, \ldots, t_k$ there are in all n digits 1. For $n < k$ there is, by Theorem 2.7, at least an integer $i \in \{1, 2, \ldots, k\}$ with $t_i = 0$ and (2.20) is true. For $n \geq k$ the distribution of the first k digits 1 leading to the least power of 2 divisor of $\prod_{i=1}^{k} t_i$ is obviously $(\ldots\ldots 1)_2, (\ldots\ldots 10)_2, (\ldots\ldots 100)_2, \ldots, (\ldots 100\ldots 0)_2$, where the last binary number has $k-1$ final digits 0. This means that $2^1 \cdot 2^2 \cdots 2^{k-1} = 2^{1+2+\cdots+(k-1)} = 2^{k(k-1)/2}$ is, as asserted, a divisor of $\prod_{i=1}^{k} t_i$. □

Note that from part (i) of the above corollary one can only draw the conclusion that exactly $k-1$ of the values $t_1, t_2, \ldots, t_k$ are even, that is 2^{k-1} is a divisor of $\prod_{i=1}^{k} t_i$. Assertion (2.20) is, indeed, a stronger property.

3 The Radius and the Centre of $S(n,k)$

In this section we shall determine the radius and the centre of $S(n,k)$.

Theorem 3.1 *Let $n, k \in \mathbb{N}$ and for $n \geq k$:*

$$\tilde{C}(n,k) := \{z \in S(n,k) \mid z = z_1 \ldots z_{k-1} z_k \ldots z_k, \{z_1, \ldots, z_k\} = \{1, \ldots, k\}\}.$$

Then

$$rad\,(S(n,k)) = \lfloor 2^{n-k+1} \cdot (2^{k-1} - 1) \rfloor = \begin{cases} 2^n - 1, & n < k \\ 2^{n-k+1} \cdot (2^{k-1} - 1), & n \geq k \end{cases} \tag{3.1}$$

$$C\,(S(n,k)) = \begin{cases} S(n,k), & n < k \\ \tilde{C}(n,k), & n \geq k \end{cases} \tag{3.2}$$

The centre of $S(n,k)$ has

$$|C\,(S(n,k))| = \begin{cases} k^n, & n < k \\ k!, & n \geq k \end{cases} \tag{3.3}$$

vertices and

$$|E(C\,(S(n,k)))| = \begin{cases} \frac{k}{2} \cdot (k^n - 1), & n < k \quad or \quad k = 1 \\ k!/2, & n \geq k \end{cases} \tag{3.4}$$

edges. In particular, for $n \geq k > 1$, the centre of $S(n,k)$ is a 1-regular graph consisting of $k!/2$ disconnected edges.

Proof. The case $k = 1$ is clear, since $S(n,1)$ is for any $n \in \mathbb{N}$ isomorphic to the complete graph K_1 on one vertex and no edges and, therefore, $rad(S(n,1)) = 0$ for any $n \in \mathbb{N}$. Since $\tilde{C}(n,1) = S(n,1) = \{\underbrace{11\ldots1}_{n}\}$, $n \in \mathbb{N}$, the centre $C(S(n,1)) = S(n,1)$ has 1 vertex and no edges.

Let $k \geq 2$ and assume $n \leq k-1$, then since the label of a vertex has length n and $k > n$ values are available, there is always an extreme vertex $ii\ldots i$, $i \in \{1,2,\ldots k\}$, with $d(v, ii\ldots i) = (11\ldots1)_2 = 2^n - 1$ for any $v \in S(n,k)$. Hence, by Lemma 2.3, $rad(S(n,k)) = 2^n - 1$. For $n \geq k$ we have by Lemma 2.1 that for any $v \in S(n,k)$ at least one value of $d(v, 11\ldots1), d(v, 22\ldots2), \ldots, d(v, kk\ldots k)$ begins with $(\underbrace{11\ldots1}_{k-1}\ldots)_2$ and by Lemma 2.3 the same is true for the eccentricity of v. The least eccentricity one can realize is

$$\begin{aligned}(\underbrace{11\ldots1}_{k-1}\underbrace{0\ldots0}_{n-k+1})_2 &= 2^{n-1} + 2^{n-2} + \cdots + 2^{n-k+1} \\ &= 2^{n-k+1} \cdot (2^{k-2} + 2^{k-2} + \cdots + 2 + 1) \\ &= 2^{n-k+1} \cdot (2^{k-1} - 1),\end{aligned}$$

that is $rad(S(n,k)) = 2^{n-k+1} \cdot (2^{k-1} - 1)$ and this proves (3.1).

To prove (3.2) assume $n \leq k-1$, then by Corollary 2.2 and by (3.1) we have $rad(S(n,k)) = 2^n - 1 = diam(S(n,k))$ and this relation means that for any $v \in S(n,k)$ the eccentricity of v is equal to the diameter and $C(S(n,k)) = S(n,k)$.

Now let $n \geq k$, then we have to solve the equation $e_{n,k}(v) = 2^{n-k+1} \cdot (2^{k-1} - 1)$. By Lemma 2.1 it is for any $z \in \tilde{C}(n,k)$: $d(z, z_k z_k \ldots z_k) = (\underbrace{11\ldots1}_{k-1}\underbrace{00\ldots0}_{n-k+1})_2 = (2^{k-1} - 1) \cdot 2^{n-k+1}$ and $d(z, ii\ldots i) < d(z, z_k z_k \ldots z_k)$ for $i \in \{1,2,\ldots,k\} \backslash \{z_k\}$, since at least one position of $z_1 z_2 \ldots z_{k-1}$ is equal to i and thus $\rho_{z_j,i} = 0 < \rho_{z_j,z_k} = 1$, $j \in \{1,2,\ldots,k-1\}$. This means that $\tilde{C}(n,k) \subset C(S(n,k))$. To show the converse inclusion $C(S(n,k)) \subset \tilde{C}(n,k)$, let $z := z_1 z_2 \ldots z_{k-1} z_k z_{k+1} \ldots z_n \in C(S(n,k))$, then by definition and by (3.1) we have $e_{n,k}(z) = rad(S(n,k)) = (\underbrace{11\ldots1}_{k-1}\underbrace{00\ldots0}_{n-k+1})_2$, i.e. by Lemma 2.1 $\max\{d(z, ii\ldots i) \,|\, i \in \{1,2,\ldots,k\}\} = (\underbrace{11\ldots1}_{k-1}\underbrace{00\ldots0}_{n-k+1})_2$. Hence, $z_k = z_{k+1} = \cdots = z_n$. From $\rho_{z_l,z_k} = 1$ we have $z_l \neq z_k$ for $l \in \{1,2,\ldots,k-1\}$. Moreover, $\{z_1, z_2, \ldots, z_k\} = \{1,2,\ldots,k\}$, since otherwise we would have $d(z, jj\ldots j) > 2^n - 1 > rad(S(n,k))$ for $j \in \{1,2,\ldots,k\} \backslash \{z_1, z_2, \ldots, z_k\}$. This means $\tilde{C}(n,k) = C(S(n,k))$, thus proving assertion (3.2).

Statement (3.3) follows immediately from (3.2).

Finally, for $n \geq k$, $z = z_1 z_2 \ldots z_{k-1} z_k z_k \ldots z_k \in C(S(n,k))$ has exactly one adjacent vertex in $C(S(n,k))$, namely $z_1 z_2 \ldots z_k z_{k-1} z_{k-1} \ldots z_{k-1}$, and since by (3.2) $C(S(n,k)) = S(n,k)$ the centre of $S(n,k)$ has, for $n < k$, $k(k^n - 1)/2$ edges, thus proving (3.4). □

An immediate consequence of Theorem 3.1 and Corollary 2.2 is the following

Corollary 3.2 *For all* $k \geq 2$

$$\lim_{n\to\infty} \frac{rad\,(S(n,k))}{diam\,(S(n,k))} = 1 - \frac{1}{2^{k-1}}. \tag{3.5}$$

In particular, for $k = 2, 3$,

$$\lim_{n\to\infty} \frac{rad\,(S(n,2))}{diam\,(S(n,2))} = \frac{1}{2}, \quad \lim_{n\to\infty} \frac{rad\,(H_3^n)}{diam\,(H_3^n)} = \frac{3}{4}. \tag{3.6}$$

Moreover,

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{rad\,(S(n,k))}{diam\,(S(n,k))} = \lim_{n\to\infty} \lim_{k\to\infty} \frac{rad\,(S(n,k))}{diam\,(S(n,k))} = 1. \tag{3.7}$$

Theorem 3.1 asserts that the greatest value of $x_1, x_2, \ldots, x_k$, where $(x_1, x_2, \ldots, x_k)$ satisfies the conditions (2.18) and (2.19), is at least equal to $\lfloor 2^{n-k+1} \cdot (2^{k-1} - 1) \rfloor$. For the least value of $x_1, x_2, \ldots, x_k$ we have the dual statement

Corollary 3.3 *The least value of* $x_1, x_2, \ldots, x_k$*, where* $(x_1, x_2, \ldots, x_k)$ *satisfies (2.18) and (2.19) is at most equal to* $(011\ldots1)_2 = 2^{n-1} - 1$.

Proof. The least number must begin with 0 and in order to get the greatest value we must have $(011\ldots1)_2 = 2^{n-1} - 1$. (Note that this value is indipendent of k.) □

Corollaries 2.2, 2.6 and 3.3 as well as Theorem 3.1 give a complete account of the complexity of Problem 1, i.e. the complexity of the distance of an arbitrary $v \in S(n,k)$ to the nearest extreme vertex and to the most distant extreme vertex as well as the average distance of $v \in S(n,k)$ to a prescribed extreme vertex and can be formulated as follows.

Corollary 3.4 *Let* $v \in S(n,k)$, $n, k \in \mathbb{N}$, *be given. Then to reach from* v *the nearest extreme vertex one needs at least* 0 *moves and at most* $2^{n-1} - 1$ *moves, whereas to reach the most distant extreme vertex one needs at least* $\lfloor 2^{n-k+1} \cdot (2^{k-1} - 1) \rfloor$ *moves and at most* $2^n - 1$. *Moreover, one needs in average* $\frac{k-1}{k} \cdot (2^n - 1)$ *moves to reach a preassigned extreme vertex from* v.

4 The Special Cases $S(n,2)$ and $S(n,3)$

In this section we shall have a closer look at the special cases $k = 2, 3$.

Case $k = 2$

For any $n \in \mathbb{N}$, $S(n,2)$ is isomorphic to the path graph P_{2^n}. By Theorem 3.1 it is for all $n \in \mathbb{N}$

$$rad\,(S(n,2)) = 2^{n-1} \tag{4.1}$$

and

$$C(S(n,2)) = \{1\underbrace{22\ldots2}_{n-1}, 2\underbrace{11\ldots1}_{n-1}\} \tag{4.2}$$

with 2 vertices and 1 edge.
Corollary 3.4 now reads as follows:

Let $v \in S(n,2) \cong P_{2^n}$, $n \in \mathbb{N}$, be given, then to reach the nearest extreme vertex one needs at least 0 moves and at most $2^{n-1} - 1$, while to reach the most distant extreme vertex one needs at least 2^{n-1} moves and at most $2^n - 1$ moves. In average one needs $\frac{1}{2} \cdot (2^n - 1)$ moves to reach a preassigned extreme vertex from v.

Case $k = 3$

For any $n \in \mathbb{N}$ the Sierpiński graphs $S(n,3)$ are isomorphic to the Hanoi graphs H_3^n with 3 pegs and n discs. By Theorem 3.1 it is for all $n \in \mathbb{N}$

$$rad\,(H_3^n) = \lfloor 3 \cdot 2^{n-2} \rfloor = \begin{cases} 2^n - 1, & n < 3 \\ 3 \cdot 2^{n-2}, & n \geq 3 \end{cases}$$

$$C\,(H_3^n) = \begin{cases} H_3^n, & n < 3 \\ \{23\underbrace{11\ldots1}_{n-2}, 32\underbrace{11\ldots1}_{n-2}, 13\underbrace{22\ldots2}_{n-2}, & \\ 31\underbrace{22\ldots2}_{n-2}, 12\underbrace{33\ldots3}_{n-2}, 21\underbrace{33\ldots3}_{n-2}\}, & n \geq 3 \end{cases} \tag{4.3}$$

with

$$|C\,(H_3^n)| = \begin{cases} 3^n, & n < 3 \\ 6, & n \geq 3 \end{cases}$$

vertices and

$$|E(C\,(H_3^n))| = \begin{cases} \frac{3}{2}(3^n - 1), & n < 3 \\ 3, & n \geq 3 \end{cases}$$

edges, namely $\{2311\ldots1, 2133\ldots3\}$, $\{3211\ldots1, 3122\ldots2\}$ and $\{1322\ldots2, 1233\ldots3\}$ for $n \geq 3$ and $E(H_3^n)$ for $n = 1, 2$.

Using the standard Hanoi notation, where the pegs are denoted by 0,1,2, the discs by $1, 2, \ldots, n$ and where for instance the state 10210 means that disc 1 is on peg 1, disc 2 on peg 0, disc 3 on peg 2, disc 4 on peg 1 and disc 5 on peg 0 we have

Corollary 4.1 *For all $n \in \mathbb{N}$*

$$rad\,(H_3^n) = \lfloor 3 \cdot 2^{n-2} \rfloor = \begin{cases} 2^n - 1, & n < 3 \\ 3 \cdot 2^{n-2}, & n \geq 3 \end{cases} \tag{4.4}$$

$$C\,(H_3^n) = \begin{cases} H_3^n, & n < 3 \\ \{\underbrace{00\ldots0}_{n-2}10, \underbrace{00\ldots0}_{n-2}20, \underbrace{11\ldots1}_{n-2}01, & \\ \underbrace{11\ldots1}_{n-2}21, \underbrace{22\ldots2}_{n-2}02, \underbrace{22\ldots2}_{n-2}12\}, & n \geq 3 \end{cases} \tag{4.5}$$

with

$$|C\,(H_3^n)| = \begin{cases} 3^n, & n < 3 \\ 6, & n \geq 3 \end{cases} \tag{4.6}$$

vertices and

$$|E(C\,(H_3^n))| = \begin{cases} \frac{3}{2} \cdot (3^n - 1), & n < 3 \\ 3, & n \geq 3 \end{cases} \tag{4.7}$$

edges, i.e. $C(H_3^n)$ consists of those states, where discs 1 to $n-2$ all stay on disc n in an arbitrary peg, while disc $n-1$ is threaded on another peg, and the edges are $\{00\ldots010, 00\ldots020\}$, $\{11\ldots121, 11\ldots101\}$ and $\{22\ldots202, 22\ldots212\}$ for $n \geq 3$ and $E(H_3^n)$ for $n = 1, 2$.

Proof. This is a direct consequence of [5, Theorem 3]. □

Corollary 3.4 now reads as follows:

Let $v \in H_3^n$, $n \in \mathbb{N}$, be given, then to reach the nearest perfect state one needs at least 0 moves and at most $2^{n-1} - 1$, while to reach the most distant perfect state one needs at least $\lfloor 3 \cdot 2^{n-2} \rfloor$ moves and at most $2^n - 1$ moves. Finally, one needs in average $\frac{2}{3} \cdot (2^n - 1)$ moves to reach a preassigned perfect state from v.

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