On (a, d)-Antimagic Labelings of Generalized Petersen Graphs $P(n, 2)$ *

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Abstract

A connected graph $G = (V, E)$ is said to be (a, d)- antimagic, for some positive integers a and d, if its edges admit a labeling by all the integers in the set $\{1, 2, \ldots, |E(G)|\}$ such that the induced vertex labels, obtained by adding all the labels of the edges adjacent to each vertex, consist of an arithmetic progression with the first term a and the common difference d. Mirka Miller and Martin Bača proved that the generalized Petersen graph $P(n, 2)$ is $(\frac{3n+6}{2}, 3)$-antimagic for $n \equiv 0 \pmod 4$, $n \geq 8$, and conjectured that $P(n, k)$ is $(\frac{5n+5}{2}, 2)$-antimagic for odd n and $2 \leq k \leq \frac{n}{2} - 1$. In this paper, we show that the generalized Petersen graph $P(n, 2)$ is $(\frac{5n+5}{2}, 2)$-antimagic for $n \equiv 3 \pmod 4$, $n \geq 7$.

Keywords: *(a, d)-antimagic labeling, Petersen graph, vertex labeling, edge labeling*

1 Introduction

Hartsfield and Ringel[2] introduced the concept of arithmetic graphs. An arithmetic graph G is a graph whose edges can be labeled with the integers $1, 2, \ldots, |E(G)|$ so that the sum of the labels at any given vertex is different

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from the sum of the labels at any other vertex, in other words, no two vertices receive the same weight, where the weight of a vertex is defined in an obvious way. Hartsfield and Ringel conjectured that every tree other than K_2 is antimagic and, more strongly, every connected graph other than K_2 is antimagic.

Bodendiek and Walther[3] defined the concept of an (a,d)-antimagic graph as a special case of an antimagic graph. Let $G=(V,E)$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p=|V(G)|$ and $q=|E(G)|$ be the numbers of vertices and edges of G, respectively. A connected graph $G=(V,E)$ is called (a,d)- antimagic if there exist positive integers a,d and a bijection $f: E \rightarrow \{1,2,\ldots,q\}$ such that the induced mapping $g_f: V \rightarrow N$, defined by $g_f(v)=\sum f(uv)$, $uv \in E(G)$, is injective and $g_f(V)=\{a,a+d,\ldots,a+(p-1)d\}$. In this case f is called an (a,d)-antimagic labeling of G.

Bodendiek and Walther[4] proved that some graphs (including even cycles, paths of even order, stars, $C_3^{(k)}$, $C_7^{(k)}$, $K_{3,3}$ and a tree with odd order $n \geq 5$ and having a vertex adjacent to at least three end vertices) are not (a,d)-antimagic. They also proved that P_{2k+1} is $(k,1)$-antimagic; C_{2k+1} is $(k+2,1)$-antimagic; if a tree of odd order $2k+1 (k>1)$ is (a,d)-antimagic, then $d=1$ and $a=k$; if K_{4k} $(k \geq 2)$ is (a,d)-antimagic, then d is odd and $d \leq (2k+1)(4k-1)+1$; if K_{2k+1} $(k \geq 2)$ is (a,d)-antimagic, then $d \leq (2k+1)(k-1)+1$. For special graphs called parachutes, (a,d)-antimagic labelings are described in [5, 6].

Let n and k be integers such that $n \geq 3$, $1 \leq k < n$ and $n \neq 2k$. For such n,k, the generalized Petersen graph $P(n,k)$ is defined by

$$V(P(n,k)) = \{u_i, v_i | 1 \leq i \leq n\},$$
$$E(P(n,k)) = \{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+k} | 1 \leq i \leq n\}$$

where and in the sequel the subscript of a vertex is computed modulo n and taken the least positive residue of n, in other words, we take u_n and v_n instead of u_0 and v_0, respectively.

Since $P(n,k)$'s form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges, it is desirable to determine which $P(n,k)$'s are (a,d)-antimagic.

Bodendiek and Walther[7] conjectured that $P(n,1)$ is $(\frac{7n+4}{2},1)$-antimagic for even n and $P(n,1)$ is $(\frac{5n+5}{2},2)$-antimagic for odd n. These conjectures were proved in [9], where it was also shown that $P(n,1)$ is $(\frac{3n+6}{2},3)$-antimagic for even n.

Mirka Miller and Martin Bača [9] proved that $P(n,2)$ is $(\frac{3n+6}{2},3)$-antimagic for $n \equiv 0 \pmod 4), n \geq 8$ and conjectured that $P(n,k)$ is $(\frac{5n+5}{2},2)$-

antimagic for odd n and $2 \leq k \leq \frac{n}{2} - 1$.

The main result of this paper is the following theorem.

Theorem 1 *$P(n,2)$ is $(\frac{5n+5}{2}, 2)$-antimagic for $n \equiv 3 \pmod 4$ and $n \geq 7$.*

2 Proof of Theorem 1

We consider two cases:

Case 1: $n \equiv 7 \pmod 8$, $n \geq 7$.

For $n = 7$, an edge labeling of $P(7,2)$ is shown in Figure 2.1.

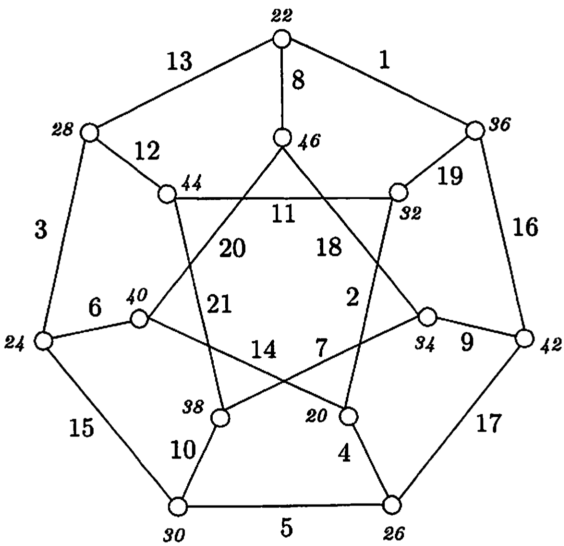

Figure 2.1 : The $(\frac{5n+5}{2}, 2)$-antimagic labeling of the graph $P(7,2)$.

According to the definition of (a,d)-antimagic labeling, it is clear that this assignment provides a $(\frac{5n+5}{2}, 2)$-antimagic labeling for $P(7,2)$.

For $n \geq 15$, we define the edge labeling f of $P(n,2)$ as follows:

$$f(u_{i-1}u_i) = \begin{cases} i & \text{if } 1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2, \\ \frac{5n-3}{2} & \text{if } i = n, \\ \frac{3n+1}{2} + i & \text{if } 2 \le i \le n-1 \text{ and } i \equiv 0 \pmod 2, \end{cases}$$

$$f(u_iv_i) = \begin{cases} n+2-i & \text{if } 1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2, \\ 3n-6 & \text{if } i = n. \\ 2n-i & \text{if } 2 \le i \le n-3 \text{ and } i \equiv 0 \pmod 2, \\ n & \text{if } i = n-1, \end{cases}$$

and

$$f(v_{i-2}v_i) = \begin{cases} \frac{5n-1}{2} + i & \text{if } 1 \le i \le \frac{n-1}{2} \text{ and } i \equiv 1 \pmod 2, \\ \frac{3n-3}{2} + i & \text{if } \frac{n+3}{2} \le i \le n-2 \text{ and } i \equiv 1 \pmod 2, \\ 2 & \text{if } i = n, \\ 3n & \text{if } i = 2, \\ \frac{5n-5}{2} + i & \text{if } 6 \le i \le \frac{n-11}{2}\ (n \ne 15) \text{ and } i \equiv 2 \pmod 4, \\ \frac{3n+1}{2} & \text{if } i = \frac{n-3}{2}, \\ \frac{n-1}{2} + i & \text{if } \frac{n+5}{2} \le i \le n-1 \text{ and } i \equiv 2 \pmod 4, \\ n+i & \text{if } 4 \le i \le \frac{n-7}{2} \text{ and } i \equiv 0 \pmod 4, \\ 3n-2 & \text{if } i = \frac{n+1}{2}, \\ 3n-4 & \text{if } i = \frac{n+9}{2}, \\ 2n+i-7 & \text{if } \frac{n+17}{2} \le i \le n-3\ (n \ne 15) \text{ and } i \equiv 0 \pmod 4. \end{cases}$$

For two integers a and b with $a \le b$, by $[a,b]$ we denote the set of consecutive integers from a to b. Set

$$A = \{f(u_{i-1}u_i) | 1 \le i \le n\},$$
$$B = \{f(u_iv_i)\ | 1 \le i \le n\},$$
$$C = \{f(v_{i-2}v_i) | 1 \le i \le n\}.$$

Then we have that $A = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{f(u_{i-1}u_i) | 1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\ &= \{i \mid 1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\ &= \{1, 3, \dots, n-2\}, \\ A_2 &= \{f(u_{i-1}u_i) | i = n\} = \{\tfrac{5n-3}{2} | i = n\} = \{\tfrac{5n-3}{2}\}, \\ A_3 &= \{f(u_{i-1}u_i) | 2 \le i \le n-1 \text{ and } i \equiv 0 \pmod 2\} \\ &= \{\tfrac{3n+1}{2} + i \mid 2 \le i \le n-1 \text{ and } i \equiv 0 \pmod 2\} \\ &= \{\tfrac{3n+1}{2} + 2, \tfrac{3n+1}{2} + 4, \dots, \tfrac{3n+1}{2} + n - 1 = \tfrac{5n-1}{2}\}, \end{aligned}$$

$B = B_1 \cup B_2 \cup B_3 \cup B_4$, where

$$\begin{aligned}
B_1 &= \{f(u_iv_i)|1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\
&= \{n+2-i|1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\
&= \{n+1, n-1, \ldots, 4\} = \{4, 6, 8, \ldots, n+1\}, \\
B_2 &= \{f(u_iv_i)|i = n\} = \{3n-6\}, \\
B_3 &= \{f(u_iv_i)|2 \le i \le n-3 \text{ and } i \equiv 0 \pmod 2\} \\
&= \{2n-i|2 \le i \le n-3 \text{ and } i \equiv 0 \pmod 2\} \\
&= \{2n-2, 2n-4, \ldots, n+3\} = \{n+3, n+5, \ldots, 2n-2\}, \\
B_4 &= \{f(u_iv_i)|i = n-1\} = \{n\},
\end{aligned}$$

and $C = \bigcup_{i=1}^{11} C_i$, where

$$\begin{aligned}
C_1 &= \{f(v_{i-2}v_i)|1 \le i \le \tfrac{n-1}{2} \text{ and } i \equiv 1 \pmod 2\} \\
&= \{\tfrac{5n-1}{2} + i|1 \le i \le \tfrac{n-1}{2} \text{ and } i \equiv 1 \pmod 2\} \\
&= \{\tfrac{5n-1}{2}+1, \tfrac{5n-1}{2}+3, \ldots, \tfrac{5n-1}{2}+\tfrac{n-1}{2}\}, \\
C_2 &= \{f(v_{i-2}v_i)|\tfrac{n+3}{2} \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\
&= \{\tfrac{3n-3}{2} + i|\tfrac{n+3}{2} \le i \le n-2 \text{ and } i \equiv 1 \pmod 2\} \\
&= \{2n, 2n+2, \ldots, \tfrac{5n-7}{2}\}, \\
C_3 &= \{f(v_{i-2}v_i)|i = n\} = \{2\}. \\
C_4 &= \{f(v_{i-2}v_i)|i = 2\} = \{3n\},
\end{aligned}$$

$$\begin{aligned}
C_5 &= \{f(v_{i-2}v_i)|6 \le i \le \tfrac{n-11}{2} \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{5n-5}{2} + i|6 \le i \le \tfrac{n-11}{2} \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{5n-1}{2}+4, \tfrac{5n-1}{2}+8, \ldots, \tfrac{5n-5}{2}+\tfrac{n-11}{2} = 3n-8\}, \\
C_6 &= \{f(v_{i-2}v_i)|i = \tfrac{n-3}{2}\} = \{\tfrac{3n+1}{2}|i = \tfrac{n-3}{2}\} = \{\tfrac{3n+1}{2}\}, \\
C_7 &= \{f(v_{i-2}v_i)|\tfrac{n+5}{2} \le i \le n-1 \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{n-1}{2} + i|\tfrac{n+5}{2} \le i \le n-1 \text{ and } i \equiv 2 \pmod 4\} \\
&= \{n+2, n+6, \ldots, \tfrac{n-1}{2}+n-1 = \tfrac{3n+1}{2}-2\}, \\
C_8 &= \{f(v_{i-2}v_i)|4 \le i \le \tfrac{n-7}{2} \text{ and } i \equiv 0 \pmod 4\} \\
&= \{n+i|4 \le i \le \tfrac{n-7}{2} \text{ and } i \equiv 0 \pmod 4\} \\
&= \{n+4, n+8, \ldots, n+\tfrac{n-7}{2} = \tfrac{3n+1}{2}-4\}, \\
C_9 &= \{f(v_{i-2}v_i)|i = \tfrac{n+1}{2}\} \\
&= \{3n-2|i = \tfrac{n+1}{2}\} = \{3n-2\}, \\
C_{10} &= \{f(v_{i-2}v_i)|i = \tfrac{n+9}{2}\} = \{3n-4|i = \tfrac{n+9}{2}\} = \{3n-4\}, \\
C_{11} &= \{f(v_{i-2}v_i)|\tfrac{n+17}{2} \le i \le n-3 \text{ and } i \equiv 0 \pmod 4\} \\
&= \{2n-7+i|\tfrac{n+17}{2} \le i \le n-3 \text{ and } i \equiv 0 \pmod 4\} \\
&= \{\tfrac{5n-1}{2}+2, \tfrac{5n-1}{2}+6, \ldots, 3n-10\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}&A_1 \cup B_4 \cup C_3 \cup B_1 \\ &\quad = \{1,3,\ldots,n-2\} \cup \{n\} \cup \{2\} \cup \{4,6,\ldots,n+1\} = [1,n+1], \\ &C_7 \cup C_8 \cup C_6 \cup A_3 \cup B_3 \cup A_2 \cup C_2 \\ &\quad = \Big[\{n+2,n+6,\ldots,\tfrac{n-1}{2}+n-1=\tfrac{3n+1}{2}-2\} \\ &\qquad \cup \{n+4,n+8,\ldots,n+\tfrac{n-7}{2}=\tfrac{3n+1}{2}-4\} \\ &\qquad \cup \{\tfrac{3n+1}{2}\} \cup \{\tfrac{3n+1}{2}+2,\tfrac{3n+1}{2}+4,\ldots,\tfrac{3n+1}{2}+n-1=\tfrac{5n-1}{2}\}\Big] \\ &\qquad \cup \Big[\{n+3,n+5,\ldots,2n-2\} \cup \{2n,2n+2,\ldots,\tfrac{5n-7}{2}\} \cup \{\tfrac{5n-3}{2}\}\Big] \\ &\quad = \{n+2,n+4,n+6,\cdots,\tfrac{5n-1}{2}\} \cup \{n+3,n+5,n+7,\cdots,\tfrac{5n-3}{2}\} \\ &\quad = [n+2,\tfrac{5n-1}{2}]\end{aligned}$$

and

$$\begin{aligned}&C_1 \cup C_{11} \cup B_2 \cup C_9 \cup C_5 \cup C_{10} \cup C_3 \\ &\quad = \{\tfrac{5n-1}{2}+1,\tfrac{5n-1}{2}+3,\ldots,3n-1\} \cup \{\tfrac{5n-1}{2}+2,\tfrac{5n-1}{2}+6,\ldots,3n-10\} \\ &\qquad \cup \{3n-6\} \cup \{3n-2\} \cup \{\tfrac{5n-1}{2}+4,\ldots,3n-8\} \cup \{3n-4\} \cup \{3n\} \\ &\quad = [\tfrac{5n-1}{2}+1,3n].\end{aligned}$$

We thus prove that $f(E(G)) = [1,3n]$

Recall that for a vertex $v \in V(G)$, $g_f(v) = \sum_{uv \in E(G)} f(uv)$. We now prove that $g_f(V) = \{g_f(v)|v \in V\} = \{a+2i|i=0,1,\ldots,2n-1\}$, where $a = \frac{5n+5}{2}$.

For convenience, define $h_f(v) = \frac{1}{2}[g_f(v)-a]$ and write

$$W = \{h_f(v)|v \in V(G)\}.$$

Then, in order to prove $g_f(V) = \{a+2i|i=0,1,\ldots,2n-1\}$ it suffices to show $W = [0,2n-1]$ or equivalently $[0,2n-1] \subseteq W$.

By definition we see that

$$\begin{aligned}h_f(u_i) &= \tfrac{1}{2}\left[f(u_{i-1}u_i)+f(u_iu_{i+1})+f(u_iv_i)-a\right],\ 1 \le i \le n, \\ h_f(v_i) &= \tfrac{1}{2}\left[f(v_{i-2}v_i)+f(v_iv_{i+2})+f(u_iv_i)-a\right],\ 1 \le i \le n.\end{aligned}$$

(1) For $i = n-2$ we have that

$$\begin{aligned}h_f(v_{n-2}) &= \tfrac{1}{2}\left[f(v_{n-4}v_{n-2})+f(v_{n-2}v_n)+f(u_{n-2}v_{n-2})-a\right] \\ &= \tfrac{1}{2}\left[(\tfrac{3n-3}{2}+n-2)+2+4-a\right] = 0.\end{aligned}$$

(2) For $1 \le i \le n-2$ and $i \equiv 1 \pmod 2$ we have

$$\begin{aligned}h_f(u_i) &= \tfrac{1}{2}\left[f(u_{i-1}u_i)+f(u_iu_{i+1})+f(u_iv_i)-a\right] \\ &= \tfrac{1}{2}\left[i+(\tfrac{3n+1}{2}+i+1)+(n+2-i)-a\right] = \tfrac{i+1}{2},\end{aligned}$$

which and (1) imply $[0, \frac{n-1}{2}] \subseteq W$.

(3) For $2 \leq i \leq n-3$ and $i \equiv 0 \pmod 2$ we have

$$h_f(u_i) = \frac{1}{2}\left[(\frac{3n+1}{2}+i)+(i+1)+(2n-i)-a\right] = \frac{n+i-1}{2},$$

which and (2) imply $[0, n-2] \subseteq W$.

(4) For $i = \frac{n-7}{2}$ we have

$$h_f(v_i) = \frac{1}{2}\left[(n+i)+\frac{3n+1}{2}+(2n-i)-a\right] = n-1,$$

which and (3) imply $[0, n-1] \subseteq W$.

(5) For $\frac{n+3}{2} \leq i \leq n-4$ and $i \equiv 1 \pmod 2$ we have

$$h_f(v_i) = \frac{1}{2}\left[(\frac{3n-3}{2}+i)+(\frac{3n-3}{2}+i+2)+(n+2-i)-a\right] = \frac{3n+2i-3}{4},$$

which and (4) imply $[0, \frac{5n-11}{4}] \subseteq W$.

(6) For $i = n-1$ we have

$$h_f(v_i) = \frac{1}{2}\left[(\frac{n-1}{2}+n-1)+(\frac{5n-1}{2}+1)+n-a\right] = \frac{5n-7}{4},$$

which and (5) imply $[0, \frac{5n-7}{4}] \subseteq W$.

(7) For $\frac{n+13}{2} \leq i \leq n-3$ and $i \equiv 0 \pmod 2$ we have

$$h_f(v_i) = \begin{cases} \frac{2n+i-8}{2}, & \text{if } i \equiv 2 \pmod 4 \\ \frac{2n+i-8}{2}, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

which and (6) imply $[0, \frac{3n-11}{2}] \subseteq W$.

(8) For $i = n$ we have

$$h_f(u_i) = \frac{1}{2}\left[(\frac{5n-3}{2})+1+(3n-6)-a\right] = \frac{3n-9}{2},$$

which and (7) imply $[0, \frac{3n-9}{2}] \subseteq W$.

(9) For $i = \frac{n+5}{2}, \frac{n+9}{2}$ we have

$$h_f(v_i) = \begin{cases} = \frac{3n-7}{2}, & \text{if } i = \frac{n+5}{2} \\ \frac{3n-5}{2}, & \text{if } i = \frac{n+9}{2} \end{cases}$$

which and (8) imply $[0, \frac{3n-5}{2}] \subseteq W$.

(10) For $\frac{n-1}{2} \leq i \leq \frac{n+1}{2}$ we have

$$h_f(v_i) = \begin{cases} \frac{3n-1}{2}, & \text{if } i = \frac{n-1}{2} \\ \frac{3n-3}{2}, & \text{if } i = \frac{n+1}{2} \end{cases}$$

which and (9) imply $[0, \frac{3n-1}{2}] \subseteq W$.

(11) For $4 \le i \le \frac{n-11}{2}$ and $i \equiv 0 \pmod 2$ we have

$$h_f(v_i) = \begin{cases} \frac{3n+i-3}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{3n+i-3}{2}, & \text{if } i \equiv 2 \pmod 4 \end{cases}$$

which and (10) imply $[0, \frac{7n-17}{4}] \subseteq W$.

(12) For $i = n$ we have

$$h_f(v_i) = \tfrac{1}{2}\left[(2 + 3n + (3n-6) - a\right] = \tfrac{7n-13}{4},$$

which and (11) imply $[0, \frac{7n-13}{4}] \subseteq W$.

(13) For $i = n-1$ we have

$$h_f(u_i) = \tfrac{1}{2}\left[\left(\tfrac{3n+1}{2} + i\right) + \tfrac{5n-3}{2} + n - a\right] = \tfrac{7n-9}{2},$$

which and (12) imply $[0, \frac{7n-9}{2}] \subseteq W$.

(14) For $i = \frac{n-3}{2}$ we have

$$h_f(v_i) = \tfrac{1}{2}\left[\tfrac{3n+1}{2} + (3n-2) + (2n-i) - a\right] = \tfrac{7n-5}{4},$$

which and (13) imply $[0, \frac{7n-5}{4}] \subseteq W$.

(15) For $i = 2$ we have

$$h_f(v_i) = \tfrac{1}{2}\left[3n + (n+i+2) + (2n-i) - a\right] = \tfrac{7n-1}{4},$$

which and (11) imply $[0, \frac{7n-1}{4}] \subseteq W$.

(16) For $1 \le i \le \frac{n-5}{2}$ and $i \equiv 1 \pmod 2$ we have

$$h_f(v_i) = \tfrac{7n+2i+1}{4},$$

which and (15) imply $[0, 2n-1] \subseteq W$. We complete the proof for Case 1.

Case 2: $n \equiv 3 \pmod 8$ and $n \ge 11$.

For $n = 11$, an edge labeling of $P(11,2)$ is shown in Figure 2.2.

According to the definition of (a,d)-antimagic labeling, it is clear that this assignment provides a $(\frac{5n+5}{2}, 2)$-antimagic labeling for $P(11,2)$.

For $n \ge 19$, we define the edge labeling f of $P(n,2)$ as follows:

$f(u_{i-1}u_i)$ and $f(u_iv_i)$ in this case agree with that in Case 1, while $f(v_{i-2}v_i)$ is defined by

$$f(v_{i-2}v_i) = \begin{cases} \frac{5n-1}{2} + i & \text{if } 1 \leq i \leq \frac{n-1}{2} \text{ and } i \equiv 1 \pmod 2, \\ \frac{3n-3}{2} + i & \text{if } \frac{n+3}{2} \leq i \leq n-2 \text{ and } i \equiv 1 \pmod 2, \\ 2 & \text{if } i = n, \\ 3n & \text{if } i = 2, \\ \frac{5n-5}{2} + i, & \text{if } 6 \leq i \leq \frac{n-15}{2}\ (n \neq 19) \text{ and } i \equiv 2 \pmod 4 \\ \frac{3n+1}{2}, & \text{if } i = \frac{n-7}{2}, \\ \frac{3n-1}{2} - 3, & \text{if } i = \frac{n+1}{2}, \\ \frac{n-1}{2} - 2 + i, & \text{if } \frac{n+9}{2} \leq i \leq n-5 \text{ and } i \equiv 2 \pmod 4), \\ n + i, & \text{if } 4 \leq i \leq \frac{n-11}{2} \text{ and } i \equiv 0 \pmod 4), \\ 3n - 4, & \text{if } i = \frac{n-3}{2}, \\ 2n - 1 + i, & \text{if } \frac{n+5}{2} \leq i \leq n-7 \text{ and } i \equiv 0 \pmod 4), \\ 3n - 2, & \text{if } i = n-3, \\ \frac{3n+1}{2} - 2, & \text{if } i = n-1. \end{cases}$$

Similarly to Case 1, we set

$$A = \{f(u_{i-1}u_i) | 1 \leq i \leq n\},$$
$$B = \{f(u_iv_i)\ | 1 \leq i \leq n\},$$
$$C = \{f(v_iv_{i+2}) | 1 \leq i \leq n\},$$

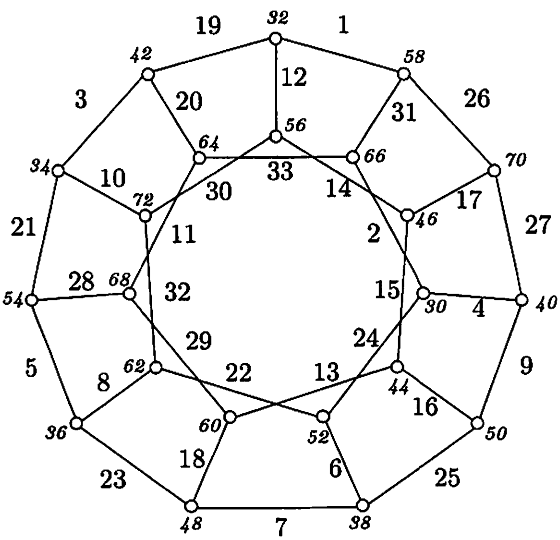

Figure 2.2 : The $(\frac{5n+5}{2}, 2)$-antimagic labeling of the graph $P(11, 2)$.

and

$$A = A_1 \cup A_2 \cup A_3, \quad B = B_1 \cup B_2 \cup B_3 \cup B_4, \text{ and } C = \bigcup_{i=1}^{13} C_i,$$

where A_i $(i = 1, 2, 3)$, B_j $(j = 1, 2, 3, 4)$ and C_k $(k = 1, 2, 3, 4)$ are as same as in Case 1, that is,

$$\begin{aligned}
A_1 &= \{1, 3, \ldots, n-2\}, \\
A_2 &= \{\tfrac{5n-3}{2}\}, \\
A_3 &= \{\tfrac{3n+1}{2} + 2, \tfrac{3n+1}{2} + 4, \ldots, \tfrac{3n+1}{2} + n - 1 = \tfrac{5n-1}{2}\}, \\
B_1 &= \{4, 6, 8, \ldots, n+1\}, \\
B_2 &= \{3n-6\}, \\
B_3 &= \{n+3, n+5, \ldots, 2n-2\}, \\
B_4 &= \{n\}, \\
C_1 &= \{\tfrac{5n-1}{2} + 1, \tfrac{5n-1}{2} + 3, \ldots, \tfrac{5n-1}{2} + \tfrac{n-1}{2} = 3n-1\}, \\
C_2 &= \{2n, 2n+2, \ldots, \tfrac{5n-7}{2}\}, \\
C_3 &= \{f(v_{i-2}v_i) | i = n\} = \{2 | i = n\} = \{2\}. \\
C_4 &= \{3n\},
\end{aligned}$$

while for others we have

$$\begin{aligned}
C_5 &= \{f(v_{i-2}v_i) | 6 \le i \le \tfrac{n-15}{2} \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{5n-5}{2} + i | 6 \le i \le \tfrac{n-15}{2} \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{5n-1}{2} + 4, \tfrac{5n-1}{2} + 8, \ldots, \tfrac{5n-5}{2} + \tfrac{n-11}{2} = 3n-10\}, \\
C_6 &= \{f(v_{i-2}v_i) | \tfrac{n+9}{2} \le i \le n-5 \text{ and } i \equiv 2 \pmod 4\} \\
&= \{\tfrac{n-5}{2} + i | \tfrac{n+9}{2} \le i \le n-5 \text{ and } i \equiv 2 \pmod 4\} \\
&= \{n+2, n+6, \ldots, \tfrac{n-5}{2} + n - 5 = \tfrac{3n+1}{2} - 8\}, \\
C_7 &= \{f(v_{i-2}v_i) | 4 \le i \le \tfrac{n-11}{2} \text{ and } i \equiv 0 \pmod 4\} \\
&= \{n + i | 4 \le i \le \tfrac{n-11}{2} \text{ and } i \equiv 0 \pmod 4\} \\
&= \{n+4, n+8, \ldots, n + \tfrac{n-11}{2} = \tfrac{3n+1}{2} - 6\}, \\
C_8 &= \{f(v_{i-2}v_i) | \tfrac{n+5}{2} \le i \le n-7 \text{ and } i \equiv 0 \pmod 4\} \\
&= \{2n - 1 + i | \tfrac{n+5}{2} \le i \le n-7 \text{ and } i \equiv 0 \pmod 4\} \\
&= \{\tfrac{5n-1}{2} + 2, \tfrac{5n-1}{2} + 6, \ldots, 3n-8\},
\end{aligned}$$

and

$$\begin{aligned}
C_9 &= \{f(v_{i-2}v_i) | i = \tfrac{n-7}{2}\} = \{\tfrac{3n+1}{2} | i = \tfrac{n-7}{2}\} = \{\tfrac{3n+1}{2}\}. \\
C_{10} &= \{f(v_{i-2}v_i) | i = \tfrac{n-3}{2}\} = \{(3n-4) | i = \tfrac{n-3}{2}\} = \{3n-4\}. \\
C_{11} &= \{f(v_{i-2}v_i) | i = \tfrac{n+1}{2}\} = \{\tfrac{3n+1}{2} - 4 | i = \tfrac{n+1}{2}\} = \{\tfrac{3n+1}{2} - 4\}. \\
C_{12} &= \{f(v_{i-2}v_i) | i = n-3\} = \{3n-2 | i = n-3\} = \{3n-2\}. \\
C_{13} &= \{f(v_{i-2}v_i) | i = n-1\} = \{\tfrac{3n-3}{2} | i = n-1\} = \{\tfrac{3n+1}{2} - 2\}.
\end{aligned}$$

Then we have that

$$\begin{aligned}
&A_1 \cup B_4 \cup C_3 \cup B_1 \\
&\quad = \{1, 3, \ldots, n-2\} \cup \{n\} \cup \{2\} \cup \{4, 6, \ldots, n+1\} = [1, n+1],
\end{aligned}$$

$$
\begin{aligned}
&C_6 \cup C_7 \cup C_{11} \cup C_{13} \cup C_9 \cup A_3 \cup B_3 \cup C_2 \cup A_2 \\
&\quad = \{n+2, n+6, \ldots, \tfrac{n-1}{2} + n - 1 = \tfrac{3n+1}{2} - 8\} \\
&\qquad \cup \{n+4, n+8, \ldots, n + \tfrac{n-7}{2} = \tfrac{3n+1}{2} - 6\} \\
&\qquad \cup \{\tfrac{3n+1}{2} - 4\} \cup \{\tfrac{3n+1}{2} - 2\} \cup \{\tfrac{3n+1}{2}\} \\
&\qquad \cup \{\tfrac{3n+1}{2} + 2, \tfrac{3n+1}{2} + 4, \ldots, \tfrac{3n+1}{2} + n - 1 = \tfrac{5n-1}{2}\} \\
&\qquad \cup \{n+3, n+5, \ldots, 2n-2\} \cup \{2n, 2n+2, \ldots, \tfrac{5n-7}{2}\} \cup \{\tfrac{5n-3}{2}\} \\
&\quad = \{n+2, n+4, n+6, \cdots, \tfrac{5n-1}{2}\} \cup \{n+3, n+5, n+7, \cdots, \tfrac{5n-3}{2}\} \\
&\quad = [n+2, \tfrac{5n-1}{2}],
\end{aligned}
$$

and

$$
\begin{aligned}
&C_1 \cup C_8 \cup C_5 \cup B_2 \cup C_{10} \cup C_{12} \cup C_4 \\
&\quad = \{\tfrac{5n-1}{2} + 1, \tfrac{5n-1}{2} + 3, \ldots, 3n-1\} \cup \{\tfrac{5n-1}{2} + 2, \tfrac{5n-1}{2} + 6, \ldots, 3n-10\} \\
&\qquad \cup \{\tfrac{5n-1}{2} + 4, \tfrac{5n-1}{2} + 8, \ldots, 3n-8\} \cup \{3n-6\} \cup \{3n-4\} \cup \{3n-2\} \\
&\quad = [\tfrac{5n-1}{2} + 1, 3n]. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup \{3n\}
\end{aligned}
$$

Hence,

$$A \cup B \cup C = [1, n+1] \cup [n+2, \tfrac{5n-1}{2}] \cup [\tfrac{5n-1}{2} + 1, 3n] = [1, 3n].$$

Therefore, f is a bijection from $E(G)$ onto $\{1, 2, \ldots, 3n\}$.

Similarly to Case 1, we define $W = \{h_f(v) | v \in V(G)\}$ and prove $[0, 2n-1] \subseteq W$. From definition we see that $h_f(u_i)$'s here agree with that in Case 1 because $f(u_iu_{i+1})$'s and $f(u_iv_i)$'s do.

(a) For $i = n-2$ we have that

$$
\begin{aligned}
h_f(v_{n-2}) &= \tfrac{1}{2}\left[f(v_{n-4}v_{n-2}) + f(v_{n-2}v_n) + f(u_{n-2}v_{n-2}) - a\right] \\
&= \tfrac{1}{2}\left[(\tfrac{3n-3}{2} + n - 2) + 2 + 4 - a\right] = 0.
\end{aligned}
$$

(b) From (2) and (3) in Case 1 we see that

$$
h_f(u_i) = \begin{cases} \frac{i+1}{2} & \text{if } 1 \le i \le n-2 \text{ and } i \equiv 1 \pmod 2, \\ \frac{n+i-1}{2} & \text{if } 2 \le i \le n-3 \text{ and } i \equiv 0 \pmod 2, \end{cases}
$$

which and (a) imply that $[0, n-2] \subseteq W$.

(c) For $i = \frac{n-11}{2}$ we have

$$h_f(v_i) = \tfrac{1}{2}\left[(n+i) + \tfrac{3n+1}{2} + (2n-i) - a\right] = n-1,$$

which and (b) imply $[0, n-1] \subseteq W$.

(d) For $\frac{n+3}{2} \le i \le n-4$ and $i \equiv 1 \pmod 2$ we have

$$h_f(v_i) = \tfrac{1}{2}\left[(\tfrac{3n-3}{2} + i) + (\tfrac{3n-3}{2} + i + 2) + (n+2-i) - a\right] = \tfrac{3n+2i-3}{4},$$

from which and (c) it follows that $[0, \frac{5n-11}{4}] \subseteq W$.

(e) For $i = n-1$ we have

$$h_f(v_i) \;=\; \tfrac{1}{2}\left[\left(\tfrac{n-1}{2}+n-1\right)+\left(\tfrac{5n-1}{2}+1\right)+n-a\right]=\tfrac{5n-7}{4},$$

from which and (d) it follows that $[0, \frac{5n-7}{4}] \subseteq W$.

(f) For $\frac{n+5}{2} \le i \le n-7$ and $i \equiv 0 \pmod 2$ we have

$$h_f(v_i) \;=\; \begin{cases} \frac{2n+i-4}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{2n+i-4}{2}, & \text{if } i \equiv 1 \pmod 4 \end{cases}$$

from which and (e) it follows that $[0, \frac{3n-11}{2}] \subseteq W$.

(g) For $i = n$ we have $h_f(u_i) = \frac{3n-9}{2}$, from which and (f) it follows that $[0, \frac{3n-9}{2}] \subseteq W$.

(h) For $i = n-5, \frac{n+1}{2}, n-3, \frac{n-1}{2}$ we have

$$h_f(v_i) \;=\; \begin{cases} \frac{3n-7}{2}, & \text{if } i = n-5 \\ \frac{3n-5}{2}, & \text{if } i = \frac{n+1}{2} \\ \frac{3n-3}{2}, & \text{if } i = n-3 \\ \frac{3n-1}{2}, & \text{if } i = \frac{n-1}{2} \end{cases}$$

from which and (g) it follows that $[0, \frac{3n-1}{2}] \subseteq W$.

(i) For $4 \le i \le \frac{n-15}{2}$ and $i \equiv 0 \pmod 2$ we have

$$h_f(v_i) \;=\; \begin{cases} \frac{3n-3+i}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{3n-3+i}{2}, & \text{if } i \equiv 1 \pmod 4 \end{cases}$$

from which and (h) it follows that $[0, \frac{7n-21}{2}] \subseteq W$.

(j) For $i = \frac{n-3}{2}, n$ we have

$$h_f(v_i) \;=\; \begin{cases} \frac{4n-10-i}{2}, & \text{if } i = \frac{n-3}{2} \\ \frac{7n-13}{4}, & \text{if } i = n \end{cases}$$

from which and (i) it follows that $[0, \frac{7n-13}{4}] \subseteq W$.

(k) For $i = n-1$ we have

$$h_f(u_i) \;=\; \tfrac{1}{2}\left[\left(\tfrac{3n+1}{2}+i\right)+\tfrac{5n-3}{2}+n-a\right]=\tfrac{7n-9}{2},$$

from which and (j) it follows that $[0, \frac{7n-9}{2}] \subseteq W$.

(l) For $i = \frac{n-7}{2}, 2, 1$ we have

$$h_f(v_i) \;=\; \begin{cases} \frac{7n-5}{2}, & \text{if } i = n-7 \\ \frac{7n-1}{4}, & \text{if } i = 2 \\ \frac{7n+3}{4}, & \text{if } i = 1 \end{cases}$$

from which and (k) it follows that $[0, \frac{7n+3}{4}] \subseteq W$.

(m) For $3 \leq i \leq \frac{n-5}{2}$ and $i \equiv 1 \pmod 2$ we have

$$h_f(v_i) \;=\; \tfrac{1}{2}\left[(\tfrac{5n-1}{2}+i)+(\tfrac{5n-1}{2}+i+2)+(n+2-i)-a\right]=\tfrac{7n+2i+1}{4},$$

from which and (l) it follows that $[0, 2n-1] \subseteq W$.

We thus prove that $P(n,2)$ is $(\frac{5n+5}{2}, 2)$-antimagic for $n \equiv 3 \pmod 4$ and $n \geq 7$. The proof is complete.

In Figure 2.3 and Figure 2.4, we give $(\frac{5n+5}{2}, 2)$-antimagic labeling for $P(19,2)$ and $P(23,2)$.

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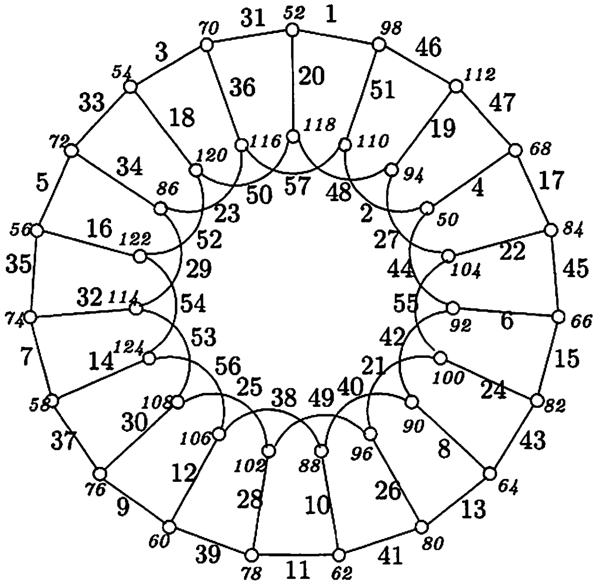

Figure 2.3 : The $(\frac{5n+5}{2}, 2)$-antimagic labeling of the graph $P(19,2)$.

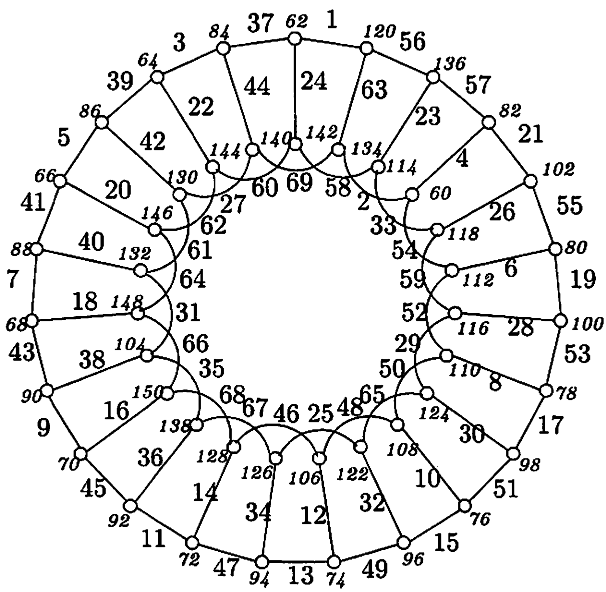

Figure 2.4 : The $(\frac{5n+5}{2}, 2)$-antimagic labeling of the graph $P(23,2)$.