The exponents of double vertex graphs*

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Abstract

Let G be a simple graph. The double vertex graph $U_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. In this paper, we consider the exponents and primitivity relationships between a simple graph and its double vertex graph. A sharp upper bound on exponents of double vertex graphs of primitive simple graphs and the characterization of extremal graph are obtained.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *walk* of length l (or *l-walk*) is a sequence $v_1 v_2 \dots v_l v_{l+1}$ of vertices such that there is a edge in G from v_i to v_{i+1} for $i = 1, 2, \dots, l$. The walk is a *path* if the vertices $v_1, \dots, v_l, v_{l+1}$ are distinct. The walk is *closed* if $v_{l+1} = v_1$, and a *cycle* is a closed walk in which $v_1, \dots, v_l$ are distinct. A cycle is a *odd cycle* (respectively, *even cycle*) if its length is odd (respectively, even). For two distinct vertices u and v, the *distance* from u to v, denoted by d_{uv}, is the minimum of lengths of walks from u to v. We agree $d_{uu} = 0$ for any vertex u. The *diameter* of G, denoted by $d(G)$, is the maximum of the distances from any vertex u to any vertex v. A graph G is *primitive* if there exists a nonnegative integer l such that for each pair v_i, v_j of vertices (not

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necessarily distinct), there exists a walk in G from v_i to v_j with length l. The *exponent* of G, $\gamma(G)$, is defined to be the minimum value of l.

For a simple graph G, in [1] the authors gave the following definition. The *double vertex graph* $U_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. Clearly, the order and the size of $U_2(G)$ are $n(n-1)/2$ and $q(n-2)$, respectively, where n is the order and q is the size of G. As examples, we have $U_2(K_2) = K_1$, $U_2(K_3) = K_3$, and $U_2(K_{1,3}) = C_6$. See Figure 1 for an example of a graph and its double vertex graph.

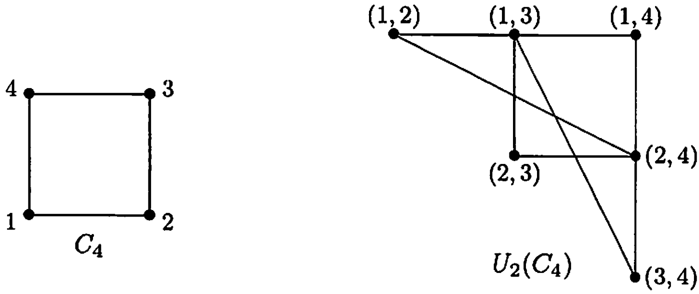

Fig. 1 A 4-cycle and its double vertex graph

Recently, there have been some papers concerning this topic, for example [1, 2, 3, 4, 5]. In [2], the authors discussed the connectivity relationships between a graph and its double vertex graph. In [4], the authors discussed the regularity, eulerian, hamiltonian, and bipartite properties of double vertex graphs. Motivated by the papers [2, 4], we consider the exponents and primitivity relationships between a simple graph and its double vertex graph. A sharp upper bound on exponents of double vertex graphs of primitive simple graphs and the characterization of extremal graph are obtained. Notations and definitions not introduced here can be found in [1, 6].

2 Basic properties of double vertex graph

In this section, we give some useful properties of double vertex graph.

Lemma 2.1 [2, 4] *The graph G is connected if and only if its double vertex graph $U_2(G)$ is connected.*

Lemma 2.2 *Let G be a simple graph, and u, v, w be three distinct vertices of G. If there is a k-walk in G from u to v, then there is a k-walk in $U_2(G)$ from (w, u) to (w, v).*

Proof If $k = 1$, then the lemma is clear. We can assume that $k \geq 2$.

Let $P = uv_1v_2 \dots v_{k-1}v$ be a k-walk in G from u to v, and denote $v_0 = u$ and $v_k = v$. We consider the following two cases.

Case 1. $w \notin \{v_1, v_2, \dots, v_{k-1}\}$. It is easy to see that the vertices (w, v_{i-1}) and (w, v_i) are adjacent in $U_2(G)$ for $i = 1, 2, \dots, k$. Thus the sequence of vertices

$$(w, v_0)(w, v_1)(w, v_2) \dots (w, v_{k-1})(w, v_k)$$

is a k-walk in $U_2(G)$ from (w, u) to (w, v).

Case 2. $w \in \{v_1, v_2, \dots, v_{k-1}\}$. Without loss of generality, we assume $w = v_j$.

Subcase 1. There are two different vertices adjacent to w in P. It is easy to see that the sequence of vertices

$$(v_j, v_0)(v_j, v_1) \dots (v_j, v_{j-1})(v_{j-1}, v_{j+1})(v_j, v_{j+1}) \dots (v_j, v_k)$$

is a k-walk in $U_2(G)$ from (w, u) to (w, v).

Subcase 2. The vertices adjacent to w in P are same, that is, $v_{j-1} = v_{j+1}$. If there is $w_1 \in V(G) \backslash \{v_{j-1}\}$ which is adjacent to w in G, then sequence of vertices

$$(v_j, v_0)(v_j, v_1) \dots (v_j, v_{j-1})(w_1, v_{j+1})(v_j, v_{j+1}) \dots (v_j, v_k)$$

is a k-walk in $U_2(G)$ from (w, u) to (w, v). If there is no vertex which is adjacent to w in G besides v_{j-1}, then must there be $w_2 \in V(G) \backslash \{w\}$ which is adjacent to v_{j-1} in G. In this case, we can replace w with w_2 in P and use case 1.

This completes the proof of the lemma. □

Analogous to Lemma 2.2, we have the following lemma.

Lemma 2.3 *Let G be a simple graph, and u, w be two distinct vertices of G. If there is a k-walk in G from u to u, then there is a k-walk in $U_2(G)$ from (w, u) to (w, u).*

Lemma 2.4 *Let G be a simple graph, and u, v, x, y be four distinct vertices of G. If there is a k-walk from u to x, and a l-walk from v to y in G, then there is a $(k + l)$-walk from (u, v) to (x, y) in $U_2(G)$.*

Proof From Lemma 2.2, there is a k-walk from (u, v) to (x, v), and a l-walk from (x, v) to (x, y) in $U_2(G)$. Thus there is a $(k + l)$-walk from (u, v) to (x, y) in $U_2(G)$. □

Lemma 2.5 *Let G be a simple graph, and u, v, x be three distinct vertices of G. If there are two walks in G from v to x with lengths k and l, respectively, then there is a $(k + l)$-walk from (u, v) to (u, v) in $U_2(G)$.*

Proof From Lemma 2.2, there is a k-walk from (u, v) to (u, x), and a l-walk from (u, x) to (u, v) in $U_2(G)$. Thus there is a $(k+l)$-walk from (u, v) to (u, v) in $U_2(G)$. The lemma follows. □

Lemma 2.6 *Let G be a simple graph, and $C = v_1 v_2 \dots v_k$ be a k-cycle of G. Then the following properties hold.*

(1) *There is a k-cycle in $U_2(G)$.*

(2) *For any vertex v_i in C and $x \in V(G)\backslash\{v_i\}$ (x may be in C), there is a k-cycle in $U_2(G)$ containing the vertex (v_i, x).*

Proof If x isn't in C, then the sequence of vertices

$$(x, v_1)(x, v_2) \dots (x, v_{k-1})(x, v_k)(x, v_1)$$

is a k-cycle in $U_2(G)$ containing the vertex (x, v_i).

If x is in C, say $x = v_j$, and $k \neq 2$, we consider the sequence of vertices

$$(v_i, v_{i+1})(v_i, v_{i+2}) \dots (v_i, v_{k-1})(v_i, v_k)(v_i, v_1) \dots (v_i, v_{i-1})(v_{i-1}, v_{i+1})(v_{i+1}, v_i).$$

It is easy to see that it is a k-cycle in $U_2(G)$ containing the vertex (x, v_i).

If x is in C, say $x = v_j$, and $k = 2$, since there is at least one vertex in $\{v_i, v_j\}$ which adjacent to two different vertices in G, without loss of generality, we assume that there is $w \in V(G)\backslash\{v_i\}$ which is adjacent to v_j in G. Then the sequence of vertices

$$(v_i, v_j)(v_i, w)(v_i, v_j)$$

is a 2-cycle in $U_2(G)$ containing the vertex (x, v_i). Then the lemma holds. □

3 The exponents and primitivity of double vertex graphs

In this section, we consider the primitivity relationships between a simple graph and its double vertex graph.

Theorem 3.1 *Let G be a simple graph of order n ($n \geq 4$). If G is primitive, then the double vertex graph $U_2(G)$ is also primitive, and $\gamma(U_2(G)) \leq \gamma(G) + d(G)$.*

Proof Let G be a primitive simple graph. Then G is connected, and the greatest common divisor of the lengths of its cycles is 1. From Lemmas

2.1 and 2.6, $U_2(G)$ is connected, and the greatest common divisor of the lengths of its cycles is 1. Thus $U_2(G)$ is primitive.

Let (u, v) and (x, y) be any two vertices of the double vertex graph $U_2(G)$. We prove that there exists a walk in $U_2(G)$ from (u, v) to (x, y) with length $\gamma(G) + d(G)$. By the primitivity of G, for each pair v_i, v_j of vertices of G and any integer $k \geq \gamma(G)$, there exists a k-walk in G from v_i to v_j. Consider the following three cases.

Case 1. $|\{u, v\} \cap \{x, y\}| = 2$. By the primitivity of G, there exists a walk in G from v to v with length $\gamma(G) + d(G)$. Then there exists a walk in $U_2(G)$ from (u, v) to (u, v) with length $\gamma(G) + d(G)$.

Case 2. $|\{u, v\} \cap \{x, y\}| = 1$. Without loss of generality, we assume $u = x$ and $v \neq y$. Since there exists a walk in G from v to y with length $\gamma(G) + d(G)$, from Lemma 2.2, there exists a walk in $U_2(G)$ from (u, v) to (x, y) with length $\gamma(G) + d(G)$.

Case 3. $|\{u, v\} \cap \{x, y\}| = 0$. Assume that $d_{ux} = l$ and there exists a path P in G from u to x with length l. Note that $l \leq d(G)$, and there exists a walk in G from v to y with length $\gamma(G) + d(G) - l$. Then, from Lemma 2.4, there exists a walk in $U_2(G)$ from (u, v) to (x, y) with length $\gamma(G) + d(G)$.

This completes the proof of the theorem. □

4 Sharp upper bound and extremal graph

In [7] B. Liu et al. proved that the exponent set of $n \times n$ $(n \geq 4)$ symmetric primitive $(0, 1)$ matrices with zero trace (the adjacency matrices of the primitive simple graphs) is $\{2, 3, \ldots, 2n-4\} \backslash S$, where S is the set of all odd numbers in $\{n-2, n-1, \ldots, 2n-5\}$. They also gave a symmetric primitive matrix A which is the adjacency matrix of the simple graph G_0 in Figure 2 that achieves this bound $2n - 4$ as its exponent.

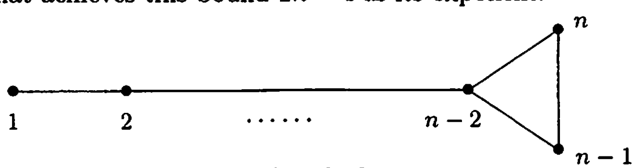

Fig. 2 Graph G_0

Lemma 4.1 [7] *Let G be a primitive simple graph of order n $(n \geq 4)$, and $E = \{m \mid m = \gamma(G)\}$. Then $E = \{2, 3, \ldots, 2n-4\} \backslash S$, where S is the set of all odd numbers in $\{n-2, n-1, \ldots, 2n-5\}$, and $\gamma(G) = 2n-4$ if and only if $G \cong G_0$.*

In this section, we give the exponent of the double vertex graph of G_0. Further, a sharp upper bound on exponents of double vertex graphs of primitive simple graphs and the characterization of extremal graph are obtained. We need some notations and technologies on graph theory.

Let G be a graph, and $L(G)$ the set of distinct cycle lengths of G. For any $x, y \in V(G)$ and $R = \{a_1, a_2, \ldots, a_r\} \subseteq L(G)$ with $\gcd(a_1, a_2, \ldots, a_r) = 1$, the *relative distance* $d_R(x, y)$ from x to y is defined to be the length of the shortest walk from x to y which meets at least one cycle of each length a_i, $i = 1, 2, \ldots, r$.

Let $\{s_1, s_2, \ldots, s_p\}$ be a set of relatively prime positive integers. The *Frobenius number*, $\phi(s_1, s_2, \ldots, s_p)$, is the least integer such that the equation $x_1s_1 + x_2s_2 + \ldots + x_ps_p = m$ has a solution in nonnegative integers $x_1, x_2, \ldots, x_p$ for all $m \geq \phi(s_1, s_2, \ldots, s_p)$. It was known to Sylvester some 150 years ago that $\phi(a_1, a_2) = (a_1 - 1)(a_2 - 1)$ if $\gcd(a_1, a_2) = 1$.

The *exponent* from vertex u to vertex v, denoted by $\gamma(u, v)$, is the least integer k such that there exists a walk of length m from u to v for all $m \geq k$.

Lemma 4.2 [8] *If G is a primitive graph, then*

$$\gamma(G) = \max_{x,y \in V(G)} \gamma(x, y).$$

Lemma 4.3 [7] *If G is a primitive graph and let $x, y \in V(G)$. If there are two walks from x to y with lengths k_1 and k_2, respectively, where $k_1 + k_2 \equiv 1 (\mathrm{mod}\ 2)$, then $\gamma(x, y) \leq \max\{k_1, k_2\} - 1$.*

Lemma 4.4 [9] *Let G be a primitive graph, $R = \{a_1, a_2, \ldots, a_r\} \subseteq L(G)$ and $\gcd(a_1, a_2, \ldots, a_r) = 1$. Then $\gamma(x, y) \leq d_R(x, y) + \phi(a_1, a_2, \ldots, a_r)$ for any $x, y \in V(G)$.*

Now we can give the exponent of $U_2(G_0)$.

Theorem 4.5 *Let G_0 be a primitive simple graph of order n $(n \geq 4)$ giving in Figure 2. Then $\gamma(U_2(G_0)) = 2n - 5$.*

Proof Clearly, $d(G_0) = n - 2$, and $\gamma(G_0) = 2n - 4$ from Lemma 4.1. Let $R = \{2, 3\}$. Then $R \subseteq L(U_2(G_0))$, each vertex of $U_2(G_0)$ is in a 2-cycle of $U_2(G_0)$ from Lemma 2.6. Further, for a vertex (u, v) of $U_2(G_0)$, if $\{u, v\} \cap \{n-2, n-1, n\} \neq \phi$, then (u, v) is in a 3-cycle of $U_2(G_0)$.

Let (u, v) and (x, y) be any two vertices of the double vertex graph $U_2(G_0)$. We now prove that $\gamma((u, v), (x, y)) \leq 2n - 5$. Take $w = n - 2$. Consider the following three cases.

Case 1. $|\{u, v\} \cap \{x, y\}| = 2$. We assume that $u = x$ and $v = y$. Without loss of generality, we assume $v \neq 1$.

If $\{u, v\} \cap \{n-2, n-1, n\} \neq \phi$, then it is clear that (u, v) is in a 3-cycle of $U_2(G_0)$, so $d_R((u, v), (u, v)) = 0$, and $\gamma((u, v), (u, v)) \leq \phi(2, 3) = 2$ from Lemma 4.4.

If $\{u,v\} \cap \{n-2, n-1, n\} = \phi$, then, by Lemma 2.2, there is a d_{vw}-walk in $U_2(G_0)$ from (u,v) to (u,w), so $d_R((u,v),(u,v)) \le 2(n-4)$ and $\gamma((u,v),(u,v)) \le 2(n-4) + \phi(2,3) = 2n-6$.

Case 2. $|\{u,v\} \cap \{x,y\}| = 1$. Without loss of generality, we assume $u = x$ and $v \neq y$.

If $\{u,v,y\} \cap \{n-2,n-1,n\} \neq \phi$, then $d_R((u,v),(u,y)) \le n-2$. If $d_R((u,v),(u,y)) \neq n-2$, it is clear that $\gamma((u,v),(u,y)) \le n-3+\phi(2,3) = n-1 \le 2n-5$. Otherwise, $v=1$, and $y = n-1$ or n, so there are two walks in $U_2(G_0)$ from (u,v) to (u,y) with lengths $n-2$ and $n-1$, respectively, from Lemma 2.2. By Lemma 4.3 $\gamma((u,v),(u,y)) \le n-2 < 2n-5$.

If $\{u,v,y\} \cap \{n-2,n-1,n\} = \phi$, then, by Lemma 2.2, there is a d_{vw}-walk in $U_2(G_0)$ from (u,v) to (u,w), and a d_{yw}-walk in $U_2(G_0)$ from (u,y) to (u,w). Then there is a $(d_{vw}+d_{yw})$-walk in $U_2(G_0)$ from (u,v) to (u,y). So $d_R((u,v),(u,y)) \le d_{vw}+d_{yw} \le 2n-7$, and $\gamma((u,v),(u,y)) \le 2n-7+\phi(2,3) = 2n-5$.

Case 3. $|\{u,v\} \cap \{x,y\}| = 0$. Without loss of generality, we assume $d_{1u} \le d_{1v}$ and $d_{1x} \le d_{1y}$.

If $\{u,v,x,y\} \cap \{n-2,n-1,n\} \neq \phi$, then, by Lemma 2.4, there is a $(d_{ux}+d_{vy})$-walk in $U_2(G_0)$ from (u,v) to (x,y), and $d_{ux}+d_{vy} \le 2n-5$. If $2n-6 \le d_{ux}+d_{vy} \le 2n-5$, then there are two walks in $U_2(G_0)$ from (u,v) to (x,y) with lengths $d_{ux}+d_{vy}$ and $d_{ux}+d_{vy}+1$, respectively, and so $\gamma((u,v),(x,y)) \le 2n-6$ from Lemma 4.3. Otherwise, $d_R((u,v),(x,y)) \le d_{ux}+d_{vy} \le 2n-7$, and $\gamma((u,v),(x,y)) \le 2n-7+\phi(2,3) = 2n-5$.

If $\{u,v,x,y\} \cap \{n-2,n-1,n\} = \phi$, let P be the shortest walk in G_0 from v to w to y. The length of P is $d_{vw}+d_{wy}$. It is not difficult to verify that $d_{vw}+d_{wy}+d_{ux} \le 2n-8$. By Lemma 2.4, there is a walk in $U_2(G_0)$ from (u,v) to (x,y) with length $d_{vw}+d_{wy}+d_{ux}$. Then $d_R((u,v),(x,y)) \le d_{vw}+d_{wy}+d_{ux} \le 2n-8$, and $\gamma((u,v),(x,y)) \le 2n-8+\phi(2,3) = 2n-6$.

On the other hand, the distance in $U_2(G_0)$ from $(1,2)$ to $(n-1,n)$ is $2n-5$. Then $\gamma((1,2),(n-1,n)) \ge 2n-5$.

This completes the proof of the theorem. □

Theorem 4.6 *Let G be a primitive simple graph of order n $(n \ge 4)$. Then $\gamma(U_2(G)) \le 2n-5$, and $\gamma(U_2(G)) = 2n-5$ if and only if $G \cong G_0$.*

Proof Let G_1 denote a primitive simple graph of order n $(n \ge 4)$ except G_0. From Theorem 4.5, we only need to prove $\gamma(U_2(G_1)) \le 2n-6$.

If $n=4$, then G_1 has a spanning subgraph G_2 in Figure 3. It is not difficult to verify that $\gamma(U_2(G_2)) = 2$, and so $\gamma(U_2(G_1)) \le 2 = 2n-6$ for $n=4$.

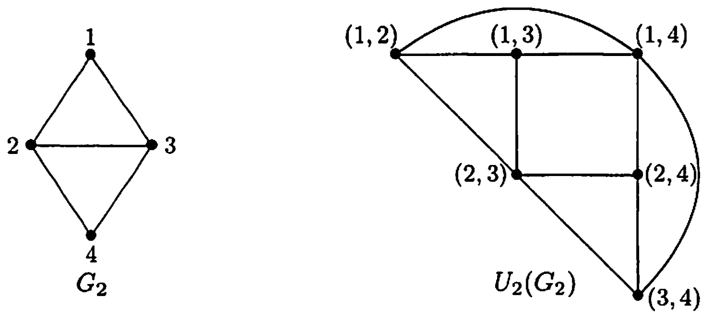

Fig. 3

In the remainder of this proof, we let $n \geq 5$. From Lemma 4.1, we have $\gamma(G_1) \leq 2n-6$, that is, for any two vertices u and v of G_1 and integer $l \geq 2n-6$, there is a l-walk in G_1 from u to v. Note that there is at least one odd cycle in G_1. Let C be the shortest odd cycle in G_1 with length r, and $R = \{2, r\}$. Clearly, $d(G_1) \leq n - r + \frac{r-1}{2}$. By Lemma 2.6, we have that $R \subseteq L(U_2(G_1))$, and each vertex of $U_2(G_1)$ is in a 2-cycle of $U_2(G_1)$. Further, for a vertex (u,v) of $U_2(G_1)$, if $\{u,v\} \cap V(C) \neq \phi$, then (u,v) is in a r-cycle of $U_2(G_1)$.

Let (u,v) and (x,y) be any two vertices of the double vertex graph $U_2(G_1)$. We now prove that $\gamma((u,v),(x,y)) \leq 2n-6$. Consider the following three cases.

Case 1. $|\{u,v\} \cap \{x,y\}| = 2$.

Since there is a l-walk in G_1 from v to v for integer $l \geq 2n-6$, it is clear that $\gamma((u,v),(u,v)) \leq 2n-6$ from Lemma 2.3.

Case 2. $|\{u,v\} \cap \{x,y\}| = 1$.

Without loss of generality, we assume $u = x$ and $v \neq y$. Since there is a l-walk in G_1 from v to y for integer $l \geq 2n-6$, we have that $\gamma((u,v),(u,y)) \leq 2n-6$ from Lemma 2.2.

Case 3. $|\{u,v\} \cap \{x,y\}| = 0$.

Subcase 1. $|\{u,v,x,y\} \cap V(C)| = 0$.

Let P and Q be the shortest paths in G_1 from u to x, and v to y, respectively. Then the lengths of P and Q are d_{ux} and d_{vy}, respectively.

(1). Either P or Q contains the vertex of C. Clearly, $d_{ux} \leq n-r+\frac{r-1}{2}$, and $d_{vy} \leq n-r+\frac{r-1}{2}$.

If $\max\{d_{ux}, d_{vy}\} = n - r + \frac{r-1}{2}$, without loss of generality, let $d_{ux} = \max\{d_{ux}, d_{vy}\}$. In this case, $d_{vy} \leq n-r-2+\frac{r-1}{2}$, and there are two paths in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. Thus there are two walks in $U_2(G_1)$ from (u,v) to (x,y) with lengths $d_{ux}+d_{vy}$ and $d_{ux}+d_{vy}+1$, respectively, and so $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy} \leq 2n-r-3 \leq 2n-6$ from Lemma 4.3.

If $\max\{d_{ux}, d_{vy}\} = n-r+\frac{r-1}{2}-1$, without loss of generality, let $d_{ux} = \max\{d_{ux}, d_{vy}\}$. If $|V(P)\cap V(C)| = \frac{r-1}{2}+1$, then $|V(G_1)\backslash(V(P)\cup V(C))| = 1$, and $d_{vy} \leq n-r-1+\frac{r-1}{2}$. Note that there are two paths in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. Thus $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy} \leq 2n-r-3 \leq 2n-6$. If $|V(P)\cap V(C)| = \frac{r-1}{2}$, then $V(G_1) = V(P)\cup V(C)$, and $d_{vy} \leq n-r-3+\frac{r-1}{2}$. Note that there are two paths in G_1 from u to x with lengths d_{ux} and $d_{ux}+3$, respectively. So $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy}+2 \leq n-r+\frac{r-1}{2}-1+n-r-3+\frac{r-1}{2}+2 = 2n-r-3 \leq 2n-6$ from Lemma 4.3.

If $\max\{d_{ux}, d_{vy}\} \leq n-r+\frac{r-1}{2}-2$, then $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy}+\phi(2,r) \leq 2(n-r+\frac{r-1}{2}-2)+r-1 = 2n-6$ from Lemma 4.4.

(2). Neither P nor Q contains the vertex of C. Let $l = \min\{d_{wz} \mid w \in V(C)$ and $z \in V(P)\cup V(Q)\}$. It is not difficult to verify that $d_{ux}+d_{vy} \leq 2(n-(r+l-1)-1)-2 = 2n-2r-2l-2$. Thus $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy}+2l+\phi(2,r) \leq 2n-r-3 \leq 2n-6$ from Lemma 4.4.

Subcase 2. $|\{u,v,x,y\}\cap V(C)| = 1$.

Without loss of generality, let $y \in V(C)$. Let $w_1 \in V(C)$ such that $d_{vw_1} = \min\{d_{vw} \mid w \in V(C)\}$, and P be the shortest path from u to x. Clearly $d_{ux} \leq n-r+\frac{r-1}{2}$, and $d_{vw_1} \leq n-r$.

(1). $d_{ux} = n-r+\frac{r-1}{2}$.

In this case, $d_{vw_1} \leq n-r-2$, and $d_{vy} \leq n-r-2+\frac{r-1}{2}$. Note that there are two walks in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. Thus $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vy} \leq n-r+\frac{r-1}{2}+n-r-2+\frac{r-1}{2} = 2n-r-3 \leq 2n-6$ from Lemma 4.3.

(2). $d_{vw_1} = n-r$.

In this case, $d_{ux} \leq n-r-2$. Note that there are two walks in G_1 from v to y with lengths $d_{vw_1}+d_{w_1y}$ and $d_{vw_1}+r-d_{w_1y}$, respectively, and d_{w_1y} and $r-d_{w_1y}$ have different parity. So $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vw_1}+r-d_{w_1y}-1 \leq d_{ux}+d_{vw_1}+r-1 \leq 2(n-r-1)+r-1 = 2n-r-3 \leq 2n-6$ from Lemma 4.3.

(3). $d_{vw_1} = n-r-1$.

In this case, $d_{ux} \leq n-r-1+\frac{r-1}{2}$. If $d_{ux} = n-r-1+\frac{r-1}{2}$, then there are two walks in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. So $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vw_1}+d_{w_1y} \leq n-r-1+\frac{r-1}{2}+n-r-1+\frac{r-1}{2} \leq 2n-r-3 \leq 2n-6$ from Lemma 4.3. If $d_{ux} \leq n-r-2+\frac{r-1}{2}$, then there are two walks in G_1 from v to y with lengths $d_{vw_1}+d_{w_1y}$ and $d_{vw_1}+r-d_{w_1y}$, respectively. So $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vw_1}+r-d_{w_1y}-1 \leq d_{ux}+d_{vw_1}+r-1 \leq n-r-2+\frac{r-1}{2}+n-r-1+r-1 = 2n-\frac{r+9}{2} \leq 2n-6$ from Lemma 4.3.

(4). $d_{ux} \leq n-r+\frac{r-1}{2}-1$, and $d_{vw_1} \leq n-r-2$.

In this case, $\gamma((u,v),(x,y)) \leq d_{ux}+d_{vw_1}+r-d_{w_1y}-1 \leq d_{ux}+d_{vw_1}+r-1 \leq n-r+\frac{r-1}{2}-1+n-r-2+r-1 = 2n-\frac{r+9}{2} \leq 2n-6$ from Lemma 4.3.

Subcase 3. $|\{u,v,x,y\}\cap V(C)|=2$.

(1). $\{x,y\}=\{u,v,x,y\}\cap V(C)$.

Without loss of generality, let $d_{ux}\geq d_{vy}$. It is clear that $d_{ux}\leq n-r+\frac{r-1}{2}$.

Let $d_{ux}=n-r+\frac{r-1}{2}$. Then $d_{vy}\leq n-r-1+\frac{r-1}{2}$, and there are two walks in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. If $d_{vy}=n-r-1+\frac{r-1}{2}$, then $r\geq 5$, and $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}\leq n-r+\frac{r-1}{2}+n-r-1+\frac{r-1}{2}=2n-r-2\leq 2n-7$ from Lemma 4.3. If $d_{vy}\leq n-r-2+\frac{r-1}{2}$, then $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}\leq n-r+\frac{r-1}{2}+n-r-2+\frac{r-1}{2}=2n-r-3\leq 2n-6$ from Lemma 4.3.

Let $d_{ux}=n-r-1+\frac{r-1}{2}$. If $|V(P)\cap V(C)|=\frac{r-1}{2}+1$, then there are two paths in G_1 from u to x with lengths d_{ux} and $d_{ux}+1$, respectively. So $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}\leq n-r-1+\frac{r-1}{2}+n-r-1+\frac{r-1}{2}=2n-r-3\leq 2n-6$ from Lemma 4.3. If $|V(P)\cap V(C)|=\frac{r-1}{2}$, and $r=3$, then $d_{vy}\leq n-r-1+\frac{r-1}{2}=n-3$, and there are two paths in G_1 from v to y with lengths $d_{vx}+d_{xy}$ and $d_{vx}+d_{xy}+1$, respectively. So $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}\leq n-r-1+\frac{r-1}{2}+n-r-1+\frac{r-1}{2}=2n-r-3=2n-6$ from Lemma 4.3. If $|V(P)\cap V(C)|=\frac{r-1}{2}$, and $r\geq 5$, noting that there are two paths in G_1 from u to x with lengths d_{ux} and $d_{ux}+3$, respectively, then $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}+2\leq n-r-1+\frac{r-1}{2}+n-r-1+\frac{r-1}{2}+2=2n-r-1\leq 2n-6$ from Lemma 4.3.

Let $d_{ux}\leq n-r-2+\frac{r-1}{2}$, then $d_{vy}\leq n-r-2+\frac{r-1}{2}$, and $\gamma((u,v),(x,y))\leq d_{ux}+d_{vy}+\phi(2,r)\leq n-r-2+\frac{r-1}{2}+n-r-2+\frac{r-1}{2}+r-1=2n-6$ from Lemma 4.4.

(2). $\{u,v\}=\{u,v,x,y\}\cap V(C)$.

Analogously to (1), we can prove $\gamma((u,v),(x,y))\leq 2n-6$.

(3). $|\{u,v\}\cap V(C)|=1$, and $|\{x,y\}\cap V(C)|=1$.

Without loss of generality, let $\{v,y\}=\{u,v,x,y\}\cap V(C)$. Then $d_{ux}\leq n-r+\frac{r-1}{2}$, and there are two walks in $U_2(G_1)$ from (u,v) to (x,y) with lengths $d_{ux}+d_{vy}$ and $d_{ux}+r-d_{vy}$, respectively. So $\gamma((u,v),(x,y))\leq d_{ux}+r-d_{vy}-1\leq n-r+\frac{r-1}{2}+r-2\leq 2n-6$ from Lemma 4.3.

Subcase 4. $|\{u,v,x,y\}\cap V(C)|=3$.

Without loss of generality, let $u\notin\{u,v,x,y\}\cap V(C)$. Clearly, $d_{ux}\leq n-r+\frac{r-1}{2}$, and there are two walks in $U_2(G_1)$ from (u,v) to (x,y) with lengths $d_{ux}+d_{vy}$ and $d_{ux}+r-d_{vy}$, respectively. So $\gamma((u,v),(x,y))\leq d_{ux}+r-d_{vy}-1\leq n-r+\frac{r-1}{2}+r-2=n+\frac{r-5}{2}$ from Lemma 4.3. If $n=5$, then $r=3$, $n+\frac{r-5}{2}=4=2n-6$. If $n\geq 6$, then $r\leq n-1$, and $n+\frac{r-5}{2}\leq 2n-6$.

Subcase 5. $|\{u,v,x,y\}\cap V(C)|=4$.

Clearly, $d_{ux}+d_{vy}\leq\frac{r-1}{2}+\frac{r-1}{2}=r-1$. Without loss of generality, we assume that $d_{ux}\leq d_{vy}$. Since there are two walks in G_1 from v to y with lengths d_{vy} and $r-d_{vy}$, respectively, we have that there are two walks in $U_2(G_1)$ from (u,v) to (x,y) with lengths $d_{ux}+d_{vy}$ and $d_{ux}+r-d_{vy}$, respectively. Then, from Lemma 4.3, $\gamma((u,v),(x,y)) \leq d_{ux} + r - d_{vy} - 1 \leq r - 1 \leq 2n - 6$.

This completes the proof of the theorem. □

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