ON THE POWER SUBGROUPS OF THE MODULAR GROUP

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ABSTRACT. We show that the power subgroups $M^{6k}(k>1)$ of the Modular group $M=PSL(2,\mathbb{Z})$ are subgroups of the groups $M'(6k,6k)$. Here the groups $M'(6k,6k)(k>1)$ are subgroups of the commutator subgroup M' of M of index $36k^2$ in M'.

1. Introduction

The Modular group M is the group consisting of all linear fractional transformations

$$t(z)=\frac{az+b}{cz+d},\ a,\ b,\ c,\ d\in\mathbb{Z}\text{ and }ad-bc=1.$$

It is well-known that M is generated by the following transformations

$$x(z)=-\frac{1}{z}\text{ and }y(z)=-\frac{1}{z+1}.$$

The Modular group is the free product of the cyclic group generated by x of order 2 and of the cyclic group generated by y of order 3. All elements of M can be considered as projective matrices $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d rational integers and $ad-bc=1$.

As a free product it is known that M can be presented as (see [1], [2] and [3])

$$M=<x,y;x^2=y^3=1>. \tag{1.1}$$

In [2], Newman investigated the group structure of the power subgroup M^n defined to be the subgroup generated by the n-th powers of all elements of M, for some positive integer n. As fully invariant subgroups, they are

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normal in M. Newman found that $M^n = M$, M^2 or M^3 if $6 \nmid n$. In particular $M^n = M$ when $(n,6) = 1$, $M^n = M^2$ when $2 \mid n$ and $(n,3) = 1$ and $M^n = M^3$ when $3 \mid n$ and $(n,2) = 1$. Also Newman found that M^2 is the free product of two cyclic groups of order 3 and of index 2, M^3 is the free product of three cyclic groups of order 2 and of index 3 and M^6 is free of rank 37 and of index 216. But in the case of $n = 6k$, $k > 1$, Newman only obtained that the power subgroups M^{6k} are free groups and $|M : M^{6k}| = \infty$ for $k \geq 72$. There are left the 70 cases M^{6k}, $2 \leq k \leq 71$ in which the index $|M : M^{6k}|$ is unknown. The exact structure of $M^{6k}(k > 1)$ is unknown.

In [2], Newman proved that $M' \supset M^6 \supset M^{6k}$ for all $k > 1$ where M' is the commutator subgroup of the Modular group. Also it is well-known that M' is a free group of rank 2 and

$$|M : M'| = 6,\ M' = \{xyxy^2, xy^2xy\}$$

and

$$M = \bigcup_{r=0}^{5} (xy)^r M'. \tag{1.2}$$

Let us write $a = xyxy^2$ and $b = xy^2xy$. In [2], Newman defined the normal subgroups $M'(p,q)$ of M' for positive integers p, q as follows:

The element $g = a^{r_1}b^{s_1}...a^{r_n}b^{s_n}$ of M' belongs to $M'(p,q)$ if and only if

$$\sum_{i=1}^{n} r_i \equiv 0(\text{mod } p),\ \sum_{i=1}^{n} s_i \equiv 0(\text{mod } q).$$

Then Newman proved that $M^6 = M'(6,6)$.

In the next section, we prove that $M^{6k} \subset M'(6k,6k)$ for $k > 1$. As a consequence, we see that $216k^2 \mid |M : M^{6k}|$ for the power subgroups $M^{6k}($ $2 \leq k \leq 71)$ and hence conclude that if these subgroups are of finite index, then they are of high index in the Modular group.

2. Power Subgroups $M^{6k}(k > 1)$ of the Modular Group

As noted in [2], we have

$$M'' \subset M'(p,q) \tag{2.1}$$

where M'' is the second commutator subgroup of the Modular group. From (2.1), we get

$$M'' \subset M'(6k, 6k) \tag{2.2}$$

for all $k \geq 1$. We use this fact in the proof of Theorem 2.1.

Theorem 2.1. *$M^{6k} \subset M'(6k, 6k)$ for $k > 1$.*

Proof. Let $g \in M$ be an arbitrary element. By (1.2), there is an integer r $(0 \leq r \leq 5)$ such that $g = (xy)^r u'$ where $u' \in M'$. Then we can write

$$\begin{aligned} g^{6k} &= \left[(xy)^r u'\right]^{6k} \\ &= \left[(xy)^r u'(xy)^{-r}\right]\left[(xy)^{2r} u'(xy)^{-2r}\right] \ldots \left[(xy)^{6kr} u'(xy)^{-6kr}\right](xy)^{6kr}. \end{aligned}$$

From [2], we know that $(xy)^6 = ab^{-1}a^{-1}b \in M''$ and therefore we get $(xy)^{6kr} \in M'' \subset M'(6k, 6k)$. If we define

$$S(w) = (xy)w(xy)^{-1}$$

for any element of M as in the proof of Theorem 6 in [2], we see that

$$g^{6k} = S^r(u')S^{2r}(u')\ldots S^{6kr}(u')(xy)^{6kr}.$$

Observe that $S^t(u') \in M'$ for every integer t and that $S^t(h_1h_2) = S^t(h_1)S^t(h_2)$ for arbitrary elements $h_1, h_2 \in M$. Since M' is abelian modulo M'', there exist integers α, β such that

$$g^{6k} = \left[S^r(a)S^{2r}(a)\ldots S^{6kr}(a)\right]^{\alpha}\left[S^r(b)S^{2r}(b)\ldots S^{6kr}(b)\right]^{\beta} u_1 \tag{2.3}$$

where $u_1 \in M'' \subset M'(6k, 6k)$. Using the fact that $(xy)^6 = ab^{-1}a^{-1}b$, we can write

$$S^{6u+t}(a) = (xy)^{6u}\left[(xy)^t a(xy)^{-t}\right](xy)^{-6u} = \left(ab^{-1}a^{-1}b\right)^u S^t(a)\left(ab^{-1}a^{-1}b\right)^{-u}$$

and

$$S^{6u+t}(b) = (xy)^{6u}\left[(xy)^t b(xy)^{-t}\right](xy)^{-6u} = \left(ab^{-1}a^{-1}b\right)^u S^t(b)\left(ab^{-1}a^{-1}b\right)^{-u},$$

where $1 \leq t \leq 5$ and u is any positive integer. For $1 \leq t \leq 5$, $S^t(a)$ and $S^t(b)$ was computed in [2] as follows:

$$\begin{aligned} S(a) &= ab^{-1}, & S(b) &= a, \\ S^2(a) &= ab^{-1}a^{-1}, & S^2(b) &= ab^{-1}, \\ S^3(a) &= ab^{-1}a^{-1}ba^{-1}, & S^3(b) &= ab^{-1}a^{-1}, \\ S^4(a) &= ab^{-1}a^{-1}b^2a^{-1}, & S^4(b) &= ab^{-1}a^{-1}ba^{-1}, \\ S^5(a) &= ab^{-1}a^{-1}baba^{-1}, & S^5(b) &= ab^{-1}a^{-1}b^2a^{-1}. \end{aligned}$$

Using these facts, if we take formula (2.3) into account, we see that $g^{6k} \in M'' \subset M'(6k, 6k)$ when $r \neq 0$ and $g^{6k} \in M'(6k, 6k)$ when $r = 0$. Hence we find that $g^{6k} \in M'(6k, 6k)$ in both cases. This implies that $M^{6k} \subset M'(6k, 6k)$. □

Remark 2.1. *Note that Theorem 2.1 holds for* $k = 1$, [2]. *Newman also proved that* $M'(6,6) \subset M^6$ *and hence obtained* $M^6 = M'(6,6)$ *by showing that*

$$M'' \subset M^6. \tag{2.4}$$

From [2], *for* $k \geq 1$ *we know that*

$$|M' : M'(6k, 6k)| = 36k^2.$$

Therefore M^6 *has index 216 in* M. *For* $2 \leq k \leq 71$, *we have the following corollary.*

Corollary 2.1. *If the power subgroups* $M^{6k}(2 \leq k \leq 71)$ *are of finite index, then we have* $216k^2 \mid \left|M : M^{6k}\right|$.

References

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