Two new classes of spectrally arbitrary sign patterns*

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Abstract

An $n \times n$ sign pattern A is a spectrally arbitrary pattern if for any given real monic polynomial $f(x)$ of degree n, there is a real matrix $B \in Q(A)$ having characteristic polynomial $f(x)$. In this paper, we give two new class of $n \times n$ spectrally arbitrary sign patterns which are generalizations of the pattern $\mathcal{W}_n(k)$ defined in [T. Britz, J.J. McDonald, D.D. Olesky, P. van den Driessche, Minimal spectrally arbitrary sign patterns, SIAM Journal on Matrix Analysis and Applications, 26(2004), 257–271].

AMS classification: 15A18
Keywords: Sign pattern; Spectrally arbitrary sign pattern; Inertially arbitrary sign pattern; Nilpotent matrix.

1 Introduction

A *sign pattern matrix* (*sign pattern*, for short) A is a matrix whose entries are in the set $\{+, -, 0\}$. Denote the set of all $n \times n$ sign patterns by Q_n. Associated with each sign pattern $A = (a_{ij}) \in Q_n$ is a class of real matrices, called the *sign pattern class* of A, defined by

$$Q(A) = \{B = (b_{ij}) \mid B \in \mathbb{R}^{n\times n}, \text{ and sign} b_{ij} = a_{ij} \text{ for all } i \text{ and } j\}.$$

A sign pattern $S = (s_{ij})$ is a *superpattern* of a sign pattern $A = (a_{ij})$ if $s_{ij} = a_{ij}$ whenever $a_{ij} \neq 0$. Each sign pattern is a superpattern of itself.

*Research supported by NNSF of China (No. 10571163) and NSF of Shanxi (No. 2007011017).

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The *inertia* of an $n \times n$ real matrix B is the ordered triple $i(B) = (i_+(B), i_-(B), i_0(B))$, in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of its eigenvalues (counting multiplicities) with positive, negative and zero real parts, respectively. The *inertia* of an $n \times n$ sign pattern A is the set of ordered triples $i(A) = \{i(B) \mid B \in Q(A)\}$.

Let $A \in Q_n$ and $n \geq 2$. Sign pattern A is an *inertially arbitrary pattern* (IAP) if $(r, s, t) \in i(A)$ for every nonnegative triple (r, s, t) with $r + s + t = n$. If, for any given real monic polynomial $f(x)$ of degree n, there is a real matrix $B \in Q(A)$ having characteristic polynomial $f(x)$, then A is a *spectrally arbitrary sign pattern* (SAP). If there is a real matrix $B \in Q(A)$ having characteristic polynomial $f(x) = x^n$ (we say that B is a nilpotent matrix), then A is *potentially nilpotent.* In particular each SAP must necessarily be both inertially arbitrary and potentially nilpotent.

The study of spectrally arbitrary and inertially arbitrary sign patterns was initiated in [1]. One of the most useful tools in finding spectrally arbitrary sign patterns is the Nilpotent-Jacobian method which is stated as Observations 10 and 15 in [1] and is proved using the Implicit Function Theorem.

Lemma 1.1 *([1]) Let A be a sign pattern of order n, and suppose there exists some nilpotent matrix $B \in Q(A)$ with at least n nonzero entries, say $b_{i_1j_1}, b_{i_2j_2}, \ldots, b_{i_nj_n}$. Let X be the matrix obtained by replacing these entries in B by variables $x_1, x_2, \ldots, x_n$ and let*

$$\det(xI - X) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1}x + \alpha_n.$$

If the Jacobian $J = \frac{\partial(\alpha_1,\alpha_2,\ldots,\alpha_n)}{\partial(x_1,x_2,\ldots,x_n)}$ is nonzero at $(x_1, x_n, \ldots, x_n) = (b_{i_1j_1}, b_{i_2j_2}, \ldots, b_{i_nj_n})$, then A is a SAP, and every superpattern of A is spectrally arbitrary.

As far as we are aware, the papers [2] and [3] give the first inertially arbitrary and spectrally arbitrary sign patterns for all orders $n \geq 2$, respectively. Recent papers [3–8] introduce some sign patterns which are spectrally arbitrary. In this paper, we give two classes of $n \times n$ spectrally arbitrary sign patterns which are generalizations of the pattern $\mathcal{W}_n(k)$ defined in [4].

2 Sign pattern $\mathcal{A}^{(1)}_{n,k,s}$

In this section, we consider the $n \times n$ sign pattern $\mathcal{A}^{(1)}_{n,k,s}$ (where $1 \leq k \leq n-2$ and $k+2 \leq s \leq n$) with positive signs throughout the first column and in the entries

$$\{(j, j+1) \mid j = 1, \ldots, k\};$$

negative signs in the entries

$$\{(j, j+1) \mid j = k+1, \ldots, n-1\}, \{(j, s) \mid j = 1, \ldots, k\}, \text{ and } (n, n);$$

and zeros elsewhere, that is,

$$\mathcal{A}^{(1)}_{n,k,s} = \begin{bmatrix} + & + & 0 & 0 & \cdots & \cdots & 0 & - & 0 & \cdots & 0 \\ + & 0 & + & 0 & & & \vdots & - & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & & \vdots \\ + & 0 & & 0 & + & \ddots & 0 & - & 0 & \cdots & 0 \\ + & 0 & & & 0 & - & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & & \ddots & - & \ddots & & & \vdots \\ \vdots & \vdots & & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & & & & & 0 & - \\ + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & - \end{bmatrix}.$$

Note that $\mathcal{A}^{(1)}_{n,k,n}$ is the sign pattern $\mathcal{W}_n(k)$ defined in [4]. It was proved that for each pair $n \geq 3$ and $1 \leq k \leq n-2$, $\mathcal{W}_n(k)$ is a SAP, and any superpattern of $\mathcal{W}_n(k)$ is spectrally arbitrary. Here we shall prove that for $1 \leq k \leq n-3$ and $k+2 \leq s \leq n-1$, $\mathcal{A}^{(1)}_{n,k,s}$ is a SAP.

Let $B \in Q(\mathcal{A}^{(1)}_{n,k,s})$ have the following form

$$B = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & 0 & & & \vdots & -2 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & & \vdots \\ a_k & 0 & & 0 & 1 & \ddots & 0 & -2 & 0 & \cdots & 0 \\ a_{k+1} & 0 & & & 0 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & \ddots & & & \vdots \\ a_{s-1} & \vdots & & & & & \ddots & -1 & \ddots & & \vdots \\ \vdots & \vdots & & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & & & & & & 0 & -1 \\ a_n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \end{bmatrix},$$

where $a_i > 0$ for $i = 1, 2, \ldots, n$. Denote the characteristic polynomial of B by $p_B(x)$. By cofactor expansion along the first column of $|xI - B|$,

$$p_B(x) = (x - a_1)x^{n-2}(x+1) - a_2 x^{n-3}(x+1) - \cdots - a_k x^{n-k-1}(x+1)$$

$$-a_{k+1}x^{n-k-2}(x+1) + a_{k+2}x^{n-k-3}(x+1) - \cdots + (-1)^{s-k-1}a_{s-1}x^{n-s}(x+1)$$

$$+[(-1)^s a_s x^{n-s-1}(x+1) + \cdots + (-1)^{n-1}a_{n-1}(x+1) + (-1)^n a_n] f(x)$$

$$= x^n + (1-a_1)x^{n-1} - \sum_{i=1}^{k}(a_i + a_{i+1})x^{n-i-1} + \sum_{i=k+1}^{s-2}(-1)^{i-k}(a_i - a_{i+1})x^{n-i-1}$$

$$+(-1)^{s-k-1}a_{s-1}x^{n-s} + [(-1)^s a_s x^{n-s} + \sum_{i=s}^{n-1}(-1)^i(a_i - a_{i+1})x^{n-i-1}] f(x),$$

where $f(x)$ is the following determinant of order $s-1$ with k (-1)'s in diagonal and k 2's in the last column:

$$f(x) = \begin{vmatrix} -1 & & & & & & 2 \\ x & \ddots & & & & & \vdots \\ & \ddots & -1 & & & & 2 \\ & & \ddots & 1 & & & 0 \\ & & & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & x & 1 \end{vmatrix}_{s-1}.$$

By cofactor expansion along the last column,

$$f(x) = (-1)^k + (-1)^s(2x^{s-k-1} + 2x^{s-k} + \cdots + 2x^{s-3} + 2x^{s-2}).$$

So $p_B(x)$

$$= x^n + (1-a_1)x^{n-1} - \sum_{i=1}^{k}(a_i + a_{i+1})x^{n-i-1} + \sum_{i=k+1}^{s-2}(-1)^{i-k}(a_i - a_{i+1})x^{n-i-1}$$

$$+(-1)^{s-k-1}a_{s-1}x^{n-s} + (-1)^{s+k}a_s x^{n-s} + \sum_{i=s}^{n-1}(-1)^{i+k}(a_i - a_{i+1})x^{n-i-1}$$

$$+2[a_s x^{n-s} + \sum_{i=s}^{n-1}(-1)^{s+i}(a_i - a_{i+1})x^{n-i-1}] \sum_{i=s-k-1}^{s-2} x^i$$

$$= x^n + (1-a_1)x^{n-1} - \sum_{i=1}^{k}(a_i + a_{i+1} - 2a_s)x^{n-i-1} + \sum_{i=k+1}^{n-1}(-1)^{i-k}(a_i - a_{i+1})x^{n-i-1}$$

$$+2\sum_{i=s}^{n-1}(-1)^{s+i}(a_i - a_{i+1})(\sum_{j=s-k-1}^{s-2} x^{n-i+j-1}).$$

When $a_1 = a_2 = \ldots = a_n = 1$, B is nilpotent. Denote $p_B(x) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1}x + \alpha_n$. Then

(1) when $n - s + 2 \le k + 1$,

$$\begin{cases} \alpha_1 = 1 - a_1, \\ \alpha_2 = -a_1 - a_2 + 2a_s, \\ \alpha_i = -a_{i-1} - a_i + 4\sum_{j=s}^{s+i-3}(-1)^{j-s}a_j + (-1)^i 2a_{s+i-2}, i = 3, \ldots, n-s+2, \\ \alpha_i = -a_{i-1} - a_i + 4\sum_{j=s}^{n-1}(-1)^{j-s}a_j + (-1)^{n-s}2a_n, \\ \qquad i = n-s+3, \ldots, k+1 \text{ (if } n-s+2 \neq k+1), \\ \alpha_i(-1)^{i-k-1}(a_{i-1} - a_i) + 2\sum_{j=s+i-k-2}^{n-1}(-1)^{j-s}a_j + 2\sum_{j=s+i-k-1}^{n}(-1)^{j-s}a_j, \\ \qquad i = k+2, \ldots, n-s+k+1, \\ \alpha_i(-1)^{i-k-1}(a_{i-1} - a_i), i = n-s+k+2, \ldots, n. \end{cases}$$

(2) when $n - s + 2 \ge k + 2$,

$$\begin{cases} \alpha_1 = 1 - a_1, \\ \alpha_2 = -a_1 - a_2 + 2a_s, \\ \alpha_i = -a_{i-1} - a_i + 4\sum_{j=s}^{s+i-3}(-1)^{j-s}a_j + (-1)^i 2a_{s+i-2}, \\ \qquad i = 3, \ldots, k+1 \text{ (if } k \ge 2), \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i) + 2\sum_{j=s+i-k-2}^{s+i-3}(-1)^{j-s}a_j + 2\sum_{j=s+i-k-1}^{s+i-2}(-1)^{j-s}a_j, \\ \qquad i = k+2, \ldots, n-s+2, \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i) + 2\sum_{j=s+i-k-2}^{n-1}(-1)^{j-s}a_j + 2\sum_{j=s+i-k-1}^{n}(-1)^{j-s}a_j, \\ \qquad i = n-s+3, \ldots, n-s+k+1 \text{ (if } k \ge 2), \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i), i = n-s+k+2, \ldots, n. \end{cases}$$

We now show that

$$J = \frac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial(a_1, a_2, \ldots, a_n)} \neq 0.$$

Consider the following four cases.

Case 1. $s = k+2$ and $n-s+2 \le k+1$.

$$J = \begin{vmatrix}
-1 & & & & & & & & & 0 & & & & & \\
-1 & -1 & & & & & & & & 2 & & & & & \\
 & -1 & -1 & & & & & & & 4 & -2 & & & & \\
 & & \ddots & \ddots & & & & & & 4 & -4 & 2 & & & \\
 & & & \ddots & \ddots & & & & & \vdots & -4 & 4 & \ddots & & \\
 & & & & \ddots & \ddots & & & & \vdots & \vdots & \vdots & \ddots & (-1)^{n-k-1}2 & \\
 & & & & & \ddots & \ddots & & & \vdots & \vdots & \vdots & \ddots & (-1)^{n-k-1}4 & (-1)^{n-k}2 \\
 & & & & & & \ddots & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots & (-1)^{n-k}2 \\
 & & & & & & & -1 & -1 & 4 & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & -1 & 3 & -4 & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & & 1 & -3 & 4 & \ddots & \vdots & \vdots \\
 & & & & & & & & & & -1 & 3 & \ddots & \vdots & \vdots \\
 & & & & & & & & & & & 1 & \ddots & (-1)^{n-k-1}4 & \vdots \\
 & & & & & & & & & & & & \ddots & (-1)^{n-k-1}3 & (-1)^{n-k}2 \\
 & & & & & & & & & & & & & (-1)^{n-k-1} & (-1)^{n-k}
\end{vmatrix}.$$

Case 2. $s = k+2$ and $n-s+2 \ge k+2$.

$$J = \begin{vmatrix}
-1 & & & & & 0 & & & & & & \\
-1 & -1 & & & & 2 & & & & & & \\
 & -1 & -1 & & & 4 & -2 & & & & & \\
 & & \ddots & \ddots & & \vdots & -4 & 2 & & & & \\
 & & & -1 & -1 & 4 & \vdots & 4 & \ddots & & & \\
 & & & & -1 & 3 & -4 & \vdots & \ddots & \ddots & & \\
 & & & & & 1 & -3 & 4 & \ddots & \ddots & (-1)^{n-k-1}2 & \\
 & & & & & & -1 & 3 & \ddots & \ddots & (-1)^{n-k-1}4 & (-1)^{n-k}2 \\
 & & & & & & & 1 & \ddots & \ddots & \vdots & \vdots \\
 & & & & & & & & \ddots & \ddots & (-1)^{n-k-1}4 & \vdots \\
 & & & & & & & & & \ddots & (-1)^{n-k-1}3 & (-1)^{n-k}2 \\
 & & & & & & & & & & (-1)^{n-k-1} & (-1)^{n-k}
\end{vmatrix}.$$

Case 3. $s \geq k+3$ and $n-s+2 \leq k+1$.

$$J = \left|\begin{array}{cccccccccccccccc}
-1 & & & & & & & & & 0 & & & & & & \\
-1 & -1 & & & & & & & & 2 & & & & & & \\
 & -1 & -1 & & & & & & & 4 & -2 & & & & & \\
 & & \ddots & \ddots & & & & & & \vdots & -4 & 2 & & & & \\
 & & & \ddots & \ddots & & & & & \vdots & \vdots & 4 & \ddots & & & \\
 & & & & \ddots & \ddots & & & & 4 & \vdots & \vdots & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & & & 2 & -4 & \vdots & \ddots & \ddots & (-1)^{n+s-1}2 & \\
 & & & & & & \ddots & \ddots & & & -2 & 4 & \ddots & \ddots & (-1)^{n+s-1}4 & (-1)^{n+s}2 \\
 & & & & & & & \ddots & -1 & & & 2 & \ddots & \ddots & \vdots & (-1)^{n+s}2 \\
 & & & & & & & & -1 & 1 & & & \ddots & \ddots & \vdots & \vdots \\
 & & & & & & & & & 1 & -1 & & & \ddots & (-1)^{n+s-1}4 & (-1)^{n+s}2 \\
 & & & & & & & & & & -1 & \ddots & & & (-1)^{n+s-1}2 & (-1)^{n+s}2 \\
 & & & & & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & & & \ddots & (-1)^{n+k-1} & \\
 & & & & & & & & & & & & & & (-1)^{n+k-1} & (-1)^{n+k}
\end{array}\right|.$$

Case 4. $s \geq k+3$ and $n-s+2 \geq k+2$.

$$J = \left|\begin{array}{cccccccccccccccc}
-1 & & & & & & & & & 0 & & & & & & \\
-1 & -1 & & & & & & & & 2 & & & & & & \\
 & -1 & -1 & & & & & & & 4 & -2 & & & & & \\
 & & \ddots & \ddots & & & & & & \vdots & -4 & 2 & & & & \\
 & & & \ddots & -1 & & & & & 4 & \vdots & -4 & \ddots & & & \\
 & & & & -1 & 1 & & & & 2 & -4 & \vdots & \ddots & \ddots & & \\
 & & & & & 1 & -1 & & & & -2 & 4 & \ddots & \ddots & (-1)^{n+s-1}2 & \\
 & & & & & & -1 & 1 & & & & 2 & \ddots & \ddots & (-1)^{n+s-1}4 & (-1)^{n+s}2 \\
 & & & & & & & 1 & \ddots & & & & \ddots & \ddots & \vdots & (-1)^{n+s}2 \\
 & & & & & & & & \ddots & \ddots & & & & \ddots & (-1)^{n+s-1}4 & \vdots \\
 & & & & & & & & & \ddots & \ddots & & & & (-1)^{n+s-1}2 & (-1)^{n+s}2 \\
 & & & & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & & & \ddots & (-1)^{n+k-1} & \\
 & & & & & & & & & & & & & & (-1)^{n+k-1} & (-1)^{n+k}
\end{array}\right|.$$

For each one of the above four cases, adding the negative ith row to the $(i+1)$th row of J, for $i = 1, 2, \ldots, n-2, n-1$, respectively, the obtained determinant is upper triangular in which all diagonal entries are 1 and -1. Thus $J \neq 0$.

By Lemma 1.1, we have the following theorem.

Theorem 2.1 *For $1 \leq k \leq n-3$ and $k+2 \leq s \leq n-1$, $\mathcal{A}^{(1)}_{n,k,s}$ is a SAP, and every superpattern of $\mathcal{A}^{(1)}_{n,k,s}$ is spectrally arbitrary.*

3 Sign pattern $\mathcal{A}^{(2)}_{n,k,s}$

In this section, we consider the $n \times n$ sign pattern $\mathcal{A}^{(2)}_{n,k,s}$ (where $1 \leq k \leq n-2$ and $k+2 \leq s \leq n-k+1$) with positive signs throughout the first column and in the entries

$$\{(j, j+1) \mid j = 1, \ldots, k\};$$

negative signs in the entries

$$\{(i, s+i-1) \mid i = 1, \ldots, k\}, \{(j, j+1) \mid j = k+1, \ldots, n-1\}, \text{ and } (n, n);$$

and zeros elsewhere, that is,

$$\mathcal{A}^{(2)}_{n,k,s} = \begin{bmatrix}
+ & + & 0 & \cdots & \cdots & 0 & - & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
+ & 0 & + & \ddots & & & \ddots & - & \ddots & & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \vdots \\
+ & 0 & & 0 & + & 0 & \cdots & \cdots & 0 & - & 0 & \cdots & 0 \\
+ & 0 & & & 0 & - & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & \ddots & - & \ddots & & & & & \vdots \\
\vdots & \vdots & & & & & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & \vdots & & & & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & & & & & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & & & & & & & & \ddots & \ddots & 0 \\
\vdots & \vdots & & & & & & & & & & 0 & - \\
+ & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -
\end{bmatrix}.$$

We shall prove that for $1 \le k \le n-2$ and $k+2 \le s \le n-k+1$, $\mathcal{A}^{(2)}_{n,k,s}$ is a SAP.

Let $B \in Q(\mathcal{A}^{(2)}_{n,k,s})$ have the following form

$$B = \begin{bmatrix}
a_1 & 1 & 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
a_2 & 0 & 1 & \ddots & & & \ddots & -2 & \ddots & & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & & & \vdots \\
a_k & 0 & & 0 & 1 & 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & 0 \\
a_{k+1} & 0 & & & 0 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & \ddots & -1 & \ddots & & & & & \vdots \\
\vdots & \vdots & & & & & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & \vdots & & & & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & & & & & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & & & & & & & & \ddots & \ddots & 0 \\
\vdots & \vdots & & & & & & & & & & 0 & -1 \\
a_n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1
\end{bmatrix},$$

where $a_i > 0$ for $i = 1, 2, \ldots, n$.

Let $s-1 \le m \le s+k-2$ and

$$D_m(x) = \begin{vmatrix}
-1 & & & & 2 & & \\
x & \ddots & & & & \ddots & \\
 & \ddots & \ddots & & & & 2 \\
 & & \ddots & -1 & & & \\
 & & & \ddots & 1 & & \\
 & & & & \ddots & \ddots & \\
 & & & & & x & 1
\end{vmatrix}_m,$$

where there are k (-1)'s in the diagonal, the entry 2 in the first row is in the $(s-1)$th column, and the entry 2 in the last column is in the $(m-s+2)$th row. By cofactor expansion along the last column, we have that

$$D_{s-1}(x) = (-1)^k + (-1)^s 2x^{s-2},$$

and

$$D_m(x) = D_{m-1}(x) + (-1)^{m-1} 2x^{s-2} \quad \text{for} \quad m > s-1.$$

Setting $D_{s-2}(x) = (-1)^k$, then

$$D_m(x) = \begin{cases} (-1)^k + (-1)^{m-1}2x^{s-2}, & \text{if } m-s \text{ is odd,} \\ (-1)^k, & \text{if } m-s \text{ is even.} \end{cases}$$

So, by cofactor expansion along the first column of $|xI - B|$,

$$p_B(x) = |xI - B| = (x-a_1)x^{n-2}(x+1) - (x+1)\sum_{i=2}^{k} a_i x^{n-i-1}$$

$$+(x+1)\sum_{i=k+1}^{s-1}(-1)^{i-k}a_i x^{n-i-1} + (x+1)\sum_{i=s}^{s+k-2}(-1)^i a_i x^{n-i-1}D_{i-1}(x)$$

$$+[(x+1)\sum_{i=s+k-1}^{n-1}(-1)^i a_i x^{n-i-1} + (-1)^n a_n]D_{s+k-2}(x)$$

Case 1. k is even.

$$p_B(x) = x^n + x^{n-1} - (x+1)\sum_{i=1}^{k} a_i x^{n-i-1} + (x+1)\sum_{i=k+1}^{n-1}(-1)^{i-k}a_i x^{n-i-1}$$

$$+(-1)^{n+k}a_n + 2(x+1)\sum_{j=0}^{\frac{k}{2}-1} a_{s+2j}x^{n-2j-3}$$

Denote $p_B(x) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1}x + \alpha_n$. Then

$$\begin{cases} \alpha_1 = 1 - a_1, & \\ \alpha_i = -a_{i-1} - a_i + 2a_{s+i-2}, & \text{for } i = 2, 4, \ldots, k, \\ \alpha_i = -a_{i-1} - a_i + 2a_{s+i-3}, & \text{for } i = 3, 5, \ldots, k+1, \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i), & \text{for } i = k+2, \ldots, n. \end{cases}$$

When $a_1 = a_2 = \ldots = a_n = 1$, B is nilpotent. And

$$J = \begin{vmatrix}
-1 & & & & & & 0 & & & & & & \\
-1 & -1 & & & & & 2 & & & & & & \\
 & -1 & -1 & & & & 2 & & & & & & \\
 & & -1 & -1 & & & & & 2 & & & & \\
 & & & -1 & -1 & & & & 2 & & & & \\
 & & & & \ddots & \ddots & & & & & \ddots & & \\
 & & & & & \ddots & \ddots & & & & & 2 & \\
 & & & & & & -1 & -1 & & & & 2 & \\
 & & & & & & & -1 & 1 & & & & \\
 & & & & & & & & 1 & -1 & & & \\
 & & & & & & & & & -1 & \ddots & & \\
 & & & & & & & & & & \ddots & (-1)^{n-1} & \\
 & & & & & & & & & & & (-1)^{n-1} & (-1)^n
\end{vmatrix}.$$

Adding the negative ith row to the $(i+1)$th row, for $i = 1, 2, \ldots, n-1$, respectively, the obtained determinant is a upper triangular determinant in which all diagonal entries are 1 and -1. Thus $J \neq 0$.

Case 2. k is odd.

$$p_B(x) = x^n + x^{n-1} - (x+1)\sum_{i=1}^{k} a_i x^{n-i-1} + (x+1)\sum_{i=k+1}^{n-1} (-1)^{i-k} a_i x^{n-i-1}$$

$$+(-1)^{n+k} a_n + 2(x+1) \sum_{j=0}^{\frac{k-1}{2}-1} a_{s+2j} x^{n-2j-3}$$

$$+2(x+1) \sum_{i=s+k-1}^{n-1} (-1)^{s+i} a_i x^{n+s-i-3} + (-1)^{n+s} 2a_n x^{s-2}$$

Denote $p_B(x) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1} x + \alpha_n$. Then

$$\begin{cases} \alpha_1 = 1 - a_1, \\ \alpha_i = -a_{i-1} - a_i + 2a_{s+i-2}, \quad \text{for } i = 2, 4, \ldots, k+1, \\ \alpha_i = -a_{i-1} - a_i + 2a_{s+i-3}, \quad \text{for } i = 3, 5, \ldots, k, \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i) + (-1)^{i-1} 2(a_{s+i-3} - a_{s+i-2}), \\ \qquad \text{for } i = k+2, \ldots, n-s+2, \\ \alpha_i = (-1)^{i-k-1}(a_{i-1} - a_i), \quad \text{for } i = n-s+3, \ldots, n. \end{cases}$$

When $a_1 = a_2 = \ldots = a_n = 1$, B is nilpotent. And $J =$

$$\begin{vmatrix}
-1 & & & & & & 0 & & & & & & & & & & \\
-1 & -1 & & & & & 2 & & & & & & & & & & \\
 & -1 & -1 & & & & 2 & & & & & & & & & & \\
 & & -1 & -1 & & & & & 2 & & & & & & & & \\
 & & & \ddots & \ddots & & & & 2 & & & & & & & & \\
 & & & & \ddots & \ddots & & & & & \ddots & & & & & & \\
 & & & & & -1 & -1 & & & & & 2 & & & & & \\
 & & & & & & -1 & 1 & & & & 2 & -2 & & & & \\
 & & & & & & & 1 & -1 & & & & -2 & 2 & & & \\
 & & & & & & & & -1 & 1 & & & & 2 & \ddots & & \\
 & & & & & & & & & 1 & -1 & & & & \ddots & (-1)^{n-s+1}2 & \\
 & & & & & & & & & & \ddots & \ddots & & & & (-1)^{n-s+1}2 & (-1)^{n-s}2 \\
 & & & & & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & & & & & \ddots & (-1)^{n-1} & \\
 & & & & & & & & & & & & & & & (-1)^{n-1} & (-1)^n
\end{vmatrix}.$$

Adding the negative ith row to the $(i+1)$th row, for $i = 1, 2, \ldots, n-1$, respectively, the obtained determinant is upper triangular in which all diagonal entries are nonzero. Thus $J \neq 0$.

By Lemma 1.1, we have the following theorem.

Theorem 3.1 *For $1 \leq k \leq n-2$ and $k+2 \leq s \leq n-k+1$, $\mathcal{A}^{(2)}_{n,k,s}$ is a SAP, and any superpattern of $\mathcal{A}^{(2)}_{n,k,s}$ is spectrally arbitrary.*

Acknowledgements

The authors thank the referee for many helpful suggestions.

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