SOME METHODS OF CONSTRUCTION OF RECTANGULAR DESIGNS

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Abstract : -
This paper describes some new methods of constructing rectangular designs from balanced incomplete block (BIB) designs and Hadamard matrices. At the end of the paper a table of rectangular designs in the range of r, k ≤ 15 is given.

1. Introduction

Rectangular designs, introduced by vartak (1955), are 3-associate PBIB designs based on a rectangular association scheme of v=mn treatments arranged in a rectangle of m rows and n columns such that, with respect to each treatment, the first associates are the other (n-1) treatments belong to the same row, the second associates are the other (m-1) treatments belong to the same column and the remaining (m-1) (n-1) treatments are the third associates.

A rectangular design is an arrangement of v=mn treatments in b blocks such that

(i) each block contains k distinct treatments $k < v$,
(ii) each treatment occurs in exactly r blocks,
(iii) the mn treatments are arranged in a rectangle of m rows and n columns such that any two treatments which are first associates occur together in λ_1 blocks, any two treatments which are second associates occur together in λ_2 blocks and two treatments which are third associates occur together in λ_3 blocks.

These designs have been studied by Bhagwandas et al. (1985), Suen (1989), Sinha (1991), Sinha et al. (1993, 1996, 1999), Kageyama and Miao (1995), Sinha et al. (2002b), and so on.

In this paper some new methods of constructing rectangular designs are given by making use of known BIB designs and Hadamard matrices. At the end a table of new designs in the range of r, k $\leq$ 15 and not found in the tables of Suen (1989), Sinha et al. (1993, 1996) and Sinha et al. (2002b)

For definition and other properties of Hadamard matrix see Hedayat and Wallis (1978).

For notations, $O_{r \times s}$ denotes an r x s matrix having all elements zero, $J_{r \times s}$ denotes a flat matrix having all elements unity, N is the incidence matrix of BIBD (v, b, r, k, λ) of appropriate order, J - N is the incidence matrix of the complementary BIB design. I_s denotes the Identity matrix of order s.

2. Construction

Some methods of construction of rectangular designs will be presented in this section.

Consider the elements of GF(p) Galois field of order p, where p is an odd prime number. This elements of GF(p) are $\propto_0 = 0, 1, 2 \ldots\ldots, p-1$. There are (p-1)/2 elements which are quadratic residues and (p-1)/2 non-quadratic residues of elements of GF(p). It is known that complete set of MOLS for any odd prime always exists. Replace 0 by O_{vxb} null matrix, all quadratic residues by N, the incidence matrix of a BIBD with v = 2k and all non-quadratic residues by J-N in the latin squares of order p. Hence we have theorem.

Theorem 2.1 :

The existence of a BIB design with parameters v = 2k and a latin square of order p, implies the existence of a rectangular design with parameters,

$v^* = pv,\ b^* = pb,\ r^* = (p-1)r,\ k^* = (p-1)\,k,\ \_{}_1^* = (p-1)\ \_,$

$\_{}^*_2 = (p-3)r/2,\ \_{}^*_3 = \{(p-1)r/2\} - \_,\ m = p,\ n = v$ ..... .(2.1)

Proof:

The parameters v^*, b^*, r^*, k^* are obvious. Among the pv treatments of a rectangular association scheme can be naturally defined as follows. These pv treatments are arranged in a rectangular array of p rows and v columns such that first associates of any treatment _ (say) are remaining (v-1) treatments contained in the same row, the second associates are other (p-1) treatments belonging to the same column and the remaining (p-1) (v-1) treatments are third associates of _.

The parameters $\_{}_i^*$ (i = 1,2,3) can be determined from the structure SS' (where S' be the transpose of S) in the following manner.

Suppose S be the (0,1) incidence matrix of order v* x b* and has row sum (column sum) as r* (k*) then,

$SS' = I_v \otimes A + (J_v - I_v) \otimes B.$ ....(2.2)

Where

$A = (p-1)/2\ NN' + (p-1)/2\ (J-N)\ (J-N)',$

$\quad = (p-1)\ \{(r-\lambda)\ I_v + \lambda J_v\}$ ....(2.3)

and

$B = (p-1)/2\ N(J-N)' + (p-3)/2\ (J-N)\ (J-N)',$

$\quad = (p-1)/2\ r\ J_v - \{(r-\lambda)\ I_v + J_v\}$ ....(2.4)

From (2.3) and (2.4) we have,

$\lambda_1^* = (p-1)\ \lambda,\ \ \lambda^*_3 = (p-1)r/2)\ - \lambda$

In order to find the value of λ_2^*, we consider the relation (2.4). We take the sum of the coefficients of I_V and J_V in terms of r only. Hence we obtain

$\lambda_2^* = (p-3)r/2$

This completes the proof.

When p=5 and p=7 in Theorem 2.1, one has the following

Corollary 2.1:

The existence of a BIB design with v=2k and a complete set of MOLS of order 5 implies the existence of a rectangular design with parameters

$v^* = 5v,\ b^* = 5b,\ r^* = 4r,\ k^* = 4k,\ \lambda^*_1 = 4\lambda,\ \ \lambda^*_2 = r,\ \lambda^*_3 = (2r-\lambda),\ m = 5,\ n = v.$

Corollary 2.2:
The existence of a BIB design with v=2k and a complete set of MOLS of order 7 implies the existence of a rectangular design with parameters,

$v^* = 7v$, $b^* = 7b$, $r^* = 6r$, $k^* = 6k$, $\lambda^*_1 = 6\lambda$, $\lambda^*_2 = 2r$, $\lambda^*_3 = (3r-\lambda)$; $m = 7$, $n = v$.

3. Construction of rectangular design using Hadamard matrices

Let H be a Hadamard matrix of order 4t in normalized form whose initial row and column having all elements +1. Now we have the following theorem.

Theorem 3.1:
The existence of a normalized Hadamard matrix of order 4t and a BIB design with v=2k implies the existence of rectangular design with parameters,
$v^* = 4tv$, $b^*=(4t-1)b$, $r^*= (4t-1)r$, $k^*= 4tk$, _$^*_1$ = (4t-1) _, _$^*_2$ = (2t-1)r, _$^*_3$ = 2tr - _; $n_1 = v - 1$, $n_2 = 4t - 1$, $n_3 = (4t - 1)(v - 1)$
By deleting first column of normalized Hadamard matrix H_{4t} and replacing +1 by N and -1 by (J-N) in the remaining matrix of 4t x (4t-1)

Proof:
The parameters v^*, b^*, r^*, k^* are obvious. Among the 4tv treatments of a rectangular association scheme can be defined as follows. These 4tv treatments are arranged in a rectangular array of 4t rows and v columns such that first associates of any treatment are (v-1) treatments other than this treatments in the same row, the second associates are other (4t-1) treatments in the same column and the remaining (4t-1)(v-1) treatments are its third associates. The parameters _$_i^*$ (i = 1,2,3) are determined following the association scheme which are as
_$_1^*$ = (4t-1) _, _$_2^*$ = (2t-1) r, _$_3^*$ = 2tr - _
This completes the proof.

Table

Rectangular designs with r, k ≤ 15

No.	v	b	r	k	λ_1	λ_2	λ_3	m	n	source
1.	20	30	12	8	4	3	5	4	5	Cor. 2.1 BIBD (4,6,3,2,1)
2.	24	22	11	12	0	5	6	12	2	Th. 3.1 BIBD (2,2,1,1,0)
3.	24	30	15	12	6	5	8	4	6	Th. 3.1 BIBD (6,10,5,3,2)
4.	24	42	14	8	6	0	4	3	8	Th. 2.1 BIBD (8,14,7,4,3)
5.	26	26	12	12	0	5	6	13	2	Th. 2.1 BIBD (2,2,1,1,0)

REFERENCES

1. Bhagwandas, S.Banerjee, S.Kageyama, *Patterned construction of partially balanced incomplete block designs,* Commun. Statist. Theor. Meth., 14(1985), 1259-1267.
2. A Hedayat and W.D.Wallis, *Hadamard matrices and their applications,* Ann. Statist. 6 (1978), 1184-1238.
3. S.Kageyama and Y.Miao, *Some methods of construction of PBIB designs,* Combinatorics Theory 1(1995), 186-198.
4. K.Sinha, S. Kageyama and M.K.Singh, *Construction of rectangular designs,* Statistics 25(1993), 63-70.
5. K.Sinha, R.K.Mitra and G.M.Saha, *Nested BIB designs, balanced bipartite weighing designs and rectangular designs,* Utilitas Math. 49(1996), 216-222.
6. K.Sinha, *A Construction of rectangular designs,* J.Comb. Math. Comb. Computing 9(1991), 199-200.
7. K.Sinha and R.K.Mitra, *Construction of nested balanced incomplete block designs, rectangular designs and q-ary codes,* Ann. Combin. 3(1999), 71-80.
8. K.Sinha, M.K.Singh, S.Kageyama and R.S.Singh, *Some series of rectangular designs,* J.Statist. Plann. Inference 106(2002b), 39-46.
9. C.Suen, *Some rectangular designs constructed by methods of differences,* J.Statist. Plann. Inference 21(1989), 273-276.
10. M.N. Vartak, *On an application of Kronecker product of matrices of statistical designs,* Ann. Math. Statist. 26(1955), 420-438.