A Note on Path-Matchings*

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Abstract

A well-known result on matchings of graphs is that the intersection of all maximal barriers is equal to "set A" in the Gallai–Edmonds decomposition. In this paper, we give a generalization of this result to the framework of path-matchings introduced by Cunningham and Geelen. Further we present a sufficient condition for a graph to have a perfect path-matching.

Keywords: Path-matching; Gallai-Edmonds-type structure theorem; Extreme set

1 Introduction

As a generalization of matchings, Cunningham and Geelen in [2] introduced the notion of path-matchings and in [2, 3] proved the polynomial-time solvability of the optimal path-matching problem via the ellipsoid method [7] by a totally dual integral polyhedral description. Then in [3] they presented an algorithm based on deterministic evaluation of the Tutte-matrix.

Let $G = (V, E)$ be a graph with vertex-set V, edge-set E and T_1, T_2 disjoint subsets of V, which are called *terminal sets*. Let $R := V \setminus (T_1 \cup T_2)$. A *path-matching* M in G is a set of edges such that every component of M is a path from $T_1 \cup R$ to $T_2 \cup R$ with internal vertices in R. Such a path is called a T_i*-half-path* ($i = 1, 2$) if one end of it is in T_i, the other in R, a (T_1, T_2)*-path* if one end of it is in T_1, the other in T_2, and *matching-edge* if it is an edge with two ends in R. The *value* of a path-matching M, denoted by $val(M)$, is the number of edges in M plus the number of matching-edges of M. (That is, each matching-edge counts twice.) The *path-matching number* of G, $val(G) = \max\{val(M) | M$ is a path-matching of $G\}$. A path-matching M of G is said to be *maximum* if $val(M) = val(G)$. The *optimal path-matching problem* is to find a maximum

*This work is supported by NSFC (grant no. 10831001).
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path-matching or to compute the path-matching number. It is easy to see that the restriction on path-matchings having no path in R of length more than 1 does not change $val(G)$. Under this restriction the path-matchings in $G = (V, \emptyset, \emptyset; E)$ correspond to the matchings in G.

Tutte [10] and Berge [1] gave a min-max theorem for the maximum cardinality of a matching in a graph.

Theorem 1.1 (Tutte-Berge formula). *Let $G = (V, E)$ be a graph. Then*

$$2\max\{|M| \mid M \text{ matching in } G\} = |V| + \min\{|X| - \mathrm{odd}_G(X) \mid X \subseteq V\},$$

where $\mathrm{odd}_G(X)$ denotes the number of odd components in $G - X$.

A set $X \subseteq V$ is called a *barrier* if the minimum is attained at X in the Tutte-Berge formula. Frank and Szegő [5] gave the path-matching number of a graph in the following theorem as a direct extension of the Berge-Tutte formula.

Theorem 1.2 ([5]). *Let $G = (V, T_1, T_2; E)$ be a graph. Then*

$$val(G) = |R| + \min\{|X| - \mathrm{odd}_G(X) \mid X \text{ is a cut in } G\}, \tag{1}$$

where a cut X is a subset of V such that there is no path between $T_1 \setminus X$ and $T_2 \setminus X$ in $G - X$ and $\mathrm{odd}_G(X)$ denotes the number of odd components disjoint from $T_1 \cup T_2$ in $G - X$.

A cut X in G is called *tight* by Spille and Szegő [9] if $\mathrm{odd}_G(X) - |X| = |R| - val(G)$. Note that, if $T_1 = T_2 = \emptyset$, then a tight cut is just a usual barrier.

The so called Gallai-Edmonds structure theorem [6, 4] describes the structure of maximum matchings based on a partition $\{A, C, D\}$ of V: D is the set of all vertices in G which are not covered by at least one maximum matching of G, A the set of vertices in $V \setminus D$ adjacent to at least one vertex in D, and $C := V \setminus (A \cup D)$.

In [8], a nice characterization of A was given in terms of maximal barriers.

Theorem 1.3 ([8]). *A is the intersection of all maximal barriers in graph G.*

Spille and Szegő [9] gave a similar partition $\{A_1, C_1, D_1\}$ of V and presented the Gallai-Edmonds-type structure concerning maximum path-matchings of a graph $G = (V, E)$. However, Spille and Szegő's decomposition theorem is considerably more complicated than Gallai-Edmonds structure theorem. It is natural to ask whether Theorem 1.3 can be extended to the situation of path-matchings or not. In this article we give a positive answer by expressing A_1 in terms of maximal tight cuts (see Theorem 3.1).

Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets, where $|T_1| = |T_2| = k$. A path-matching M is called *perfect* in G, if M consists of k disjoint

(T_1, T_2)-paths and a perfect matching of the vertices of R not in the paths. It is easy to see that for a graph G, it has a perfect path-matching if and only if $val(G) = |R| + k$. In [5], Frank and Szegő presented a characterization theorem of graphs with perfect path-matchings which generalizes Tutte's theorem [10]. In the last section of this paper, we present a sufficient condition for a graph to have a perfect path-matching as a simple application of Gallai-Edmonds-type structure theorem for path-matchings.

2 Preliminaries and extreme set

As preliminaries, in this section we extend the concept of "extreme set" on matchings to path-matchings by Theorem 2.1 and present some useful properties. We can see that the extended notion of extreme sets will be much more difficult to handle than the previous.

We first define the *deficiency of G with respect to path-matchings* as $\mathrm{def}'(G) := |R| - val(G)$. We can see that its degenerated case of $T_1 = T_2 = \emptyset$ is the usual deficiency in matching theory. Then, by Theorem 1.2 $\mathrm{def}'(G) = \max_{\text{cut } X}(\mathrm{odd}_G(X) - |X|)$. Since the path-matching number is equal to the number of vertices in R covered by a maximum path-matching M having no path in R of length more than 1 plus the number of (T_1, T_2)-paths in M, $\mathrm{def}'(G)$ equals the number of vertices in R uncovered by M minus the number of (T_1, T_2)-paths in M.

For any $x \in V$, we write the set of vertices adjacent to x in G by $N_G(x)$. For $S \subseteq V$, $N_G(S) := \bigcup_{x \in S} N_G(x) \setminus S$, $\nabla(S)$ denotes the set of edges with exactly one end-vertex in S, and $G[S]$ the subgraph of G induced by S. Other terminology used and undefined in this article is standard and can be found in textbooks.

Theorem 2.1. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. For any set X of vertices in G,*

$$\mathrm{def}'(G - X) \leq \mathrm{def}'(G) + |X|. \tag{2}$$

Proof. Let M be a maximum path-matching in G. M is partitioned into two parts: $M \cap E(G - X)$ and $M \cap (E(G[X]) \cup \nabla(X))$. It is evident that each part is a path-matching in G and each matching-edge of M is still matching-edge of $M \cap E(G - X)$ or $M \cap (E(G[X]) \cup \nabla(X))$. Hence

$$val(M) \leq val(M \cap E(G - X)) + val(M \cap (E(G[X]) \cup \nabla(X))). \tag{3}$$

Since $M \cap E(G - X)$ is also a path-matching in $G - X$ with terminal sets $T_1 \setminus X$ and $T_2 \setminus X$,

$$val(M \cap E(G - X)) \leq val(G - X). \tag{4}$$

On the other hand, there are at most two edges of M traversing a vertex in $X \cap R$ and one edge traversing a vertex in $X \setminus R$. (Note that one matching-edge can be considered as two edges.) We have

$$val(M \cap (E(G[X]) \cup \nabla(X))) \leq 2|X \cap R| + |X \setminus R|. \qquad (5)$$

Combining inequalities (3), (4) and (5), we obtain

$$val(M) \leq val(G - X) + 2|X \cap R| + |X \setminus R|.$$

Thus we have

$$\begin{aligned} \mathrm{def}'(G) &= |R| - val(M) \\ &\geq |R| - val(G - X) - 2|X \cap R| - |X \setminus R| \\ &= |R \setminus X| - val(G - X) - |X \cap R| - |X \setminus R| \\ &= \mathrm{def}'(G - X) - |X|, \end{aligned}$$

and the proof is complete. □

$X \subseteq V$ is called an *extreme set* in graph G if equality holds in (2), that is, $\mathrm{def}'(G - X) = \mathrm{def}'(G) + |X|$. For convenience we use the same name here.

Using the proof of Theorem 2.1, we can obtain the following result.

Proposition 2.2. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. Let X be an extreme set in G. Then the intersection of any maximum path-matching M in G and the edge-set of $G - X$ is a maximum path-matching on $G - X$ and any $v \in X$ is either covered by a matching-edge of M, or by a (T_1, T_2)-path, or by a T_i-half-path of M but v is not the R-end-vertex, and X induces no edge of M.*

Note that the converse of Proposition 2.2 does not hold. For example, let P_5 be a path $v_1v_2v_3v_4v_5$ and $T_1 = \{v_1\}$, $T_2 = \{v_5\}$. We see that P_5 has a unique perfect path-matching, P_5 itself. Thus the conclusion of Proposition 2.2 holds for $X = \{v_2, v_5\}$. However, X is not an extreme set in P_5 since $\mathrm{def}'(P_5 - X) = 0$ and $\mathrm{def}'(P_5) = -1$.

Proposition 2.3. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. Let X be an extreme set and M a maximum path-matching in G. Then each matching-edge in the intersection of M and the edge-set of $G - X$ is also a matching-edge in M.*

Proof. If there exists a matching-edge in $M \cap E(G - X)$ which is not matching-edge in M, then strict inequality in (3) holds. Thus X can not be an extreme set in G, a contradiction. This completes the proof. □

Proposition 2.4. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets.*

A tight cut in G is an extreme set.

Proof. Let X be a tight cut in G. Then by definition we have $\mathrm{odd}_G(X) = \mathrm{def}'(G)+|X|$ and $\mathrm{def}'(G-X) \geq \mathrm{odd}_{G-X}(\emptyset) = \mathrm{odd}_G(X)$. Thus $\mathrm{def}'(G-X) \geq \mathrm{def}'(G)+|X|$. On the other hand, by Theorem 2.1 $\mathrm{def}'(G-X) \leq \mathrm{def}'(G)+|X|$. Hence the equality holds and X is extreme set. □

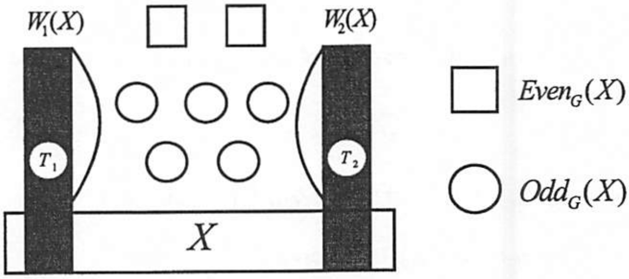

Figure 1. Cut X, $Odd_G(X)$, $Even_G(X)$, $W_1(X)$, $W_2(X)$.

Spille and Szegő gave the Optimality Criteria of a maximum path-matching and a generalization of the Gallai-Edmonds structure theorem in [9]. Recall that a graph G is *factor-critical* if $G-v$ has a perfect matching for every vertex v in G. A matching of G is called *near-perfect* if it does not cover precisely one vertex. Let $Odd_G(X)$ ($Even_G(X)$) denote the union of odd (even) components of $G-X$ which are disjoint from $T_1 \cup T_2$. For $i = 1, 2$, $W_i(X)$ is the union of components in $G-X$ joint T_i. (See Figure 1 [5, 9].)

Theorem 2.5 (Optimality Criteria [9]). *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. Let M be a path-matching in G such that any path in M of length more than one has at least one end in $T_1 \cup T_2$, and X a cut in G. Then M is maximum and X is tight if and only if* (O1)-(O7)*hold.*

(O1) *M induces a perfect matching on $Even_G(X)$,*

(O2) *for $i = 1, 2$, M induces T_i-half-paths and matching-edges on $W_i(X)$ covering all the vertices of $W_i(X) \setminus T_i$,*

(O3) *for any component K in $Odd_G(X)$, either M induces a matching and an even path on K covering all vertices of K or M induces a near-perfect matching on K,*

(O4) *for any vertex $v \in X$, v is either covered by a matching-edge of M, or by a (T_1, T_2)-path of M, or by a T_i-half-path of M but v is not the R-end-vertex $(i = 1, 2)$,*

(O5) *X induces no edge of M,*

(O6) *for any R-end-vertex v of a T_i-half-path of M, $v \in Odd_G(X) \cup W_i(X), (i = 1, 2)$,* (O7) *for any* $v \in R$ *not covered by* M, $v \in Odd_G(X)$.

Theorem 2.6 (Structure Theorem for Path-Matchings [9]). *Let* $G = (V, E)$ *be a graph and* T_1 *and* T_2 *two terminal sets. Define the decomposition* A_1, C_1, D_1 *of* V *by*

$F_1 := \{v \in T_1 \cup R : \exists$ *maximum path-matching not covering* $v\}$,

$H_1 := \{v \in R : \exists$ *maximum path-matching such that* v *is* R*-end-vertex of* T_1*-half-path*$\}$,

$D_1 := F_1 \cup H_1$,

$A_1 := \{u \in V \setminus D_1 : uv \in E$ *for some* $v \in D_1\} \cup (T_1 \setminus D_1)$,

$C_1 := V \setminus (D_1 \cup A_1)$.

Then A_1 *is a tight cut,*

$$D_1 = W_1(A_1) \cup Odd_G(A_1), \qquad C_1 = Even_G(A_1) \cup W_2(A_1),$$

and every component in $Odd_G(A_1)$ *is factor-critical.*

Now we show that the restriction on maximum path-matchings having no path in R of length more than one in Theorem 2.6 does not change the Gallai-Edmonds-type decomposition. Since D_1 can determine the decomposition in a given graph, we only need to verify that this restriction does not change D_1. Let $F_1' := \{v \in T_1 \cup R : \exists$ maximum path-matching with this restriction not covering $v\}$ and $H_1' := \{v \in R : \exists$ maximum path-matching with this restriction such that v is R-end-vertex of T_1-half-path$\}$. Clearly $F_1' \cup H_1' \subseteq D_1$. For any vertex $v \in D_1$, there is a maximum path-matching M such that v is missed or v is R-end-vertex of T_1-half-path in M. Then we can get another maximum path-matching M' having no path in R of length more than one by replacing all paths in R with the respective maximum matchings on these paths from M. Since either v is still missed M' or v is R-end-vertex of T_1-half-path in M', $v \in F_1' \cup H_1'$. Thus $D_1 = F_1' \cup H_1'$.

Proposition 2.7. *Let* $G = (V, E)$ *be a graph and* T_1 *and* T_2 *two terminal sets. An extreme set in* G *contains no vertex in* D_1.

Proof. Let X be an extreme set in G. For any $x \in X$ we have that x is covered by any maximum path-matching and x is not the R-end-vertex of an T_i-half-path by Proposition 2.2. These imply that $x \notin F_1$ and $x \notin H_1$. Hence $x \notin D_1$. □

To establish the main result of this paper we need an important property of maximal tight cuts which was obtained by Frank and Szegő in [5].

Theorem 2.8 ([5]). *Let* $G = (V, E)$ *be a graph and* T_1 *and* T_2 *two terminal sets. Let* X *be a maximal tight cut in* G. *Then every component of* $G - X$

disjoint from $T_1 \cup T_2$ is factor-critical.

For a tight cut X of G, let $D_1(X) := Odd_G(X) \cup W_1(X)$. The following theorem shows that A_1 is the tight cut X such that the set $D_1(X)$ is minimal.

Theorem 2.9. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. For every tight cut X in G, $D_1 = D_1(A_1) \subseteq D_1(X)$.*

Proof. Let x be any vertex in D_1, then by definition there is a maximum path-matching M such that x is R-end-vertex of some T_1-half-path in M or x is missed by M. If the former happens, by Theorem 2.5 (O6) $x \in Odd_G(X) \cup W_1(X) = D_1(X)$. If x is missed and $x \in R$, then $x \in Odd_G(X) \subseteq D_1(X)$ by Theorem 2.5 (O7). Otherwise, we have $x \in T_1 \subseteq X \cup W_1(X)$. Since X is tight cut, $x \notin X$ by Propositions 2.4 and 2.7. Hence $x \in W_1(X) \subseteq D_1(X)$. Thus the proof is complete. □

3 An expression for $A_1 \cup W_1(A_1)$

Now we describe the main result as follows.

Theorem 3.1. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets. Then*

$$A_1 \cup W_1(A_1) = \bigcap_{\substack{\text{maximal}\\ \text{tight cut } X}} (X \cup W_1(X)) \setminus W_2(A_1)$$

Proof. First we show that $A_1 \cup W_1(A_1) \subseteq \bigcap_{\substack{\text{maximal}\\ \text{tight cut } X}} (X \cup W_1(X)) \setminus W_2(A_1)$. Since $A_1 \cup W_1(A_1)$ is disjoint from $W_2(A_1)$, it suffices to show that $A_1 \cup W_1(A_1) \subseteq X \cup W_1(X)$ for every maximal tight cut X in G. Suppose, to the contrary, there exists a vertex $x \in A_1 \cup W_1(A_1)$ such that $x \notin X \cup W_1(X)$. By the maximality of X, there is no even component in $G - X$, so we have $x \in W_2(X) \cup Odd_G(X)$. There are two cases to be distinguished.

Case 1: $x \in A_1$.

Since $x \in W_2(X) \cup Odd_G(X)$, $x \notin T_1$. Hence, by the definition of A_1, x is adjacent to some $y \in D_1$. Then $y \notin X$ by Propositions 2.4 and 2.7.

If $x \in W_2(X)$, then $y \in W_2(X)$. But $y \in D_1 \subseteq W_1(X) \cup Odd_G(X)$ by Theorem 2.9, a contradiction.

Now suppose $x \in Odd_G(X)$. Let K be the component in $Odd_G(X)$ containing x. Then y lies in K and by Theorem 2.8 we know that K is factor-critical. Since $y \in D_1$, there exists a maximum path-matching M in G such that y is missed by M or y is R-end-vertex of some T_1-half-path. If the former happens, by Theorem 2.5 (O3) M induces a near-perfect matching on K. Furthermore, by Proposition 2.3 every edge in the near-perfect matching is matching-edge in

M. Then replacing the near-perfect matching with another near-perfect matching missing x on K, we get a new maximum path-matching in G which misses x. Hence $x \in D_1$, which contradicts $x \in A_1$. So suppose that y is R-end-vertex of some T_1-half-path in M.

Claim 1. There is only one edge in M traversing K.

Suppose there exist two edges uu' and vv' in M such that u, v are vertices in K and u', v' are not ($u' \neq v'$). By Theorem 2.5 (O3) we know that M induces an even path P (a single vertex is considered as a trivial even path.) and a matching M_k covering all vertices on K. By Propositions 2.4 and 2.3 every edge in M_k is a matching-edge in M, so u and v are not incident with any edge in M_k. Then u, v are the end-vertices of P. Thus no vertex in K is R-end-vertex of some T_1-half-path in M, a contradiction. The claim is proved.

Let y' be the first vertex on the T_1-half-path after entering K. Since K is factor-critical, there are perfect matchings $M_{y'}$ and M_x of $K - y'$ and $K - x$ respectively. Then $M_{y'} \cup M_x$ induces a component, an even path $P_{y',x}$ between y' and x. Hence $M^* := M_x \cup E(P_{y',x})$ consists of a matching of K and the even path $P_{y',x}$, covering all vertices of K. Replacing the edges of M in K by M^*, we get a new maximum path-matching in G such that x ends a T_1-half-path. So $x \in D_1$, which contradicts $x \in A_1$.

Case 2: $x \in W_1(A_1)$.

If $x \in W_1(A_1) \cap W_2(X)$, then by Theorems 2.6 and 2.9 we have $x \in W_1(A_1) \subseteq D_1 \subseteq W_1(X) \cup Odd_G(X)$. This contradicts $x \in W_2(X)$.

Otherwise, $x \in W_1(A_1) \cap Odd_G(X)$.

Let K be the component in $Odd_G(X)$ containing x. Clearly, $N_G(K)$ is a subset of X.

Claim 2. $K \subseteq W_1(A_1)$.

By Case 1, we have $A_1 \cap Odd_G(X) = \emptyset$, and $K \subseteq V \setminus A_1$. Furthermore $K \subseteq W_1(A_1)$ since $x \in K \cap W_1(A_1)$. The claim is proved.

Claim 2 together with $K \cap T_1 = \emptyset$ imply that $N_G(K)$ contains at least one vertex in $W_1(A_1)$, i.e. $N_G(K)$ intersects $W_1(A_1)$.

Claim 3. $X \cap W_1(A_1) = \emptyset$.

Since X is tight, we know that X is an extreme set in G by Proposition 2.4, and X contains no vertex in $D_1 = W_1(A_1) \cup Odd_G(A_1)$ by Theorem 2.6 and Proposition 2.7. So Claim 3 follows.

Thus, Claim 3 together with the fact that $N_G(K) \subseteq X$ imply that $N_G(K) \cap W_1(A_1) = \emptyset$. That is a contradiction.

In the following we will show that $\bigcap_{\substack{\text{maximal} \\ \text{tight cut } X}} (X \cup W_1(X)) \subseteq A_1 \cup W_1(A_1) \cup W_2(A_1)$. It suffices to show that, for any vertex $x \in Odd_G(A_1) \cup Even_G(A_1)$, $x \notin \bigcap_{i=1}^{n} (X_i \cup W_1(X_i))$ for some maximal tight cuts $X_1, \cdots, X_n$ in G.

Case 1: $x \in Odd_G(A_1)$.

Let X be a maximal tight cut containing A_1. By Propositions 2.4 and 2.7, $X \cap D_1 = \emptyset$. Since $Odd_G(A_1) \subseteq D_1$ (Theorem 2.6), $X \cap Odd_G(A_1) = \emptyset$. Hence $x \in Odd_G(A_1) \subseteq Odd_G(X)$. Furthermore $x \notin X \cup W_1(X)$.

Case 2: $x \in Even_G(A_1)$.

Let K be the component in $Even_G(A_1)$ containing x and let $Y_1, \cdots, Y_n$ be all maximal barriers of K. By Theorems 2.5 (O1) and 2.6 K contains a perfect matching. By Theorem 1.3 we have $\bigcap_{i=1}^{n} Y_i = \emptyset$. Note that $A_1 \cup Y_i$, $i = 1, \cdots, n$, are tight cuts in G. (Since K has a perfect matching, we have $|Y_i| = \text{odd}_K(Y_i)$, and $\text{def}'(G) = \text{odd}_G(A_1 \cup Y_i) - |A_1 \cup Y_i|$.) Let $X_1, \cdots, X_n$ be maximal tight cuts of G containing $Y_1 \cup A_1, \cdots, Y_n \cup A_1$ respectively.

Claim 4. $X_i \cap K = Y_i, i = 1, \cdots, n$.

Let $X_i \cap K := Y_i'$. Then $Y_i \subseteq Y_i'$. Clearly $X_i \setminus Y_i'$ is a cut in G, and

$$\text{odd}_G(X_i \setminus Y_i') - |X_i \setminus Y_i'| \leq \text{def}'(G). \tag{6}$$

On the other hand, since K can be considered as a graph with empty terminal sets, we have

$$\text{odd}_K(Y_i') - |Y_i'| \leq 0. \tag{7}$$

Adding (6) and (7), we obtain

$$\begin{aligned} \text{def}'(G) &\geq \text{odd}_G(X_i \setminus Y_i') + \text{odd}_K(Y_i') - |X_i \setminus Y_i'| - |Y_i'| \\ &= \text{odd}_G(X_i) - |X_i|. \end{aligned}$$

Since X_i is a tight cut in G, we infer that equality must hold throughout. Thus Y_i' is a barrier in G containing Y_i. Hence $Y_i' = Y_i$, since Y_i is a maximal barrier.

Claim 4 implies that $x \notin \bigcap_{i=1}^{n} X_i$. Since $A_1 \subseteq X_i$, $W_1(X_i) \subseteq W_1(A_1), i = 1, \cdots, n$. Hence x is not in any one of $W_1(X_1), W_1(X_2), \cdots, W_1(X_n)$ by $x \in Even_G(A_1)$. So $x \notin \bigcap_{i=1}^{n}(X_i \cup W_1(X_i))$. The proof is complete. □

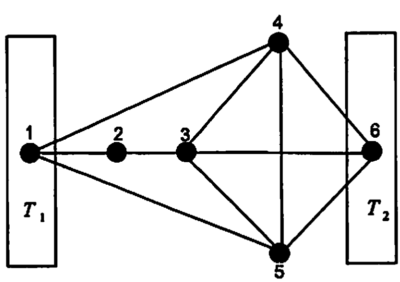

Figure 2. Graph $G = (V, E)$ with two terminal sets T_1 and T_2.

If $T_1 = T_2 = \emptyset$, then $W_1(X), W_1(A_1)$ and $W_2(A_1)$ are empty. By Theorem 3.1 we can get Theorem 1.3. Notice that $\bigcap_{\text{maximal tight cut } X} (X \cup W_1(X))$ may contain some vertices in $W_2(A_1)$. For example, consider graph $G = (V, T_1, T_2; E)$ in Figure 2. It has two maximal tight cuts $X_1 = \{1,3\}$ and $X_2 = \{6\}$, and $W_1(X_1) = \emptyset$, $W_1(X_2) = \{1,2,3,4,5\}$. It is easy to see that the intersection of $W_2(A_1) = \{2,3,4,5,6\}$ and $(X_1 \cup W_1(X_1)) \cap (X_2 \cup W_1(X_2))$ is $\{3\}$.

4 A sufficient condition for graphs to have a perfect path-matching

Wang and Hao in [11] obtained a sufficient condition for perfect matchings on extreme set as follow.

Theorem 4.1. *Let G be a simple graph containing an independent set of size i (≥ 2). If all independent sets of size i in G are extreme sets, then G has a perfect matching.*

In this section, we present a generalization to path-matchings as a simple application of Gallai-Edmonds-type structure theorem. The following equivalent conditions for graphs with perfect path-matchings is needed.

Theorem 4.2. *Let $G = (V, E)$ be a graph and T_1 and T_2 two terminal sets with $|T_1| = |T_2| = k$. Then the following are equivalent:*
(i) *G has a perfect path-matching;*
(ii) $\mathrm{def}'(G) = -k$;
(iii) $D_1 = \emptyset$.

Proof. (i)$\Rightarrow$(iii). Trivial.

(iii)$\Rightarrow$(ii). Since $D_1 = \emptyset$, $Odd_G(A_1) = \emptyset$ and $A_1 = T_1$ by Theorem 2.6. Thus $\mathrm{def}'(G) = \mathrm{odd}_G(A_1) - |A_1| = -k$.

(ii)$\Rightarrow$(i). This is immediate by definitions of perfect path-matching and deficiency. □

Theorem 4.3. *Let $G = (V, E)$ be a graph and T_1 and T_2 terminal sets such that $|T_1| = |T_2| = k$ and T_1 is an extreme set and R contains an independent set of size i. If every independent set X of size i in R is an extreme set, then G has a perfect path-matching.*

Proof. Suppose that G has no perfect path-matching. Then $D_1 \neq \emptyset$ and $\mathrm{def}'(G) \geq -k+1$ by Theorem 4.2. By definitions we have that $T_1 \subseteq A_1 \cup D_1$. Since T_1 is an extreme set, $T_1 \cap D_1 = \emptyset$ by Proposition 2.7. Hence $T_1 \subseteq A_1$ and $D_1 = Odd_G(A_1)$.

Case 1: $\mathrm{odd}_G(A_1) \geq i$.

Let $G_1, \cdots, G_i$ be components in $Odd_G(A_1)$ and choose a vertex v_j from G_j for all $j = 1, \cdots, i$. Then we get an independent set $X = \{v_1, \cdots, v_i\}$ and $X \subseteq R$. So by assumption X is an extreme set. But $X \cap D_1 = \emptyset$ by Proposition

2.7, which is a contradiction.

Case 2: $1 \leq \text{odd}_G(A_1) \leq i-1$.

Let X be an independent set in R of size i. By Proposition 2.7 X contains no vertex in D_1 and so $X \subseteq A_1 \cup C_1$. Now suppose $|X \cap A_1| = s$, and then $|X \cap C_1| = i - s$. Since $T_1 \subseteq A_1$, we have

$$|A_1| \geq s + k. \tag{8}$$

On the other hand, since A_1 is a tight cut in G (Theorem 2.6), $-k+1 \leq \text{def}'(G) = \text{odd}_G(A_1) - |A_1|$ so

$$|A_1| \leq \text{odd}_G(A_1) + k - 1. \tag{9}$$

Combining inequalities (8) and (9), we obtain $\text{odd}_G(A_1) \geq s+1$. Since $\text{odd}_G(A_1) \leq i-1$, we get $s \leq i-2$.

If $X \cap A_1 = \emptyset$, then $X \subseteq C_1$. Replacing any vertex in X with one in D_1, since there is no edge between C_1 and D_1 by Theorem 2.6, the resulting set is an independent set in R of size i, which leads to a similar contradiction with Case 1.

Hence $1 \leq s \leq i-2$. Let $G_1, \cdots, G_s$ be components in $Odd_G(A_1)$ and choose one vertex v_j from each component G_j, $j = 1, \cdots, s$. Replacing the vertices of X in A_1 with $v_1, \cdots, v_s$, then we get a new independent set X' of size i in R which contains at least one vertex in D_1. But by assumption X' is an extreme set, which is a contradiction by Proposition 2.7. Thus our proof is complete. □

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